Fuzzballs and Random Matrices

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24 February 2023, DTU Da Nang

- Based on arXiv:2208.14744, arXiv:2301.11780 with Suman Das, Sumit Garg, Preetham Kumar, Arnab Kundu

- The nature of the black hole horizon has been a source of confusion ever since the invention/discovery of the Schwarzschild metric.
- This debate has gone through various iterations, and has recently been revived in the context of the information paradox [Hawking, Page, Mathur, AMPS].
- Today, we will view this as a question about the meaning and interpretation of black hole microstates in string theory.
- These have the same macroscopic charges as a black hole in the bulk, but at least in the supergravity approximation, candidate microstate solutions [Lunin-Mathur, Bena-Warner-et-al, ...] cap off before the horizon and are completely regular.
- They provide the foundation for the fuzzball program.

There are a few facts about these solutions that I find remarkable:

- It is non-trivial that solutions with a throat that caps off before the horizon exist in string theory, even though in classical general relativity they are ruled out by no hair theorems and the like.
- Far from the throat they are locally indistinguishable from the black hole – they have the same charges as the black hole. These solutions can be black hole mimickers all the way up to the horizon, with the “hair” only in the cap region deep in the throat. In this, they are quite distinct from say stars or even neutron stars. This qualitative feature would be surprising, if these solutions had no significance for black holes.

- There are whole moduli spaces of such solutions. This is necessary if they are to be interpreted as microstates of the black hole (in the classical limit).
- In some limited cases, these spaces can be fully determined and successfully geometrically quantized to yield the entropy of the black hole [Rychkov] including the precise numerical coefficient [Avinash-CK].

But there are also flies in the ointment:

- Modulo mior caveats, microstate solutions are only known for BPS black holes which are at zero temperature.
- An obvious criticism regarding supergravity microstates is that the more complicated profile functions that capture generic fuzzballs will have high (presumably Planckian) curvature and are not reliable in supergravity or even tree level string theory. But once one is in the regime of quantum string theory, there is an operational lack of clarity (at least in my opinion) about what it means to say that spacetime caps off at the horizon for individual microstates.

- It has been argued [Sen] that the fuzzball solutions should be *added* to the entropy of the black hole, depending on the duality frame. See also counter-arguments against this by [Mathur-Turton].
- In the 3-charge case, the solution spaces discovered so far ("superstrata") only account for a subleading fraction of the black hole entropy.

- Any claim that microstates cap off at the horizon raises various dynamical/thermodynamic questions, which have relatively simple (or at least well-known) explanations in terms of the black hole picture, but are very challenging in terms of microstates. The most basic of these are questions of smooth infall, the nature of Kruskal coordinates and the interpretation of Hawking's original computation of Hawking radiation.
- More generally, the problem of understanding various aspects of horizon physics in a dynamical setting via an ensemble of horizonless microstates, is clearly an outstanding challenge.

Fuzzballs are one take on black holes in quantum gavity. There is also an alternate approach that I will call the semi-classical approach.

- It draws inspiration from semi-classical gravity and (holographic) entanglement entropy.
- Interesting results: horizon physics involves chaos, scrambling and random matrices [Sekino-Susskind, Stanford-Shenker, MSS, Cotler-et-al.]
- A recent success: Page curve of an evaporating black hole can be reproduced from ideas about holographic entanglement entropy that were introduced for other reasons. [Penington, Almheiri-et-al.]

- Challenges: While the final Page curve in the island paradigm is compatible with unitarity, the detailed emergence of unitarity at each epoch of Hawking radiation has raised numerous questions related to ensemble averaging and factorization [Shenker & co.].
- The calculation is ultimately Euclidean, the Lorentzian time evolution is put in by hand, in the epoch by epoch nature of the calculation [Vyshnav-CK].
- There is evidence that semi-classical gravity should be viewed as an ergodic proxy for a time average during each epoch of Hawking radiation [Rozali et al, Liu-Vardhan, Vyshnav-CK, more recently many others].
- In both fuzzball approach and the semi-classical approch, the emergence of a smooth horizon still remains mysterious.

- A natural goal: what is one to make of this situation?
- Perhaps a synthesis of both approaches is necessary? [Das-CK-Kumar-Kundu]
- Not simply because we wish to be reconcilatory, but because both camps have (certainly) had interesting results and (arguably) successes, but also they both face profound challenges.
- How might one go about trying to synthesize the two viewpoints?

- Can one reproduce the successes of one approach using the tools of the other?
- We will take a step in this direction.
- Specifically, can we reproduce the level spacing distribution and spectral form factor expected for black holes from semi-classical arguments [Cotler et al.], ...
- ... but from fuzzballs?
- These are supposed to be random matrix diagnostics. If the spectrum exhibits level repulsion, it is expected to show a linear ramp.

- Definition of SFF:

$$g(\beta,t) = \frac{|Z(\beta,t)|^2}{|Z(\beta,0)|^2}. \quad (1)$$

- For a given quantum mechanical system

$$Z(\beta,t) = \mathrm{Tr}\left[e^{-(\beta-it)H}\right] = \sum_{\omega} e^{-(\beta-it)\omega} \quad (2)$$

where β, t and H are inverse temperature, time and the Hamiltonian respectively.

- In RMT systems, SFF has a distinct linear ramp on a log-log plot:

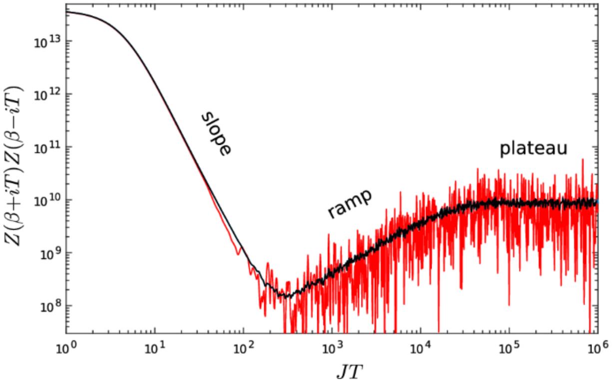

Figure: A typical RMT SFF taken from google.

- Level spacing distribution of chaotic vs integrable systems.

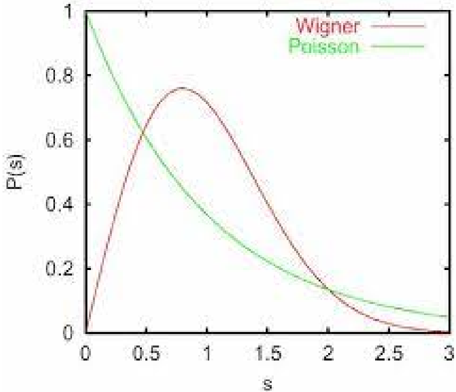

Figure: Again taken from google.

- The expectation is that fuzzballs cannot do these, because the spectrum of normal modes of a capped off geometry is roughly evenly spaced (think of standing waves in global AdS), and so we expect something akin to an integrable system.
- in other words, we expect – level spacing distribution without repulsion, and spectral form factor without linear ramp.
- Explicit fuzzball “geometries” are inaccessible at finite temperature horizons. So this is a heuristic expectation.

- But perhaps, one does not need explicit constructions in string theory to test this. [Das-CK-Kumar-Kundu]
- The features we are looking for are expected to be generic, robust features of black hole microstates.
- The ramp is also a semi-qualitative feature.
- So perhaps we can simply chop off the geometry before the horizon as a toy model for a fuzzball, and see how the modes behave?

- This would mean that we would be computing the normal modes of a black hole instead of quasi-normal modes, by introducing a stretched horizon.
- We can impose Dirichlet boundary conditions at the stretched horizon (akin to the brick wall model of ’t Hooft).
- We will do this for a horizon where the wave equation is reasonably tractable: BTZ.

- Consider a scalar field Φ on the BTZ black hole

$$ds^2 = -(r^2 - r_h^2)dt^2 + \frac{dr^2}{(r^2 - r_h^2)} + r^2 d\psi^2 \tag{3}$$

where $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \psi < 2\pi$.

$$\Box\Phi = m^2\Phi \tag{4}$$

This can be solved by

$$\Phi(t, r, \psi) \sim \sum e^{iJ\psi} e^{-i\omega t} \phi_{\omega,J}(r) \tag{5}$$

where J is integer (due to periodicity in ψ direction).

- The radial part takes the form

$$(r^2-1)^2\frac{d^2\phi(r)}{dr^2}+2r(r^2-1)\frac{d\phi(r)}{dr}+\omega^2\phi(r)-V(r)\phi(r)=0 \qquad (6)$$

- Demand normalizable boundary conditions at infinity.
- Demand Dirichlet boundary conditions at horizon: $\phi(r=r_0)=0$, where r_0 is the location of the stretched horizon.
- (I don't believe either condition is important for the linearity of the ramp - what is important is that one b.c is close to the horizon, and the other is far from it. A caveat, later.)

- Two conditions are enough to fix the spectrum and we find $\omega(n, J)$.
- When we compute this, we find a wrinkle in the conventional expectation.
- While it is true that the spectrum has linear dependence on the n quantum number, the dependence on J is non-trivial.
- So when we compute the (truncated) SFF ...

$$Z(\beta, t) = \sum_{\omega} e^{-(\beta - it)\omega} = \sum_{J=-J_{cut}}^{J_{cut}} \sum_{n=1}^{n_{cut}} e^{-(\beta - it)\omega_{n,J}}. \tag{7}$$

- ...we find,

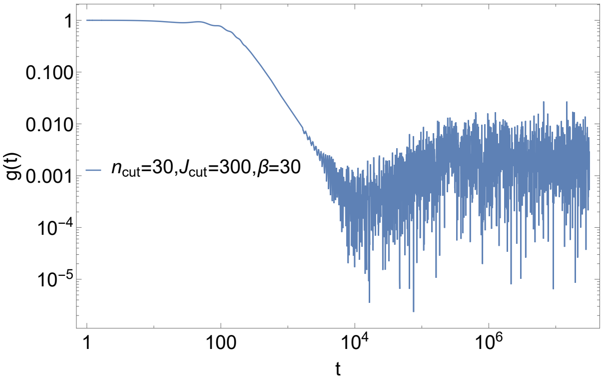

Figure: A typical SFF for BTZ.

Key Observation:

- The spectrum in the J-direction is key for getting the ramp. If the sum is over enough J's, we always see the ramp, it doesn't matter how many n's we keep. Often we will work with $n = 1$.
- By plotting a slope one line through the ramp, we can convince oneself that it is consistent with ~ 1.

- We can see this better by taking an ensemble average over a small number of stretched horizon radii:

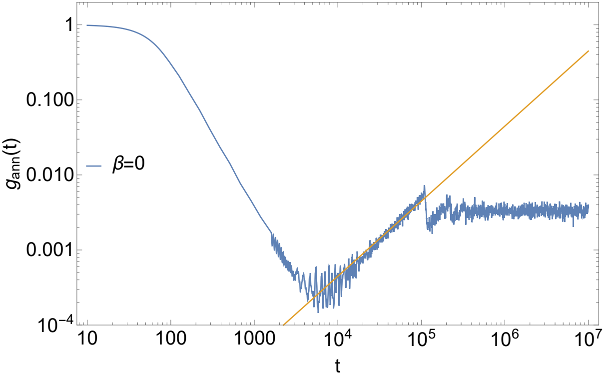

Figure: Ensemble averaging is optional, and only for clarity.

- A ramp of slope 1 in the SFF is usually attributed to RMT.
- Typical integrable systems have no ramps.
- One may find a ramp whose slope is not ~ 1 by doing an average over random couplings in integrable systems [Lau-Ma-Murugan-Tezuka].
- The surprsing thing here is that it seems our system does not have conventional level repulsion and is therefore not technically RMT, and yet there is a ramp.

- The spectrum has interesting structure ...

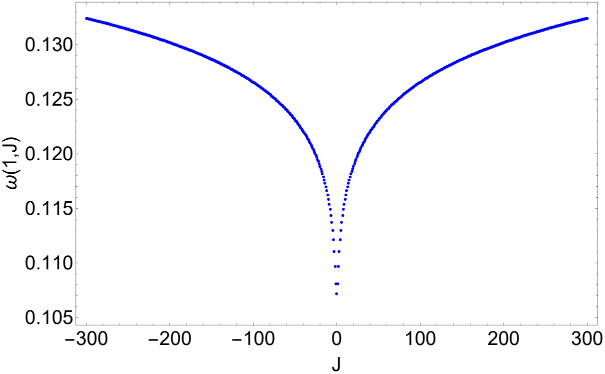

Figure: $\omega(n, J)$ for fixed n.

- ...and the level spacing of the full spectrum looks Poisson...

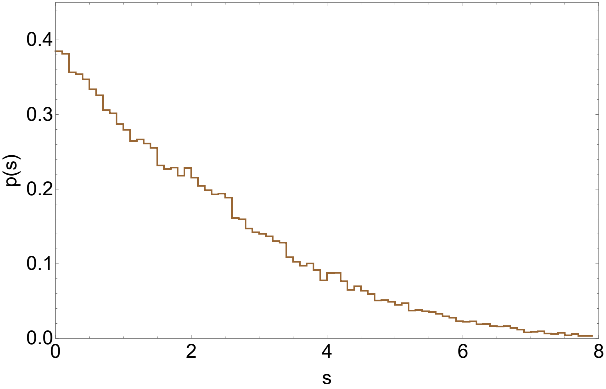

Figure: Level spacing for moderately large n and J.

- ...but the level spacing for fixed n looks like an extreme version of a Wigner-Dyson curve, if you squint at it.

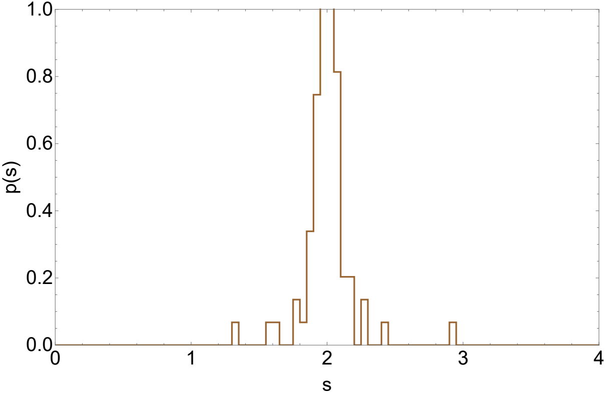

Figure: Level spacing as a function of J, for fixed n.

Aside:

- The compact directions are crucial. This is a feature that one often sees in discussions of fuzzballs and how they evade various no-go results.
- The J-dependence of the spectrum getting pulled down due to the horizon was key, for the emergence of the ramp. In AdS as well as flat space with a hole/box, the spectrum has linear dependence on J and this is what prevents the ramp.
- We expect that this is a universal feature of horizons, as long as there are compact extra dimensions. The results are qualitatively identical for Rindler $\times X$, where also the wave equation is solvable in terms of well-studied special functions.

Aside:

- In the semi-classical approach, the ramp in SFF is thought of as the result of including replica wormholes in the path integral. This leads to a smooth ramp.
- It is expected that the true quantum gravity microstates should reproduce the ramp, but with fluctuations.
- Our calculation reproduces both the ramp and the fluctuations – somehow putting a stretched horizon seems to know some aspects of quantum gravity.

- Crucial Question:
- Can we get conventional (Wigner-Dyson-like) level repulsion together with this linear ramp?
- Yes! [Das-Garg-CK-Kundu]
- Key idea: We take hint from the profile functions of conventional BPS fuzzballs, and instead of $\phi = 0$ at the stretched horizon, demand

$$\phi = \phi_0(\theta) \tag{8}$$

  for some choice of "generic" profile.
- Eg: take the Fourier coefficients of $\phi_0(\theta)$ to be Gaussian distributed.

- As the variance is tuned from zero to large, the LSD goes over from “extreme" RMT to more conventional WD spectra to eventually, Poisson.
- The linear ramp also goes away as we increase the variance.

- Zero Variance:

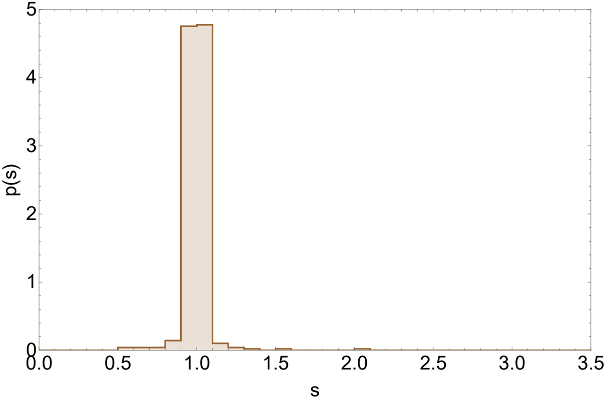

Figure: “Extreme" RMT level spacing.

- Small Variance:

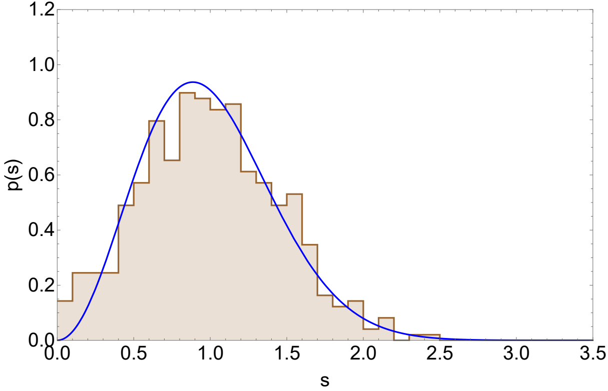

Figure: Blue curve is LSD for GUE

- Large Variance:

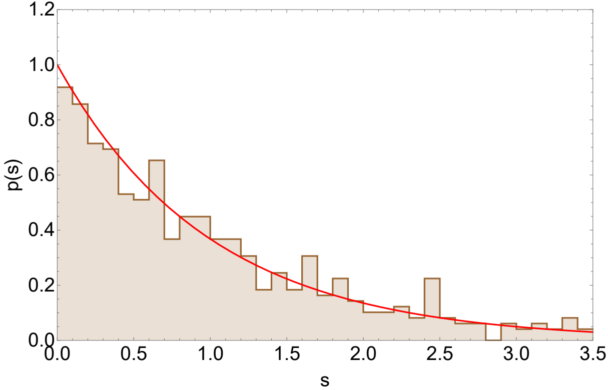

Figure: Red curve is Poisson LSD

- The ramp also goes away for large variance...

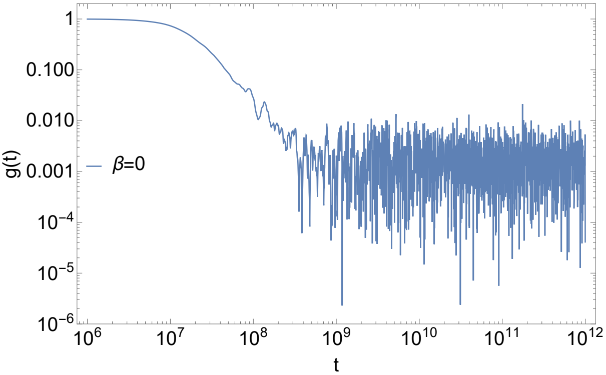

Figure: SFF for identical parameters as previous slide.

Some RMT Takeaways:

- RMT/chaos folks tell me that getting a linear ramp cannot happen without the three Wigner-Dyson classes (or perhaps their seven Altland-Zernbauer generalizations).
- Our approach provides a simple and general way to generate counter-examples to this. “Experimentally”, we have some sufficiency conditions for when a deterministic sequence of real numbers (eigenvalues), can give rise to ramps.
- Some intriguing observations about the simple harmonic oscillator. Steeper vs shallower ramp. Noisy spectra.
- The linear ramp may be a more robust diagnostic of underlying chaos than level spacing data.

Black Hole Takeaways:

- Our result can be viewed as encouraging for the fuzzball program. But the real challenge (in my view) for fuzzballs, is that it is ill-defined as a proposal for the full quantum microstates. The trouble is not that the statement is wrong, but that there is no statement. All we know are some classical BPS solutions.
- But unlike the opponents of fuzzballs, I find the heuristic idea of "geometry stopping at the horizon" stimulating.
- Perhaps our calculation is a hint, as to how to give an operational definition of a bulk quantum microstate at finite temperature.
- Many many more open questions/directions...

Many others, but for now ...

- Thank You For Your Attention!