Comments on Krylov Complexity

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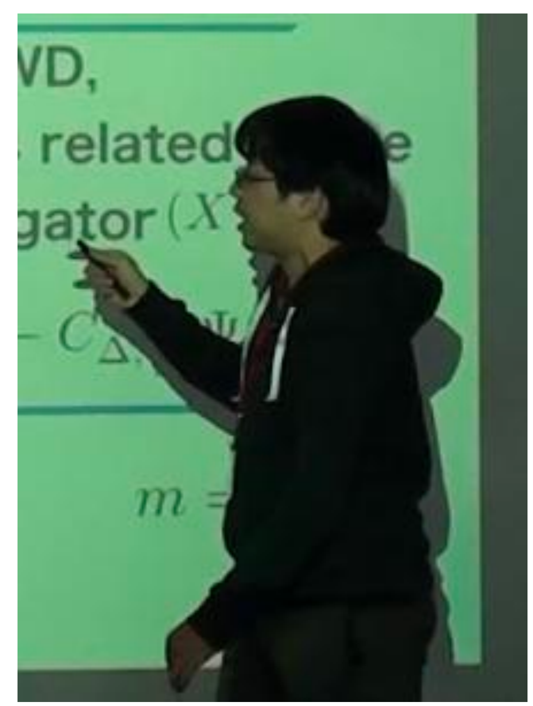
arXiv > hep-th > arXiv:2212.14702

High Energy Physics – Theory

[Submitted on 30 Dec 2022 (v1), last revised 25 Jan 2023 (this version, v2)]

Krylov Complexity in Free and Interacting Scalar Field Theories with Bounded Power Spectrum

Hugo A. Camargo, Viktor Jahnke, Keun-Young Kim, Mitsuhiro Nishida



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arXiv > hep-th > arXiv:2212.14429

High Energy Physics – Theory

[Submitted on 29 Dec 2022]

Krylov complexity in quantum field theory, and beyond

Alexander Avdoshkin, Anatoly Dymarsky, Michael Smolkin

A Universal Operator Growth Hypothesis

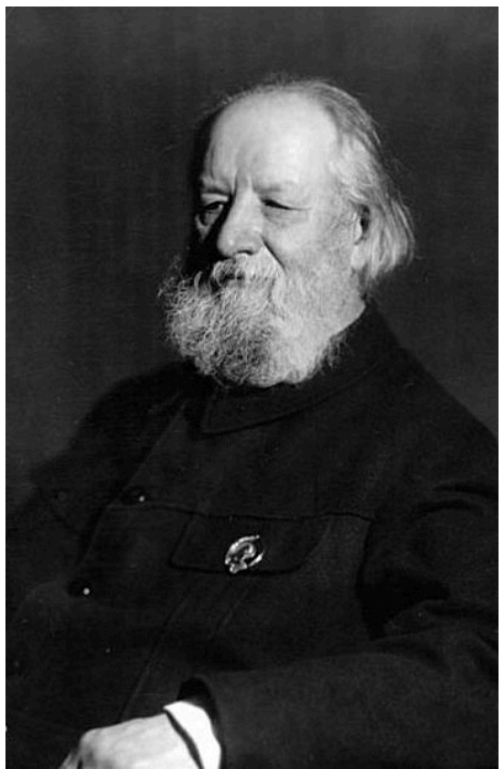
#1

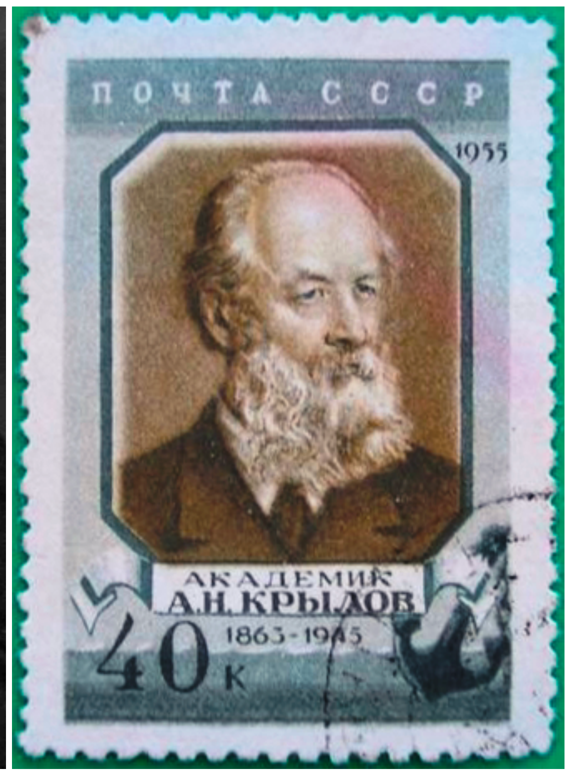
Daniel E. Parker (UC, Berkeley), Xiangyu Cao (UC, Berkeley), Alexander Avdoshkin (UC, Berkeley), Thomas Scaffidi (UC, Berkeley), Ehud Altman (UC, Berkeley) (Dec 20, 2018)

Published in: *Phys.Rev.X* 9 (2019) 4, 041017 • e-Print: 1812.08657 [cond-mat.stat-mech]

Comments on Krylov Complexity

Aleksey Nikolaevich Krylov (1863 –1945)
a Russian naval engineer, applied mathematician and memoirist.





Complexity Equals Anything

By Shan-Ming Ruan

Holographic complexity ⟷ QFT complexity

Ambiguity   Ambiguity

Not well-defined!!!

Complexity

FO What is complexity? **ChatGPT**

Complexity refers to the level of difficulty involved in understanding, analyzing, or managing a system, problem, or process. It often refers to the number of components, interconnections, interactions, or variables involved in a particular system or process.

(Computational) complexity [Computer science] quantifying the difficulty of carrying out a task.

(Circuit) complexity Quantum Computer ~ Quantum Circuit

Minimal number of gates for the transformation from the reference to target state

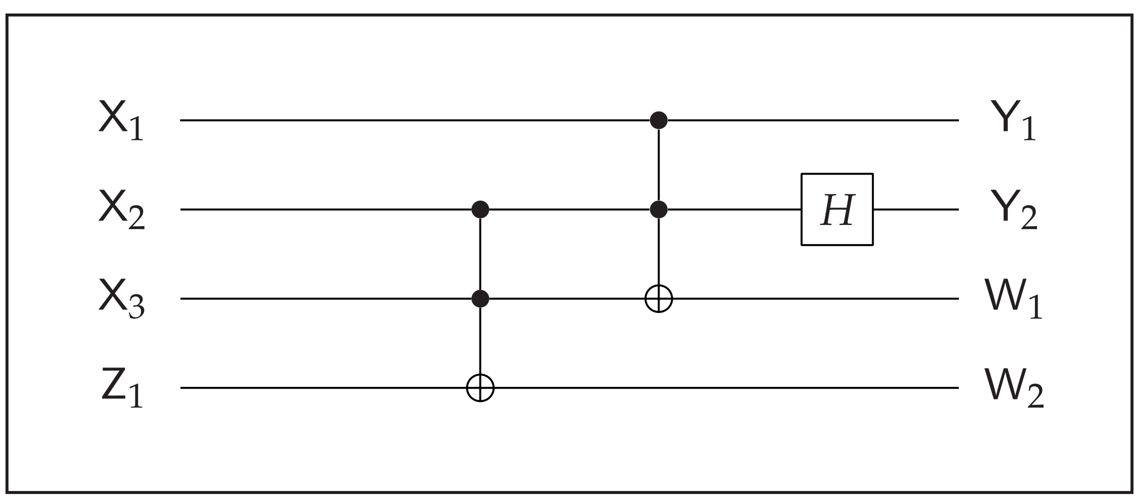
$$|\psi_T\rangle = U|\psi_R\rangle = g_n g_{n-1} \cdots g_2 g_1 |\psi_R\rangle$$

Universal gate sets = {a,b,c,d,e,f}

$$G = dbe$$
$$G = ceab$$
$$G = abefa$$

complexity = 3

ex)



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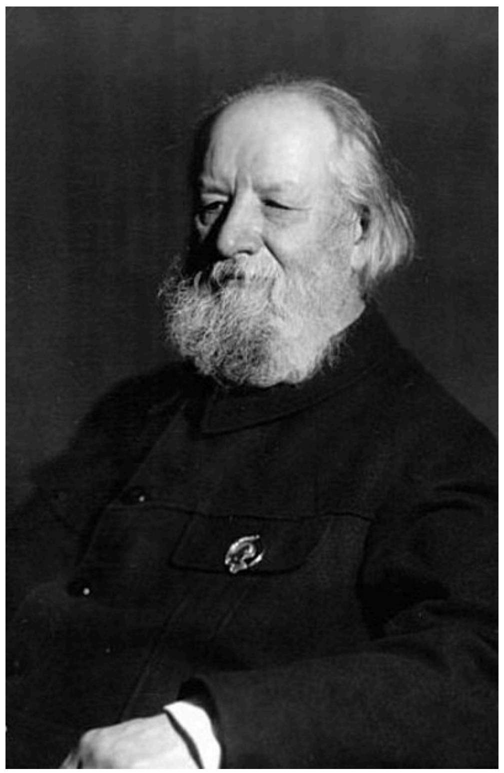

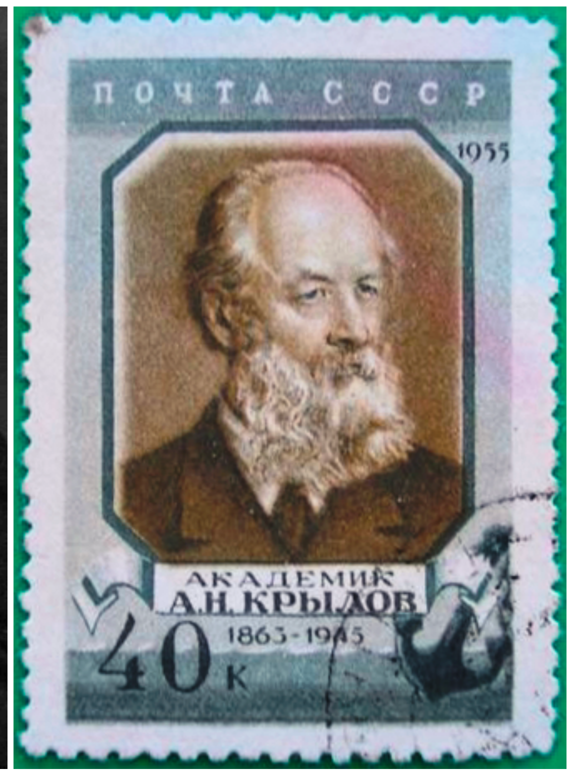

Complexity Equals Anything

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Not well-defined!!!

Holographic complexity ⟷ QFT complexity

Ambiguity Ambiguity

Complexity: how much things are complex

Chaos: how fast things get complex
~ fast time evolution of complexity

"Krylov complexity" is a well-defined concept
proposed as a diagnose of quantum chaos (which is not-well defined)

Comments on Krylov Complexity

- Quantum Chaos
- Krylov complexity as a diagnosis of quantum chaos
  - Krylov space (Hamiltonian, operator)
  - Operator growth
  - Lanczos coefficient
- Examples for lattice systems
  - Universal operator growth hypothesis
- Towards field theory
  - Subtlety in field theory
  - Revision of universal operator growth hypothesis

Cornelius (Cornel) Lanczos (1893-1974):
a Hungarian-American and later Hungarian-Irish mathematician and physicist.

Quantum Chaos

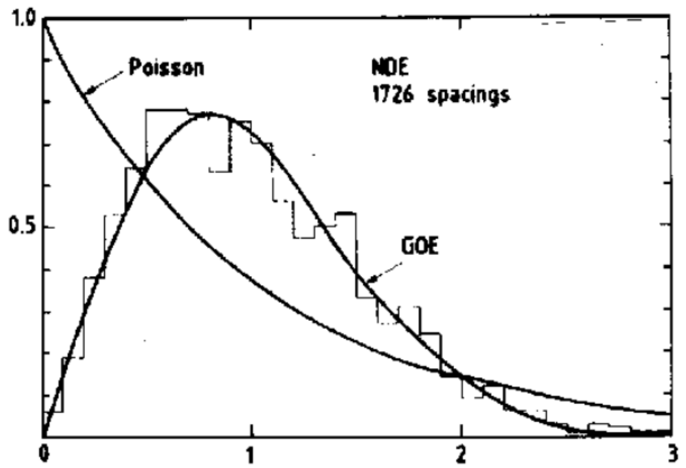
$$\left|\{q^i(t), p^j(0)\}_{PB} = \left|\frac{\partial q^i(t)}{\partial q^j(0)}\right| \sim e^{\lambda t}\right.$$

$$-\langle[q^i(t), p^j(0)]^2\rangle_{\beta},$$

$$-\langle[V(t), W(0)]^2\rangle_{\beta}.$$

Out-of-time-order correlator (OTOC)

Level spacing statistics



Eingenstate Thermalization Hypothesis (ETH)

FO

What is quantum chaos?

ChatGPT

Quantum chaos is a field of physics that studies the behavior of quantum systems that are classically chaotic. Classically chaotic systems are those that exhibit sensitive dependence on initial conditions, meaning small changes in the initial conditions lead to vastly different outcomes over time.

In quantum mechanics, chaotic behavior can manifest in the statistical properties of the system's energy spectrum or in the time evolution of its wave function. The study of quantum chaos is concerned with understanding how the behavior of the system changes as classical chaos is introduced or increased, and how this behavior can be observed and measured experimentally.

Quantum chaos has applications in various fields, including solid-state physics, quantum computing, and quantum information theory. It has also led to new insights into the fundamental nature of quantum mechanics and the connections between classical and quantum physics.

Chethan Krishnan: Fuzzballs and Random Matrices   Black hole physics

The time evolution of an operator O by a time independent Hamiltonian H

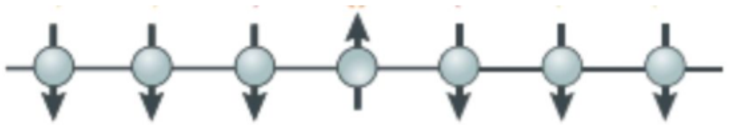
$$\partial_t \mathcal{O}(t) = i\,[H, \mathcal{O}(t)]$$

$$\mathcal{O}(t) = e^{itH}\,\mathcal{O}(0)\,e^{-itH}$$

Baker-Campbell-Hausdorff (BCH) formula $e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{\mathcal{L}_X^n Y}{n!}$

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \tilde{\mathcal{O}}_n \qquad \mathcal{O}(t) = \mathcal{O}_0 + \frac{it}{\hbar}[H, \mathcal{O}] + \frac{(it)^2}{2!\,\hbar^2}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!\,\hbar^3}[H,[H,[H,\mathcal{O}]]] + \cdots \, .$$

$$\tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \qquad \mathcal{L} := [H, \cdot\,] \qquad \mathcal{O}(t) = e^{i\mathcal{L}t}\,\mathcal{O}(0)$$

ex) 1D spin chain 

$$H = -\sum (Z_i Z_{i+1} + g X_i + h Z_i)$$

$$Z_1(t) = Z_1 - it[H, Z_1] - \frac{t^2}{2!}[H,[H,Z_1]] + \frac{it^3}{3!}[H,[H,[H,Z_1]]] + \ldots$$

$$[H, Z_1] \sim Y_1$$

$$[H,[H,Z_1]] \sim Y_1 + X_1 Z_2$$

$$[H,[H,[H,Z_1]]] \sim Y_1 + Y_2 X_1 + Y_1 Z_2$$

$$[H,[H,[H,[H,Z_1]]]] \sim X_1 + Y_1 + Z_1 + X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 + X_1 Z_2 + Z_3 Y_1 + Y_1 Z_2 Y_2 + Z_1 X_2 X_1 + X_2 Z_3 X_1$$

Krylov space: Lanczos algorithm

1812.08657: Parker et al.

The time evolution of an operator O by a time independent Hamiltonian H

$$\partial_t \mathcal{O}(t) = i\,[H, \mathcal{O}(t)] \qquad \partial_t|\mathcal{O}(t)) = i\mathcal{L}|\mathcal{O}(t))$$

$$\mathcal{O}(t) = e^{itH}\,\mathcal{O}(0)\,e^{-itH}$$

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(i\,t)^n}{n!}\tilde{\mathcal{O}}_n \qquad \mathcal{O}(t) = \mathcal{O}_0 + \frac{it}{\hbar}[H,\mathcal{O}] + \frac{(it)^2}{2!\,\hbar^2}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!\,\hbar^3}[H,[H,[H,\mathcal{O}]]] + \cdots .$$

$$\tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \qquad \mathcal{L} := [H, \cdot\,] \qquad \mathcal{O}(t) = e^{i\mathcal{L}t}\,\mathcal{O}(0)$$

- The set of operators $\{\tilde{\mathcal{O}}_n\}$ defines a basis of the so-called *Krylov space* associated to the operator $\mathcal{O}$
- Regard the operator as a state $\mathcal{O} \to |\mathcal{O})$ in the Hilbert space of operators

Inner product: Wightman inner product

$$(A|B) := \langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B\rangle_\beta = \frac{1}{\mathcal{Z}_\beta}\mathrm{Tr}(e^{-\beta H/2}A^\dagger e^{-\beta H/2}B) \qquad \mathcal{Z}_\beta := \mathrm{Tr}\left(e^{-\beta H}\right)$$

Krylov basis

$(\mathcal{O}_m|\mathcal{O}_n) = \delta_{mn}$ (Lanczos algorithm: Gram–Schmidt procedure)

$\{b_n\}$: Lanczos coefficients

$$|\mathcal{O}_0) := |\tilde{\mathcal{O}}_0) := |\mathcal{O}(0))$$

$$|\mathcal{O}_1) := b_1^{-1}\mathcal{L}|\tilde{\mathcal{O}}_0) \qquad b_1 := (\tilde{\mathcal{O}}_0\mathcal{L}|\mathcal{L}\tilde{\mathcal{O}}_0)^{1/2}$$

$$|\mathcal{O}_n) := b_n^{-1}|A_n) \qquad b_n := (A_n|A_n)^{1/2}$$

$$|A_n) := \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$$

$$L_{nm} := (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

$$= b_n\delta_{m,n-1} + b_{n+1}\delta_{m,n+1}$$

Discrete "Schrodinger equation"

$$\partial_t \mathcal{O}(t) = i\,[H, \mathcal{O}(t)]$$

$$\partial_t |\mathcal{O}(t)) = i\mathcal{L}|\mathcal{O}(t))$$

"probability amplitudes"

$$|\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \varphi_n(t) |\mathcal{O}_n) \qquad \varphi_n(t) := i^{-n}(\mathcal{O}_n|\mathcal{O}(t)) \qquad \sum_{n=0}^{\infty} |\varphi_n(t)|^2 = 1$$

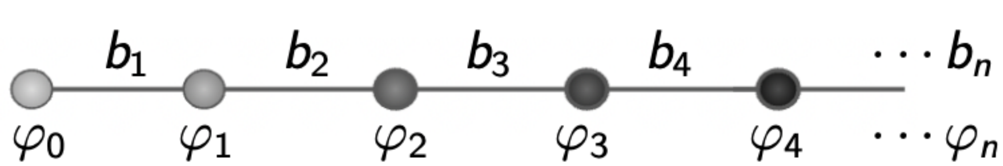
$$L_{nm} := (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} = b_n \delta_{m,n-1} + b_{n+1}\delta_{m,n+1}$$

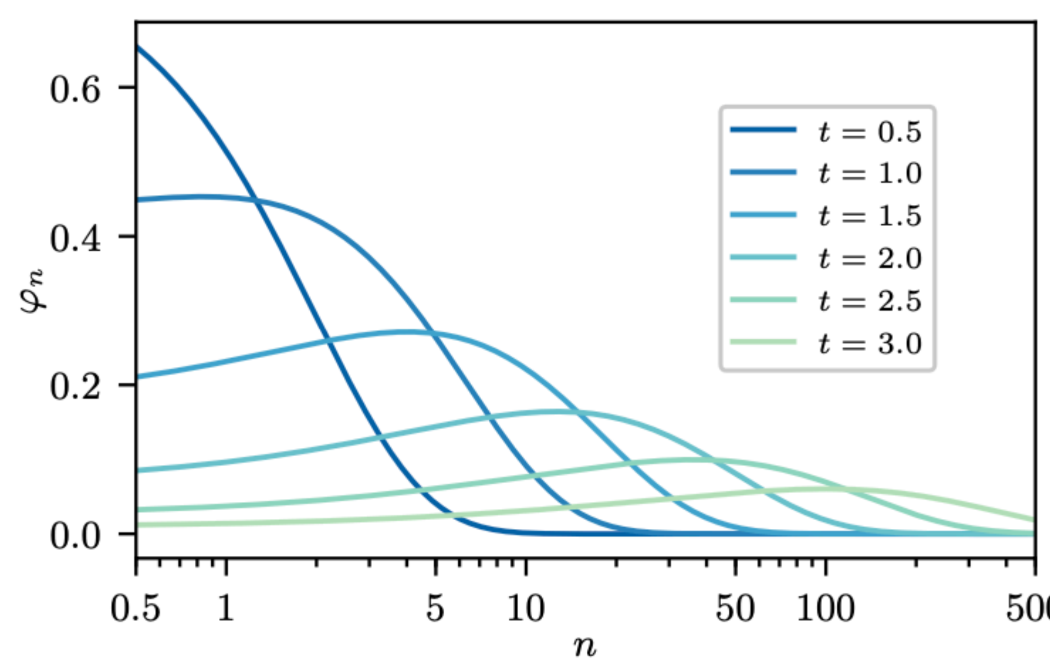
$$\frac{\mathrm{d}\,\varphi_n(t)}{\mathrm{d}t} = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t) \qquad \varphi_n(0) = \delta_{n,0} \qquad \varphi_{-1}(t) \equiv 0 \equiv b_0$$

$$\dot{\varphi}_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1 \varphi_1(t)$$
$$\dot{\varphi}_1(t) = b_1 \varphi_0(t) - b_2 \varphi_2(t)$$
$$\vdots$$
$$\dot{\varphi}_n(t) = b_n \varphi_{n-1}(t) - b_{n+1} \varphi_{n+1}(t)$$

a quantum-mechanical particle on a 1- dimensional chain.

b_n = hopping amplitudes





Krylov complexity

average position over the chain

$$K_{\mathcal{O}}(t) := (\mathcal{O}(t)|n|\mathcal{O}(t)) = \sum_{n=0}^{\infty} n|\varphi_n(t)|^2$$

K-complexity from the auto-correlation function

1812.08657: Parker et al.

Auto-correlation function

$C(t) = \Pi^W(t) = \varphi_0(t)$

$$C(t) := (\mathcal{O}(t)|\mathcal{O}(0)) = \varphi_0(t)$$
$$= \langle e^{i(t-i\beta/2)H}\mathcal{O}^\dagger(0)e^{-i(t-i\beta/2)H}\mathcal{O}(0)\rangle_\beta$$
$$= \langle \mathcal{O}^\dagger(t-i\beta/2)\mathcal{O}(0)\rangle_\beta =: \Pi^W(t)\ .$$

$$\langle\cdots\rangle_\beta = \mathrm{Tr}(e^{-\beta H}\cdots)/\mathrm{Tr}(e^{-\beta H})$$

Power spectrum

$f^W(\omega)$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

Moments

μ_{2n}

$$\Pi^W(t) := \sum_{n=0}^{\infty}\mu_{2n}\frac{(it)^{2n}}{(2n)!} \qquad \mu_{2n} := \frac{1}{i^{2n}}\frac{d^{2n}\Pi^W(t)}{dt^{2n}}\bigg|_{t=0}$$

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

Lanczos coefficients from moments

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$$\mu_2 = b_1^2\,,\quad \mu_4 = b_1^4 + b_1^2 b_2^2\,,\quad \cdots\,.$$

Hankel matrix constructed from the moments.

$$H_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix}$$

$$b_n = \sqrt{M_{2n}^{(n)}}\,,\qquad M_{2l}^{(j)} = \frac{M_{2l}^{(j-1)}}{b_{j-1}^2} - \frac{M_{2l-2}^{(j-2)}}{b_{j-2}^2} \quad \text{with} \quad l = j,\ldots,n\,,$$
$$M_{2l}^{(0)} = \mu_{2l}\quad,\quad b_{-1}\equiv b_0 := 1\quad,\quad M_{2l}^{(-1)} = 0\,.$$

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega)$$

$$f^W(\omega) \longrightarrow \mu_{2n}$$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n} \cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

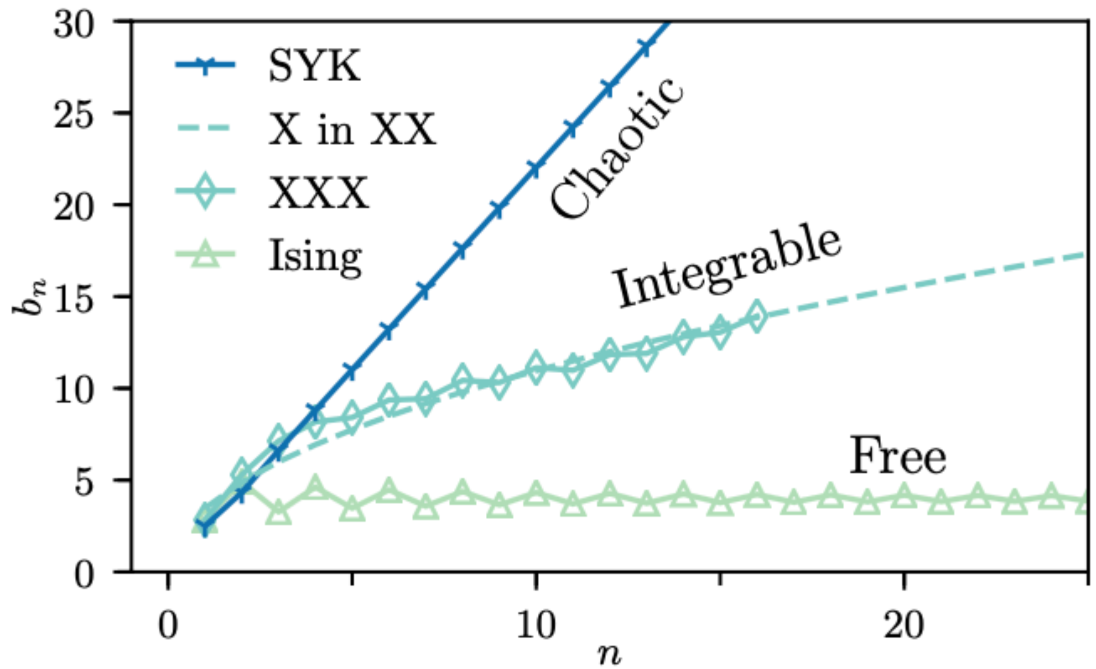
$$C(t) = \Pi^W(t) = \varphi_0(t) \qquad b_n$$

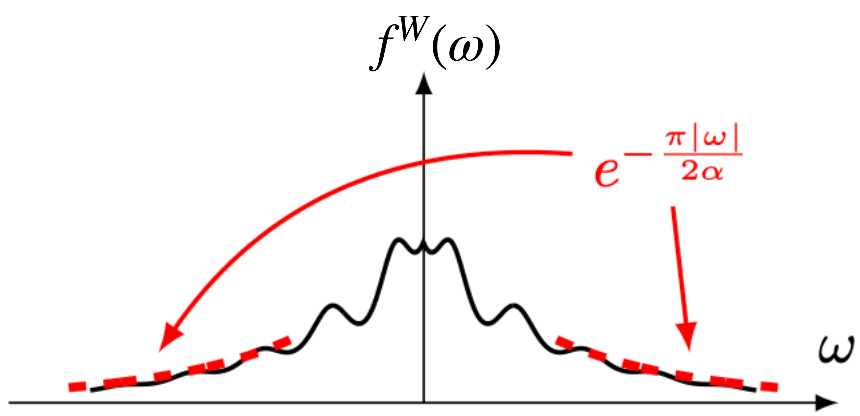
K-complexity

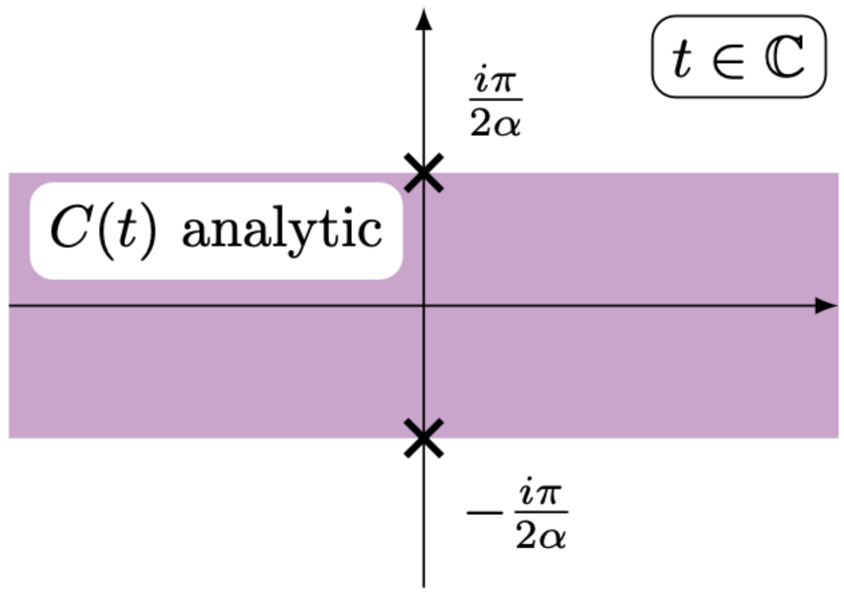
$$\dot{\varphi}_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$
$$\dot{\varphi}_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$
$$\vdots$$
$$\dot{\varphi}_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2, \quad n_{\max} = 200.$$

$$b_n \sim n^\delta \Longleftrightarrow f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$
$$\delta \leq 1$$







Universal operator growth hypothesis

In a chaotic quantum system

Lanczos coefficients $\{b_n\}$ grow as fast as possible

$$b_n \sim \alpha n$$

the slowest possible decay of the power spectrum

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

Krylov complexity grows exponentially

$$K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \quad \Longleftrightarrow \quad b_n \sim \alpha n \quad \Longleftrightarrow \quad K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Wightman 2-point function

$$\Pi^W(t,\mathbf{x}) := \langle \phi(t - i\beta/2, \mathbf{x})\phi(0,\mathbf{0})\rangle_\beta \quad \left(t = \frac{i\beta}{2}\right)$$

Power spectrum

$$C(t) = \Pi^W(t,\mathbf{0})$$

$$f^W(\omega) := \int \mathrm{d}t\, C(t) e^{i\omega t} = \int \mathrm{d}t\, \Pi^W(t,\mathbf{0}) e^{i\omega t}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}} \quad \left(\alpha = \frac{\pi}{\beta}\right)$$

~~In a chaotic quantum system~~ In general QFT
Lanczos coefficients $\{b_n\}$ grow as fast as possible??

$$b_n \sim \alpha n \sim \frac{\pi}{\beta} n$$

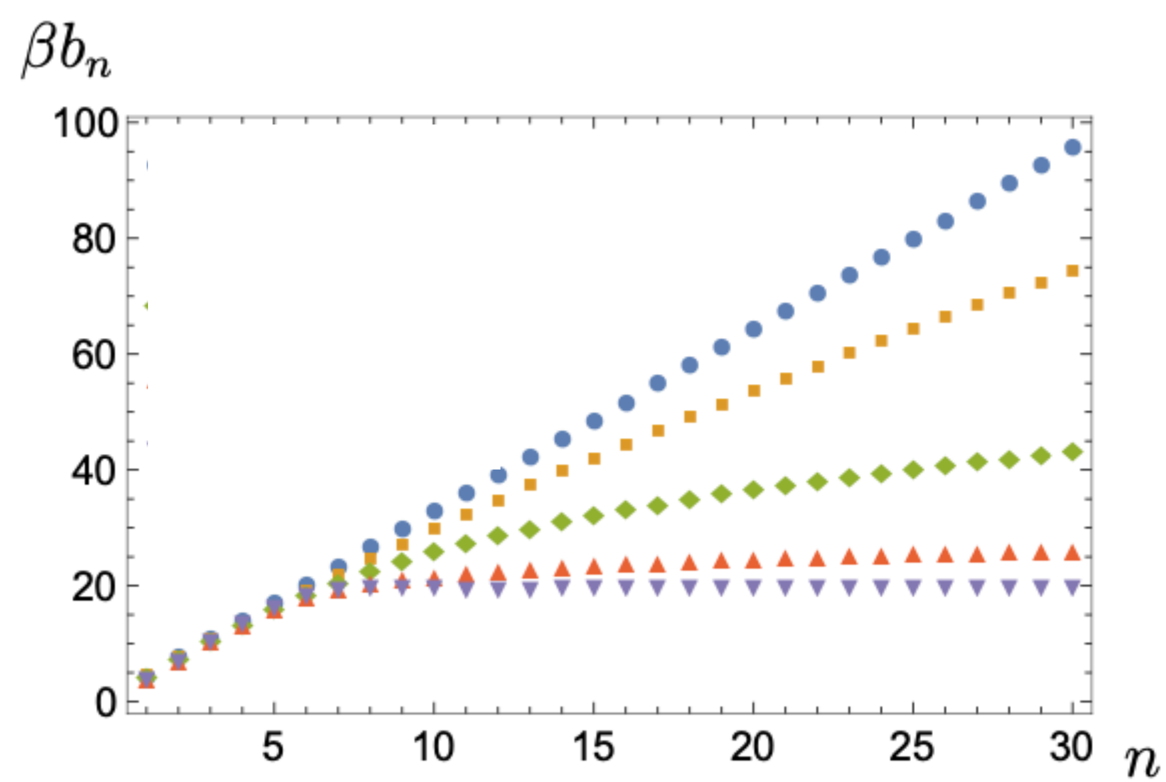
$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Only if b_n is a smooth function of n
Otherwise

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \not\Leftrightarrow b_n \not\sim \alpha n \not\Leftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

m=0, d=4 Free theory is chaotic?

$$f^W(\omega) = \frac{\beta^2\omega}{\pi \sinh(\frac{\beta\omega}{2})}$$



2212.14429: Avdoshikin, Dymarsky, Smolkin
2212.14702: Camargo, Jahnke, KYK, Nishida

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Wightman 2-point function

$$\Pi^W(t,\mathbf{x}) := \langle \phi(t - i\beta/2, \mathbf{x})\phi(0,\mathbf{0})\rangle_\beta \qquad \left(t = \frac{i\beta}{2}\right)$$

Power spectrum

$$C(t) = \Pi^W(t,\mathbf{0})$$

$$f^W(\omega) := \int \mathrm{d}t\, C(t) e^{i\omega t} = \int \mathrm{d}t\, \Pi^W(t,\mathbf{0}) e^{i\omega t}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}} \qquad \left(\alpha = \frac{\pi}{\beta}\right)$$

~~In a chaotic quantum system~~ In general QFT

Lanczos coefficients $\{b_n\}$ grow as fast as possible??

$$b_n \sim \alpha n \sim \frac{\pi}{\beta} n$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Only if b_n is a smooth function of n

Otherwise

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \not\Leftrightarrow b_n \not\sim \alpha n \not\Leftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

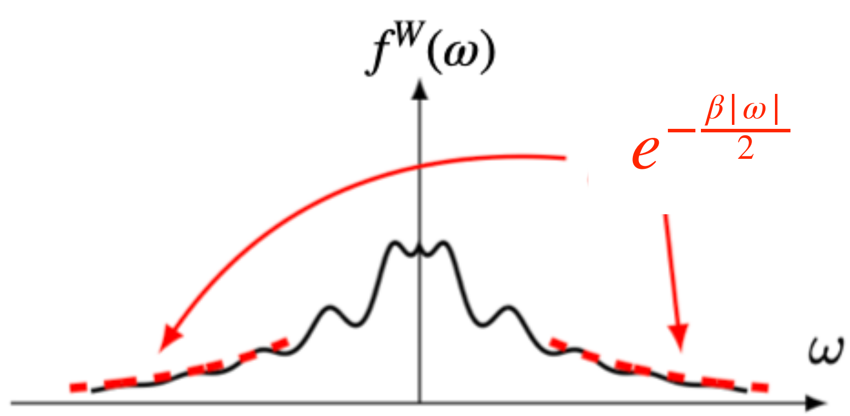
High frequency tail of the power spectrum

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \qquad f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}}$$

Dynamical info (Lattice) vs Kinematical info (QFT)

Need to take into account

- Low frequency behavior
- Sub leading behavior



Free massive scalar in d-dimensions (IR-cutoff)

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Wightman 2-point function

$$\Pi^W(t,\mathbf{x}) := \langle \phi(t-i\beta/2,\mathbf{x})\phi(0,\mathbf{0})\rangle_\beta \ ,$$

$$\Pi^W(\omega,\mathbf{k}) := \int \mathrm{d}t \int \mathrm{d}^{d-1}\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\, \Pi^W(t,\mathbf{x})$$

$$\Pi^W(\omega,\mathbf{k}) = \frac{1}{\sinh[\beta\omega/2]}\rho(\omega,\mathbf{k}).$$

$$\rho(\omega,\mathbf{k}) = \frac{N}{\epsilon_k}[\delta(\omega-\epsilon_k) - \delta(\omega+\epsilon_k)],$$

$$\epsilon_k := \sqrt{|\mathbf{k}|^2 + m^2}.$$

Power spectrum

$$C(t) = \Pi^W(t,\mathbf{0})$$

$$f^W(\omega) := \int \mathrm{d}t\, C(t) e^{i\omega t} = \int \mathrm{d}t\, \Pi^W(t,\mathbf{0}) e^{i\omega t} = \int \frac{\mathrm{d}^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \Pi^W(\omega,\mathbf{k}).$$

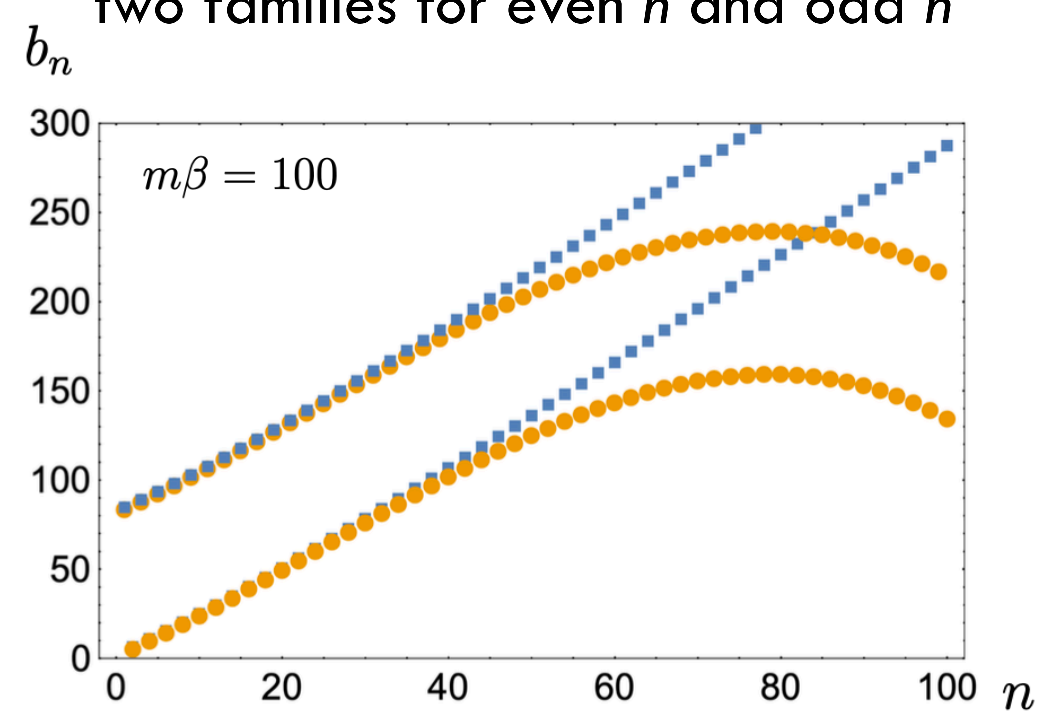
$$f^W(\omega) = N(m,\beta,d)\,\frac{(\omega^2-m^2)^{(d-3)/2}}{|\sinh\left(\frac{\beta\omega}{2}\right)|}\Theta(|\omega|-m)$$

$$\int \frac{\mathrm{d}\omega}{2\pi} f^W(\omega) = 1$$

$$f^W(\omega) \longrightarrow \mu_{2n} \longrightarrow b_n$$

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega) \qquad b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

Staggering: two families for even n and odd n



Non-trivial mass (IR-cutoff) effect: staggering

Power spectrum

$\beta m \gg 1$

$$f^W(\omega) \approx N(m,\beta,d)\, e^{-\beta|\omega|/2} \left(\omega^2 - m^2\right)^{(d-3)/2} \Theta(|\omega| - m) \qquad N(m,\beta,d) = \frac{\pi^{3/2}\beta^{(d-2)/2}}{2^{d-2}\, m^{(d-2)/2} K_{\frac{d-2}{2}}\left(\frac{m\beta}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}$$

$K_n(z)$ is the modified Bessel function of the second kind

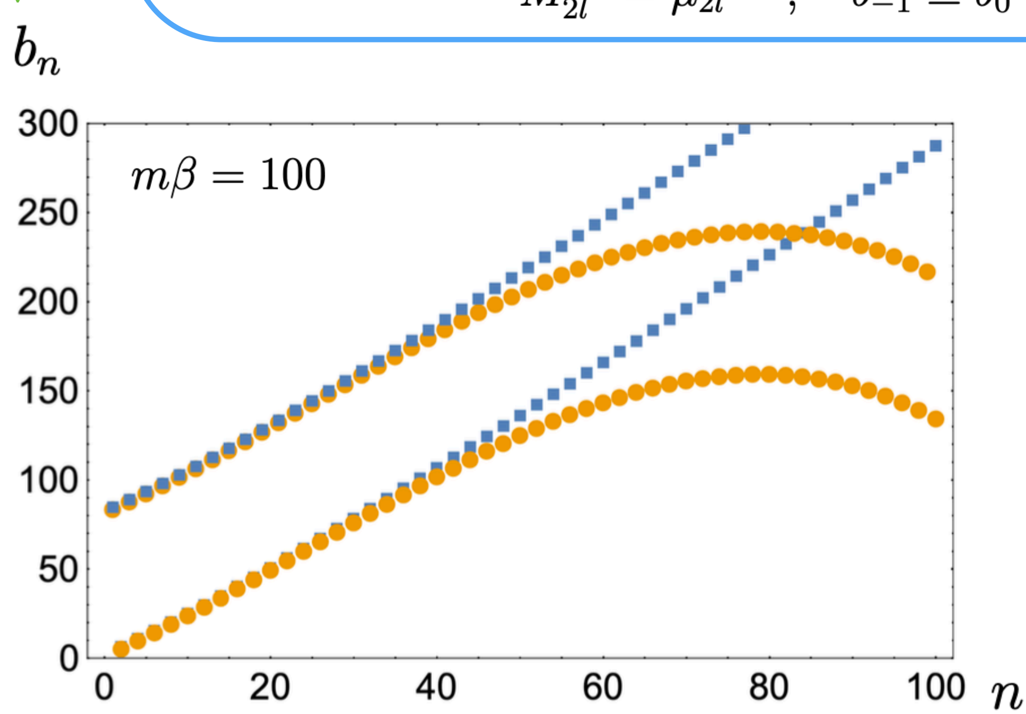
Moments to Lanczos coefficients (d=5)

$\tilde{\Gamma}(n,z)$ is the incomplete Gamma function.

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega) = \frac{2^{-2}e^{\frac{m\beta}{2}}}{2+m\beta}\left(\frac{2}{\beta}\right)^{2n}\left[-m^2\beta^2\,\tilde{\Gamma}\left(2n+1, \frac{m\beta}{2}\right) + 4\tilde{\Gamma}\left(2n+3, \frac{m\beta}{2}\right)\right]$$

$$b_n = \sqrt{M_{2n}^{(n)}}\,, \qquad M_{2l}^{(j)} = \frac{M_{2l}^{(j-1)}}{b_{j-1}^2} - \frac{M_{2l-2}^{(j-2)}}{b_{j-2}^2} \quad \text{with} \quad l = j, \dots, n\,,$$

$$M_{2l}^{(0)} = \mu_{2l} \quad , \quad b_{-1} \equiv b_0 := 1 \quad , \quad M_{2l}^{(-1)} = 0\,.$$



$$\beta^2 b_n^2 = m^2\beta^2 \begin{cases} 1 + 4\frac{1+n}{m\beta} + 8\frac{(n+1)^2}{m^2\beta^2} + 12\frac{(n+1)^3}{m^3\beta^3} + \cdots\,, & \text{for } n \text{ odd}\,, \\ 4\frac{n(n+2)}{m^2\beta^2} + 8\frac{n(n+1)(n+2)}{m^3\beta^3} + \cdots\,, & \text{for } n \text{ even}\,, \end{cases}$$

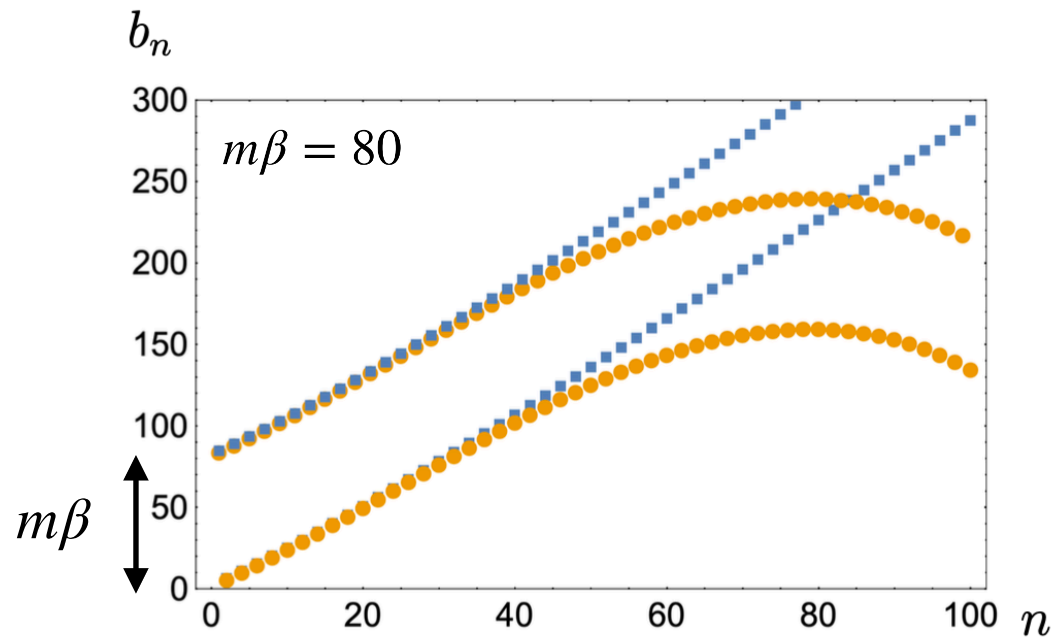
Staggering: two families for even *n* and odd *n*

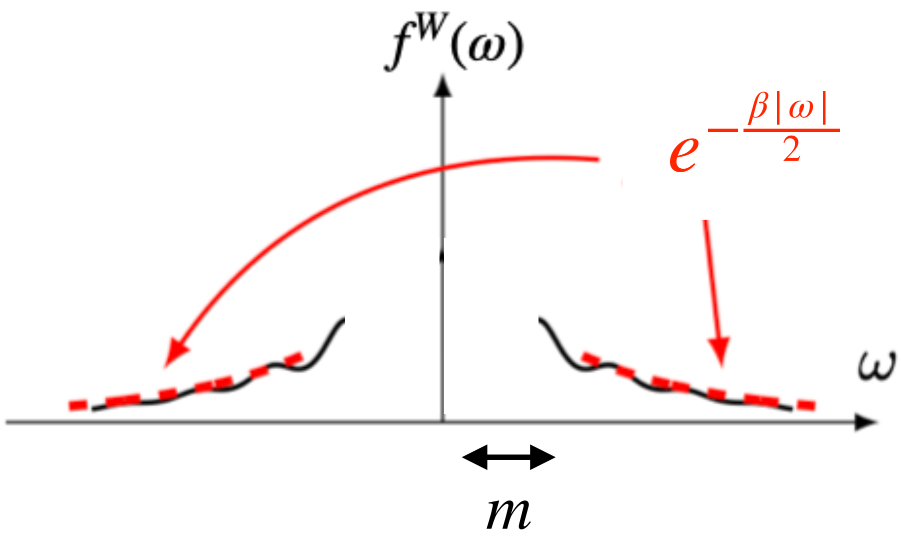
Because b_n is not a smooth function of n

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \not\Leftrightarrow b_n \sim \alpha n \not\Leftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Non-trivial mass (IR-cutoff) effect: staggering

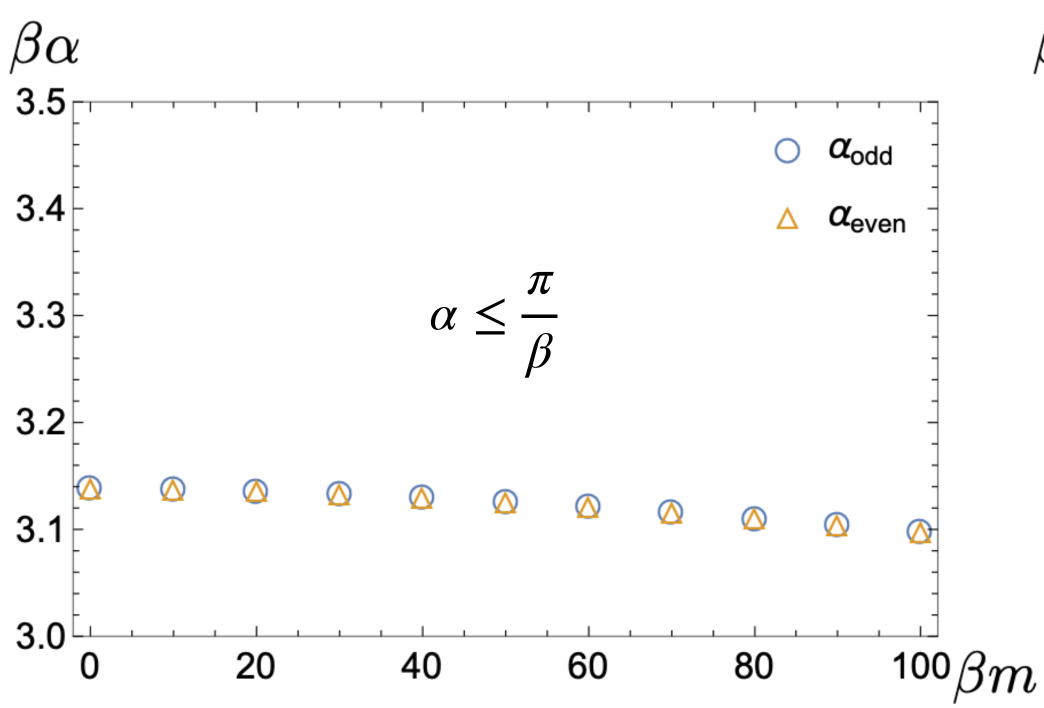
2212.14702: Camargo, Jahnke, KYK, Nishida



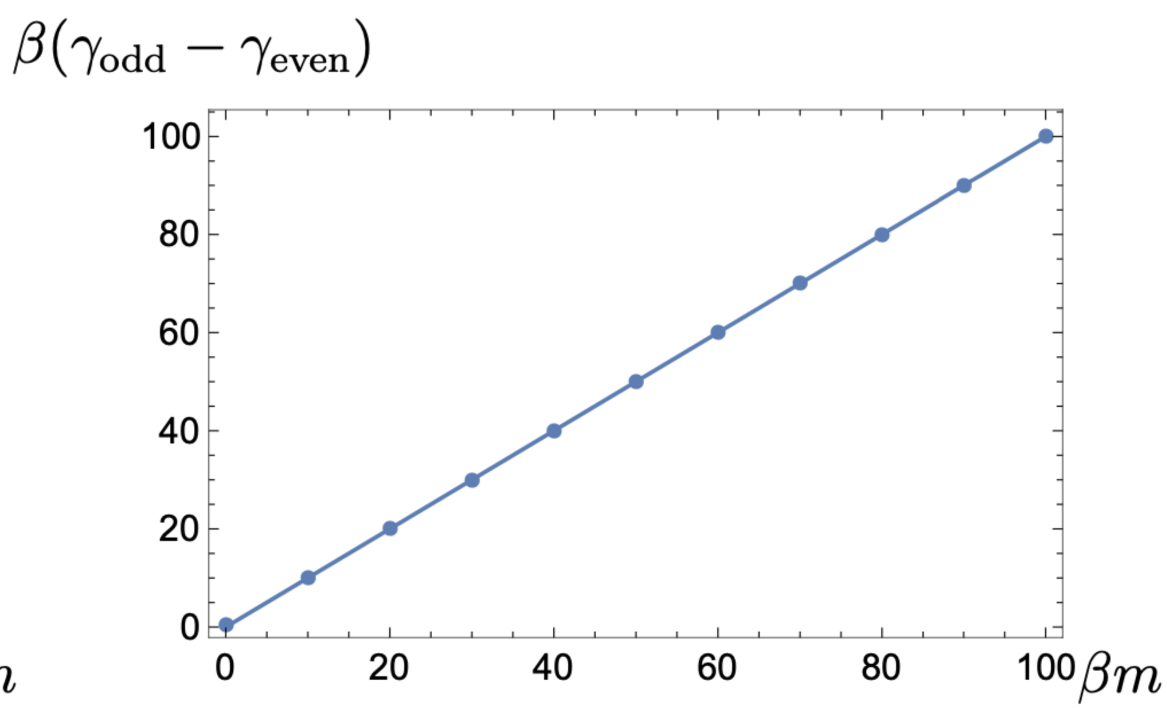


Staggering

$$b_n \sim \alpha_{\rm odd}\, n + \gamma_{\rm odd} \quad (\text{odd } n)$$
$$b_n \sim \alpha_{\rm even}\, n + \gamma_{\rm even} \quad (\text{even } n)$$



(a) Mass-dependence of $\alpha_{\rm odd}$ and $\alpha_{\rm even}$



(b) Mass-dependence of $\gamma_{\rm odd} - \gamma_{\rm even}$

Non-trivial mass (IR-cutoff) effect: K-complexity

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

$f^W(\omega) \longrightarrow \mu_{2n}$

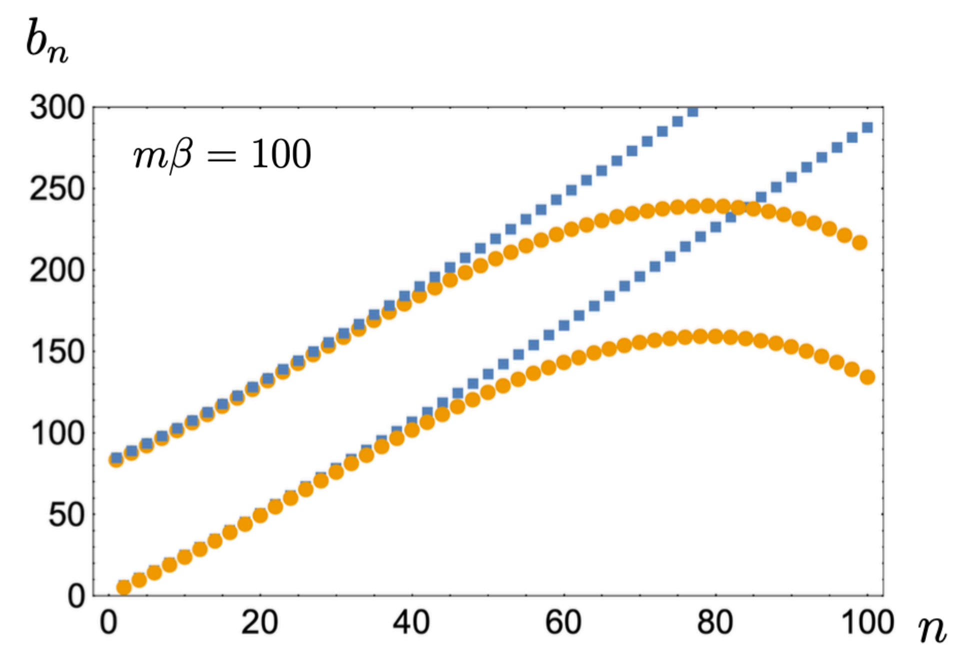
$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$\mu_{2n} \longrightarrow b_n$

$$C(t) = \Pi^W(t) = \varphi_0(t)$$

$$C^{(d)}(t) \equiv \varphi_0^{(d)}(t) = c_1^{(d)}(t)\left(c_2^{(d)}(t)\sin(m\,t) + c_3^{(d)}(t)\cos(m\,t)\right)$$

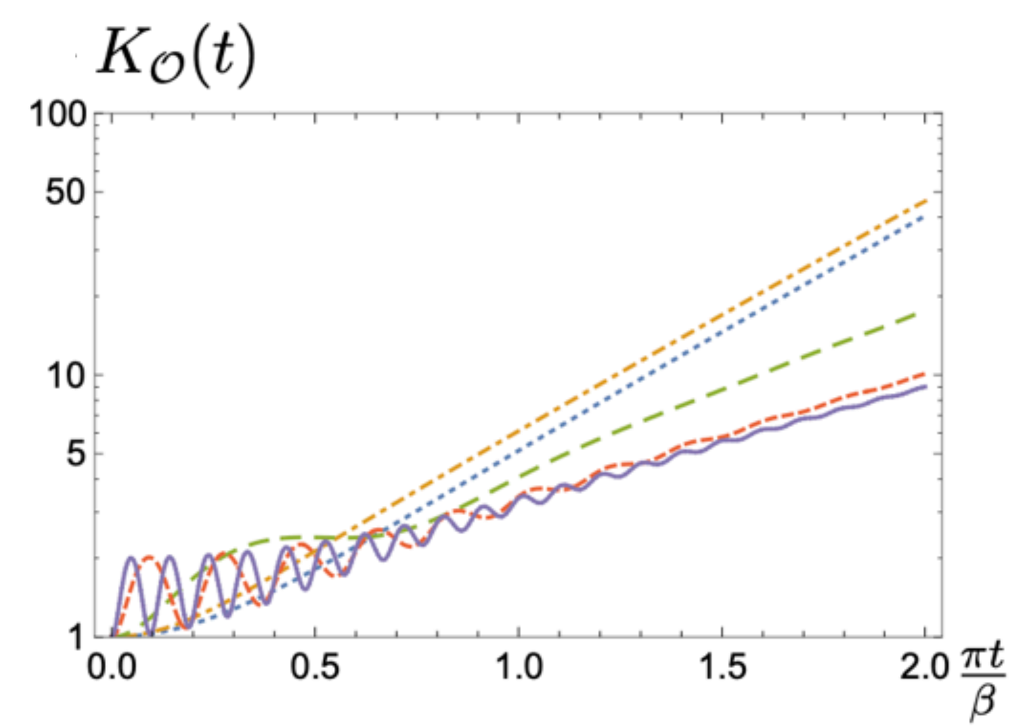


K-complexity

$$\dot\varphi_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$
$$\dot\varphi_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$
$$\vdots$$
$$\dot\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2\,, \quad n_{\max} = 200.$$

$K_O(t)=(d-2)\sinh^2(\pi t/\beta)$
$K_O(t)$ for $\beta m=0$
$K_O(t)$ for $\beta m=10$
$K_O(t)$ for $\beta m=50$
$K_O(t)$ for $\beta m=100$



Non-trivial mass (IR-cutoff) effect: K-complexity

2212.14702: Camargo, Jahnke, KYK, Nishida

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

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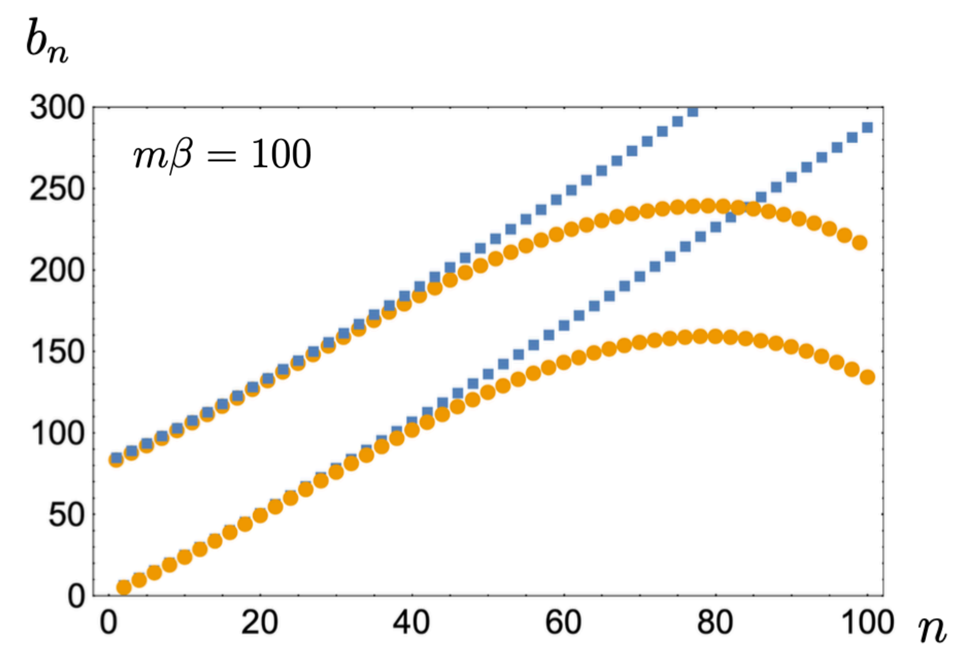
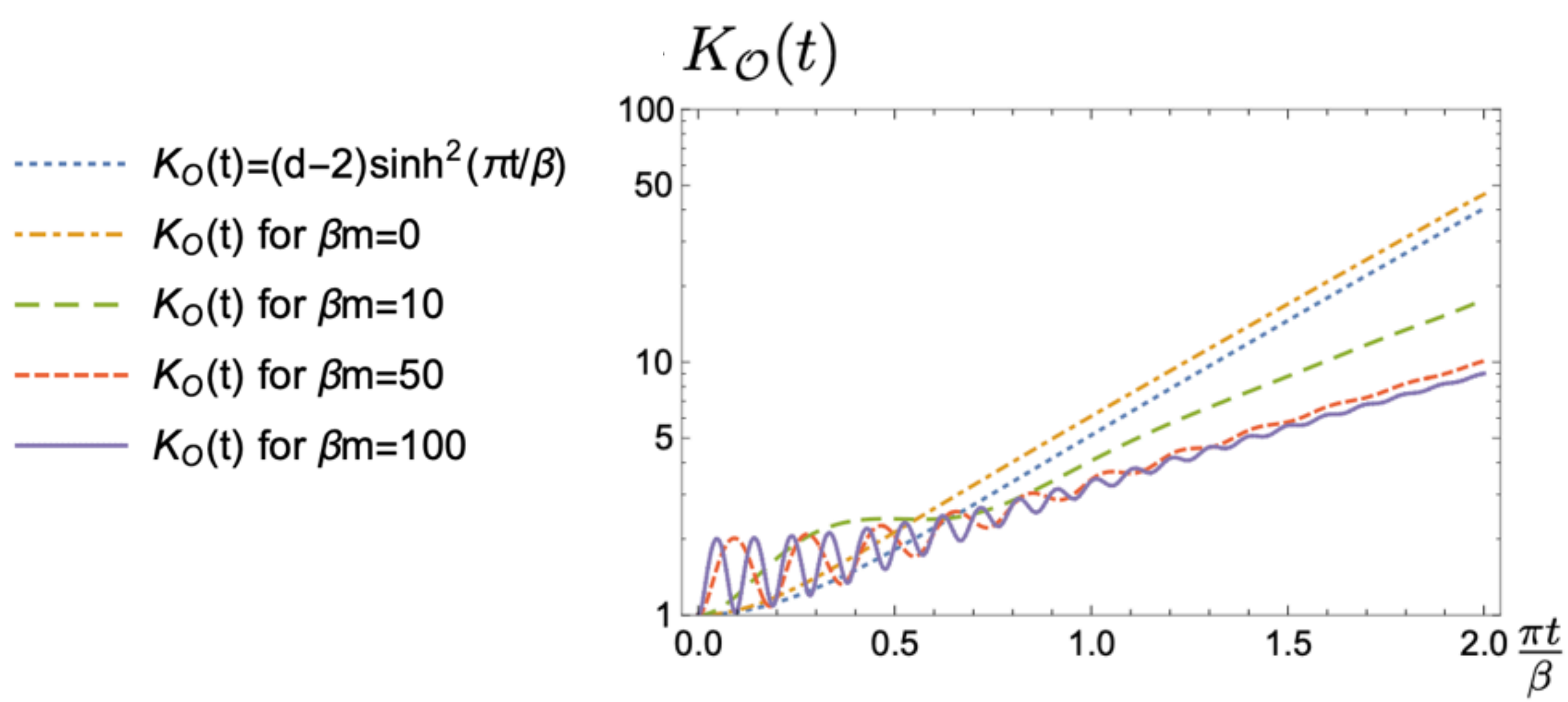
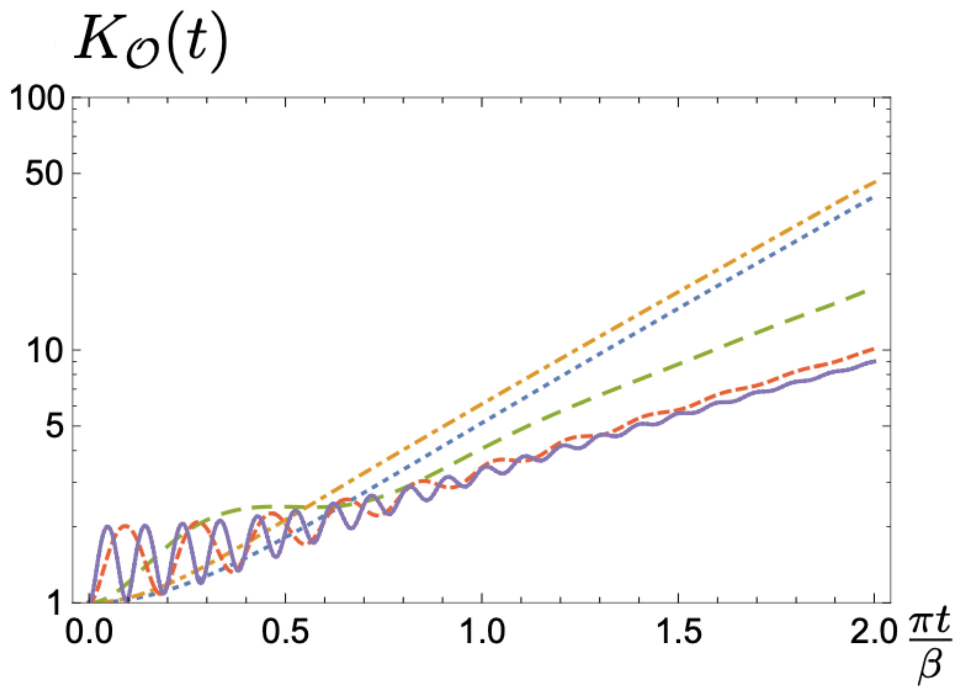


K-complexity

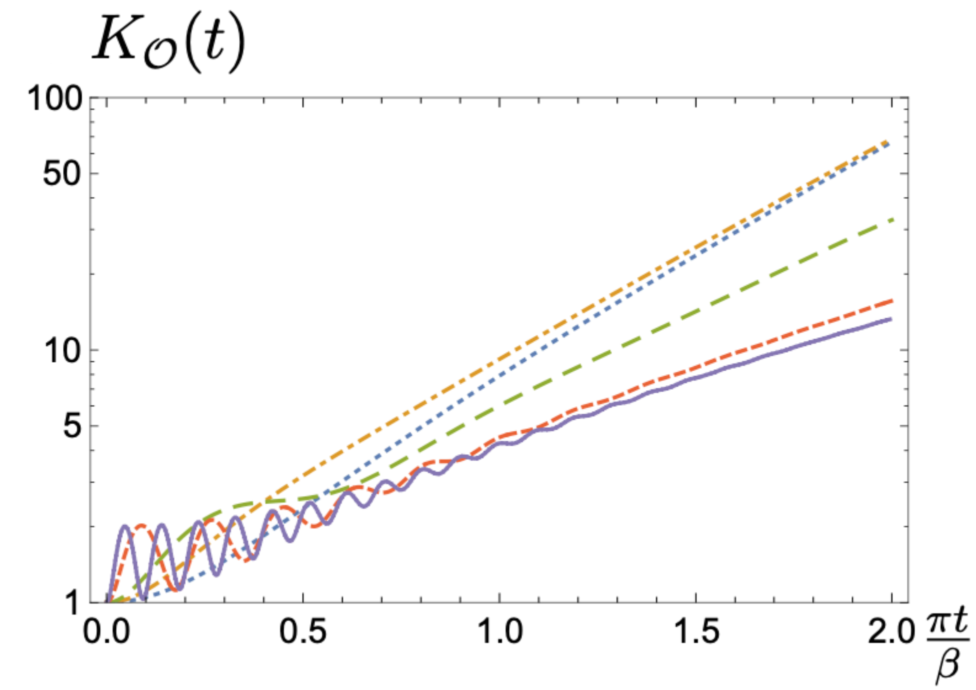
- Early time: oscillation:
  - larger m, shorter period
- Late time: oscillation disappears
  - cancelation due to large n
- Exponential increase
  - larger m, slower increase
  - staggering effect



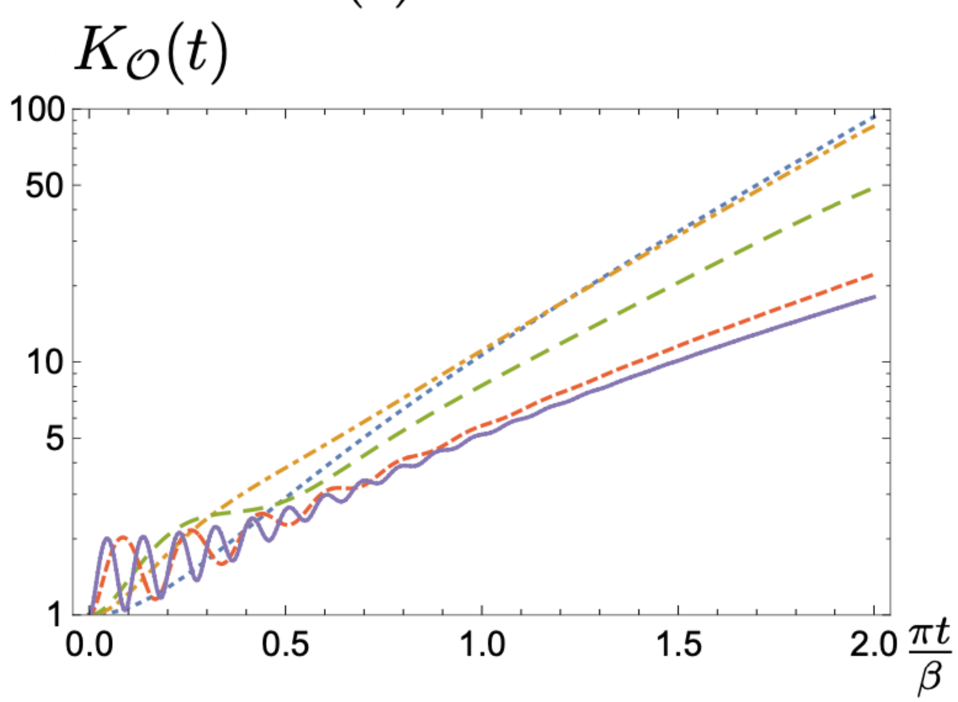
2212.14702: Camargo, Jahnke, KYK, Nishida



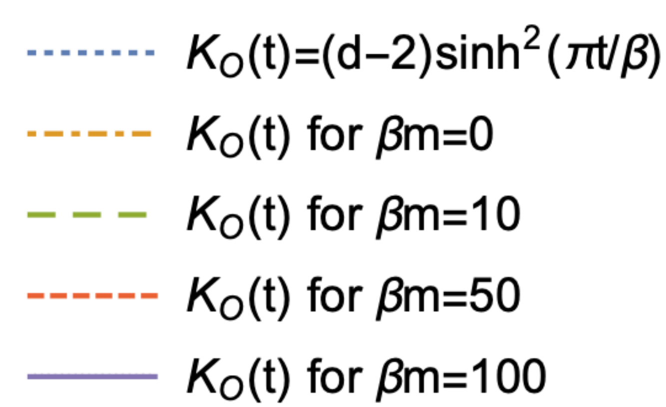
(a) $d = 5$



(b) $d = 7$

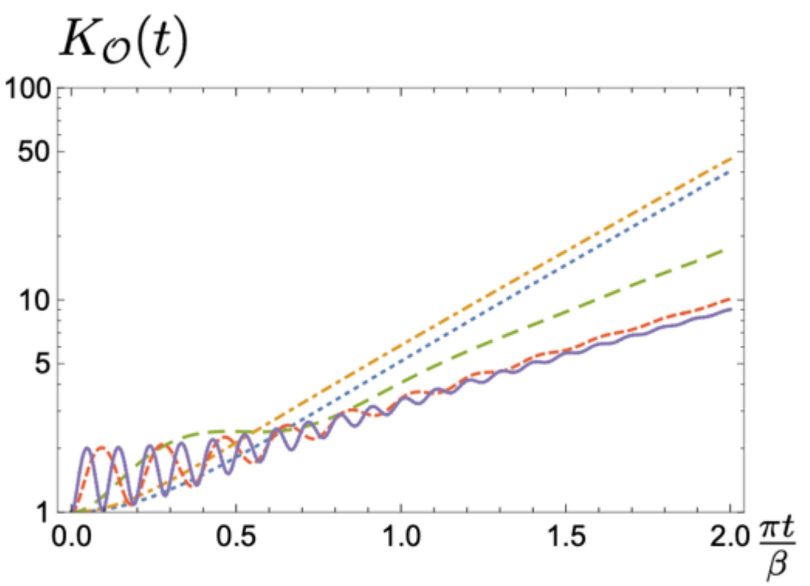


(c) $d = 9$

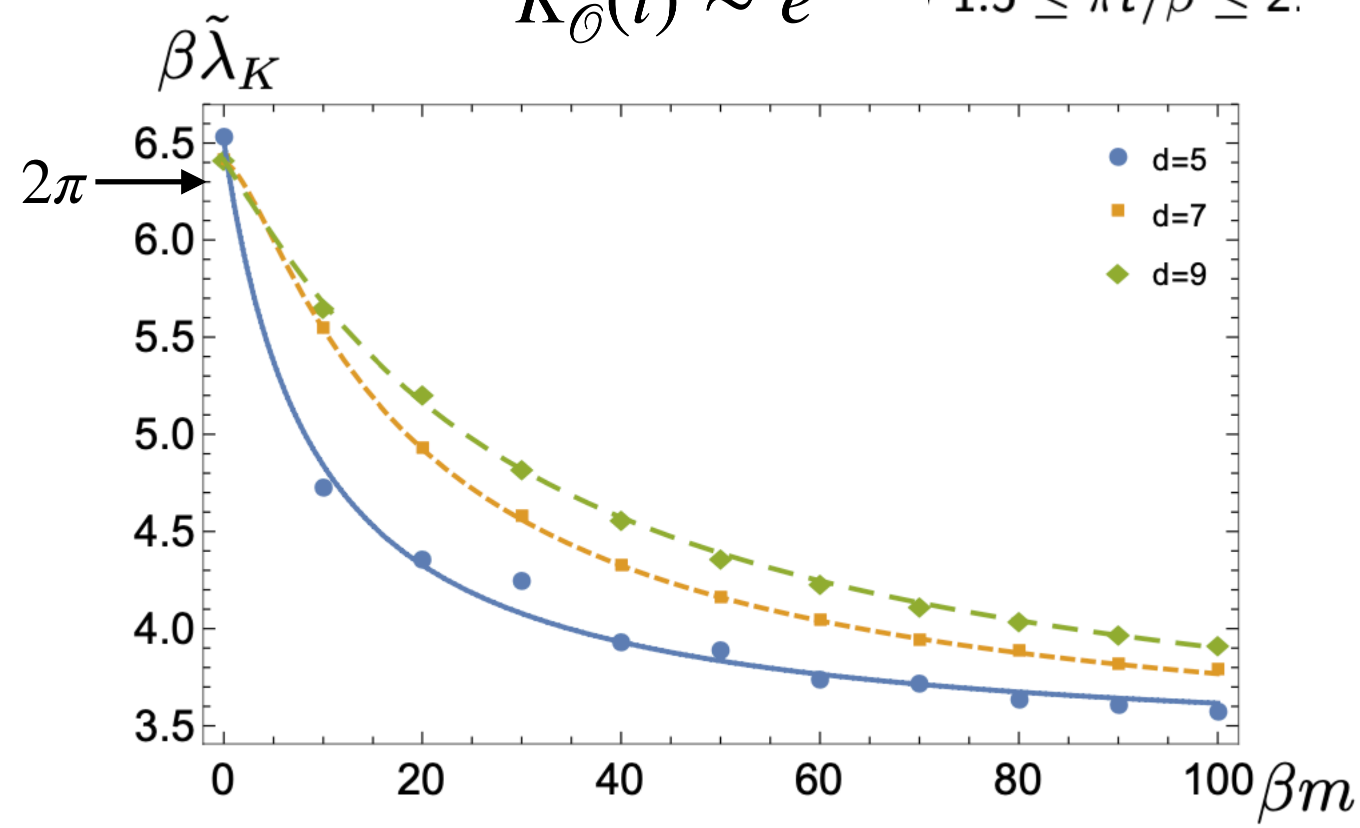


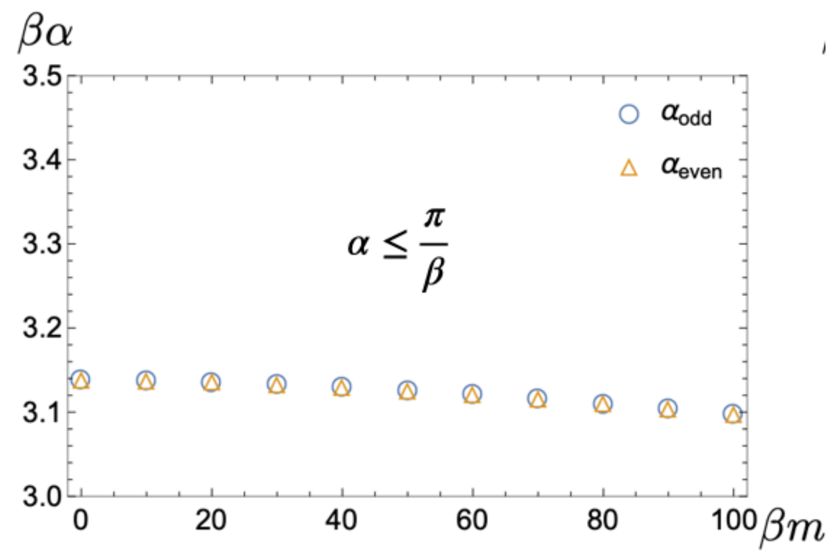
Non-trivial mass (IR-cutoff) effect: K-complexity

2212.14702: Camargo, Jahnke, KYK, Nishida

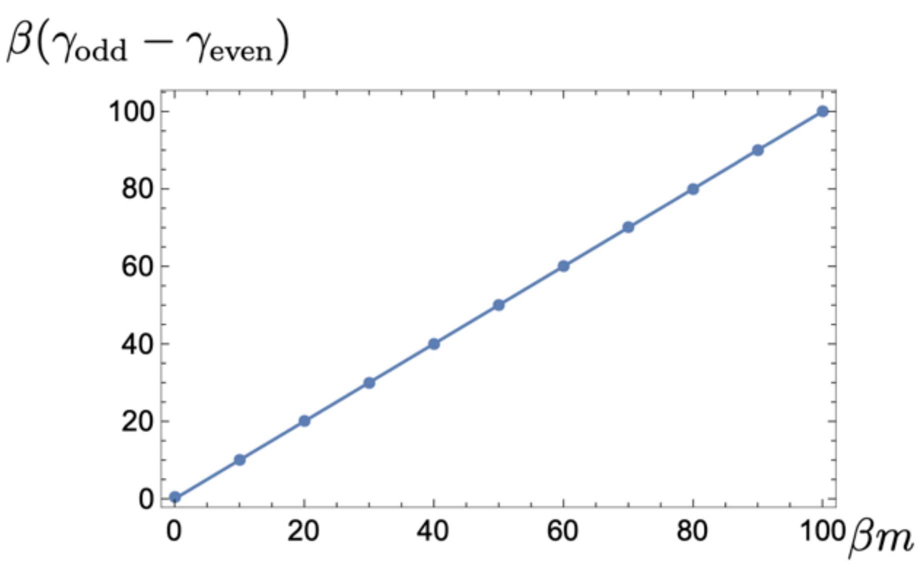


$K_{\mathcal{O}}(t) \sim e^{\tilde{\lambda} t}$, $1.5 \leq \pi t/\beta \leq 2$.



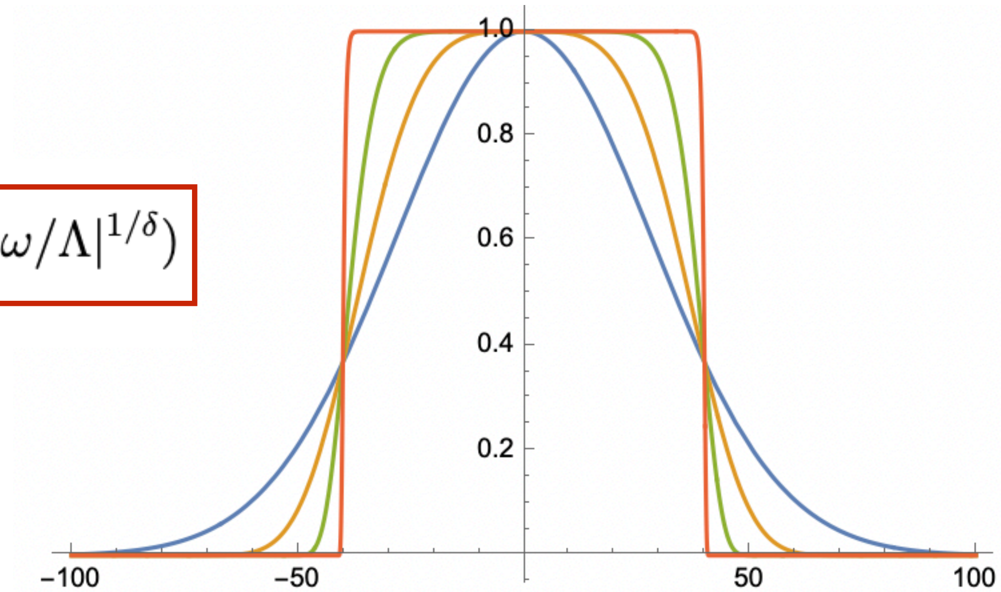


$$\beta\tilde{\lambda}_K^{(d)} = \beta(\alpha_{\text{odd}} + \alpha_{\text{even}}) + k_2^{(d)} \left(\frac{1}{k_3^{(d)} + \beta|\gamma_{\text{odd}} - \gamma_{\text{even}}|} - \frac{1}{k_3^{(d)}} \right) + k_4^{(d)} \left(\frac{1}{\left(k_3^{(d)} + \beta|\gamma_{\text{odd}} - \gamma_{\text{even}}|\right)^2} - \frac{1}{(k_3^{(d)})^2} \right),$$



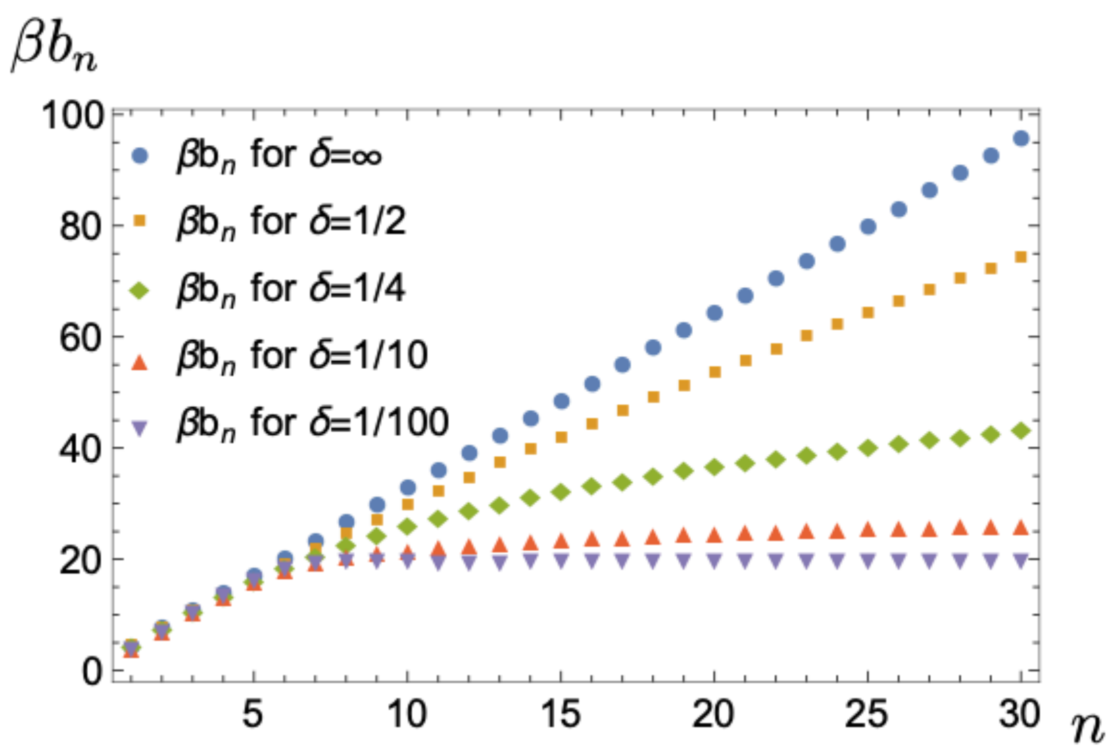
m=0, d=4

$$f^W(\omega) = N(\beta, \Lambda, \delta)\frac{\omega}{\sinh(\frac{\beta\omega}{2})}\exp(-|\omega/\Lambda|^{1/\delta})$$

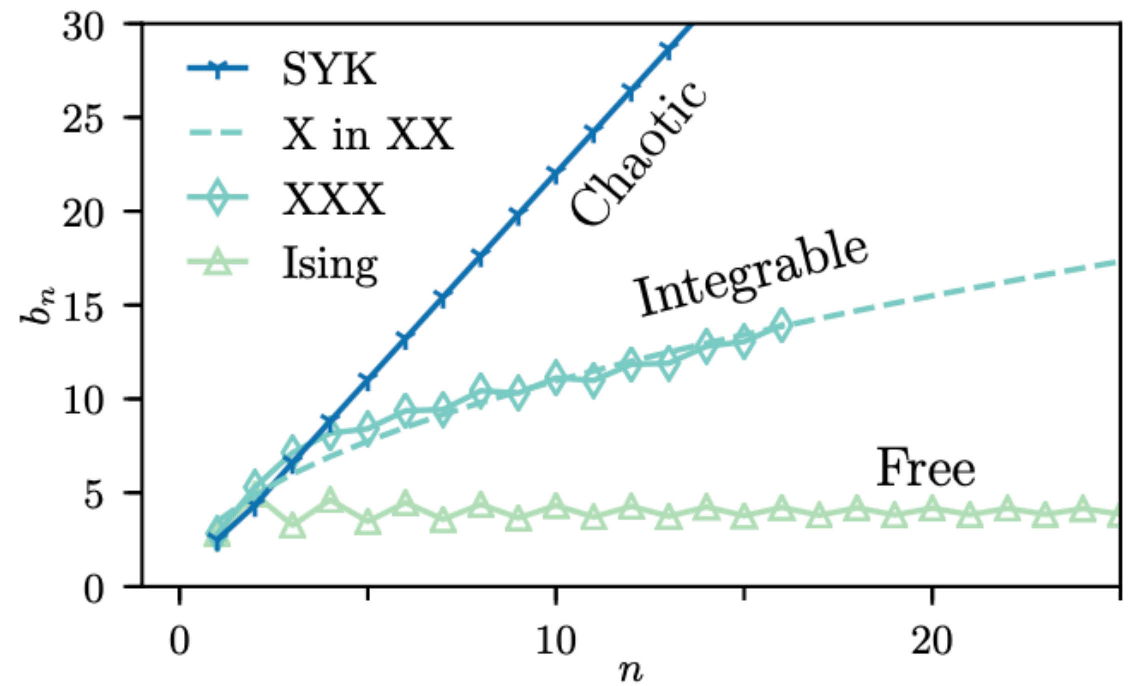


$$b_n \sim n^\delta \iff f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$

$\delta \leq 1$



$\beta\Lambda = 40.$



Summary

- Is it possible to extract the chaos-info from a C(t) or the spectral function?

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

$f^W(\omega) \longrightarrow \mu_{2n}$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$$C(t) = \Pi^W(t) = \varphi_0(t) \qquad b_n$$

K-complexity

$$\dot\varphi_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$
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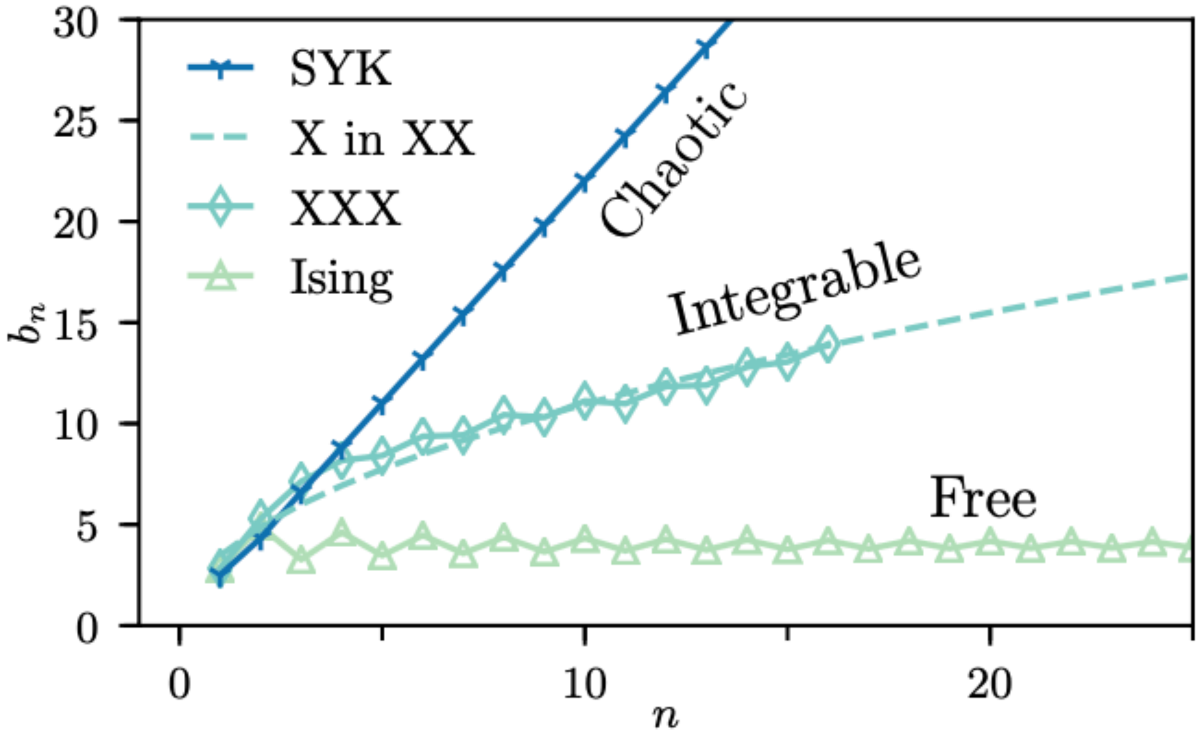
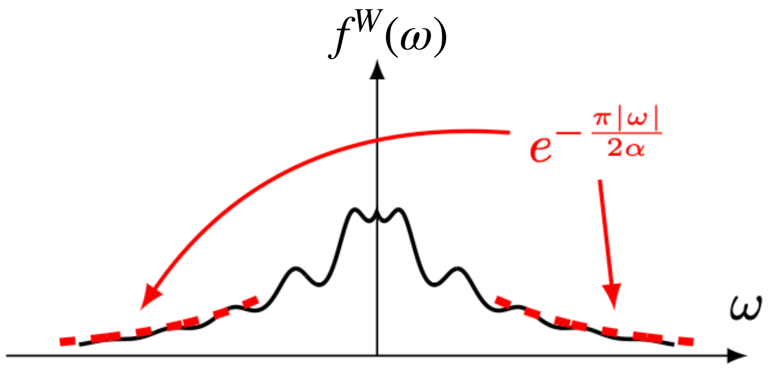
$$\vdots$$
$$\dot\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

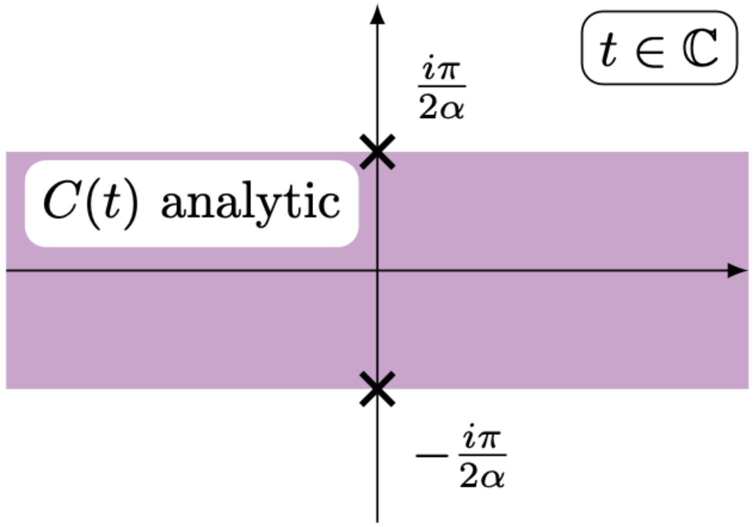
$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2, \quad n_{\max} = 200.$$

Summary

- Is it possible to extract the chaos-info from a C(t) or the spectral function?
  - Seems to be possible







Universal operator growth hypothesis

In a chaotic quantum system

Lanczos coefficients $\{b_n\}$ grow as fast as possible

$$b_n \sim \alpha n$$

the slowest possible decay of the power spectrum

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

Krylov complexity grows exponentially

$$K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

- Supporting evidences and counter examples
- Subtleties of QFT and refinements of the hypothesis

Summary

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Wightman 2-point function

$$\Pi^W(t,\mathbf{x}) := \langle \phi(t - i\beta/2, \mathbf{x})\phi(0,\mathbf{0})\rangle_\beta \quad \left(t = \frac{i\beta}{2}\right)$$

Power spectrum

$$C(t) = \Pi^W(t,\mathbf{0})$$

$$f^W(\omega) := \int \mathrm{d}t\, C(t) e^{i\omega t} = \int \mathrm{d}t\, \Pi^W(t,\mathbf{0}) e^{i\omega t}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}} \quad \left(\alpha = \frac{\pi}{\beta}\right)$$

~~In a chaotic quantum system~~ In general QFT

Lanczos coefficients $\{b_n\}$ grow as fast as possible??

$$b_n \sim \alpha n \sim \frac{\pi}{\beta} n$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Only if b_n is a smooth function of n

Otherwise

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \not\Leftrightarrow b_n \not\sim \alpha n \not\Leftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

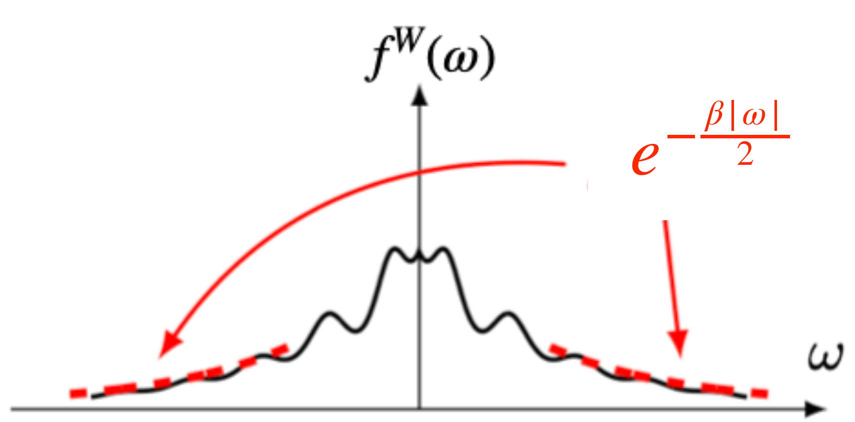
High frequency tail of the power spectrum

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \qquad f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}}$$

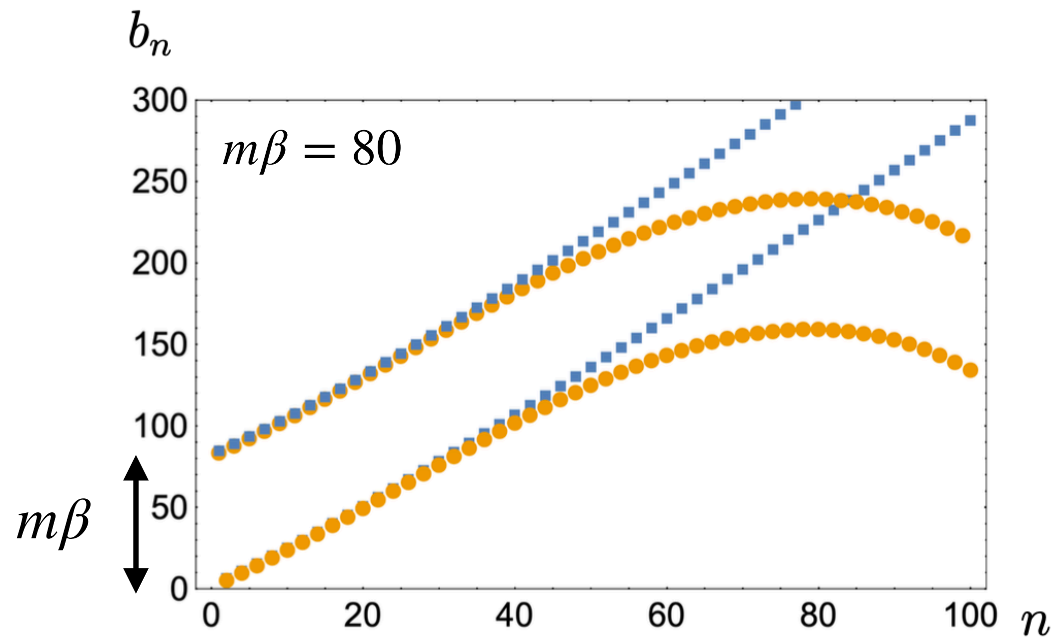
Dynamical info (Lattice) vs Kinematical info (QFT)

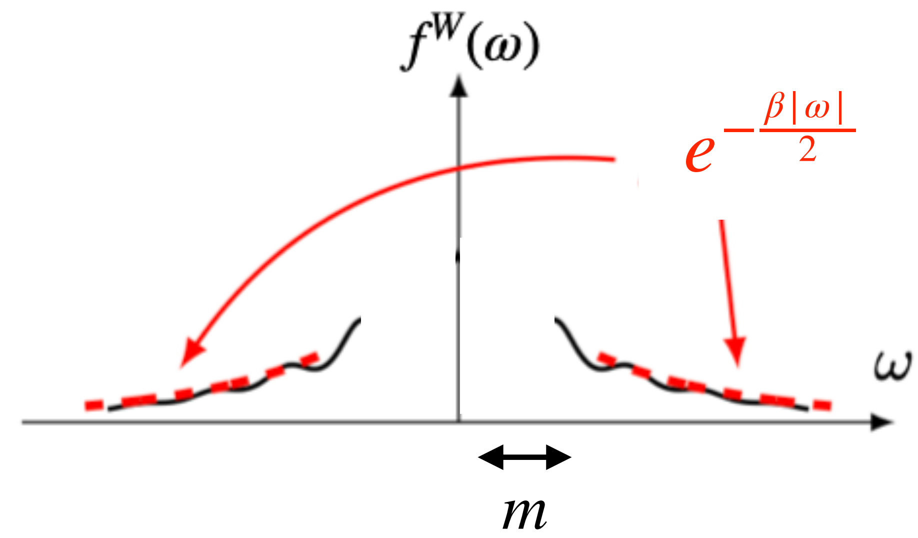
Need to take into account

- Low frequency behavior
- Sub leading behavior



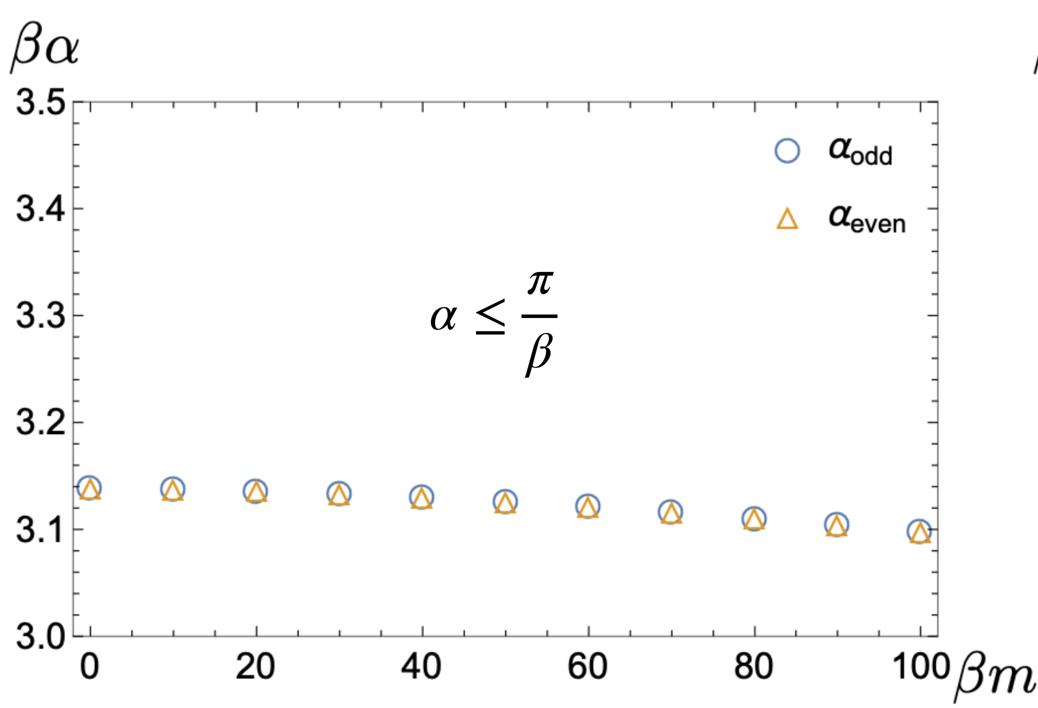
Summary



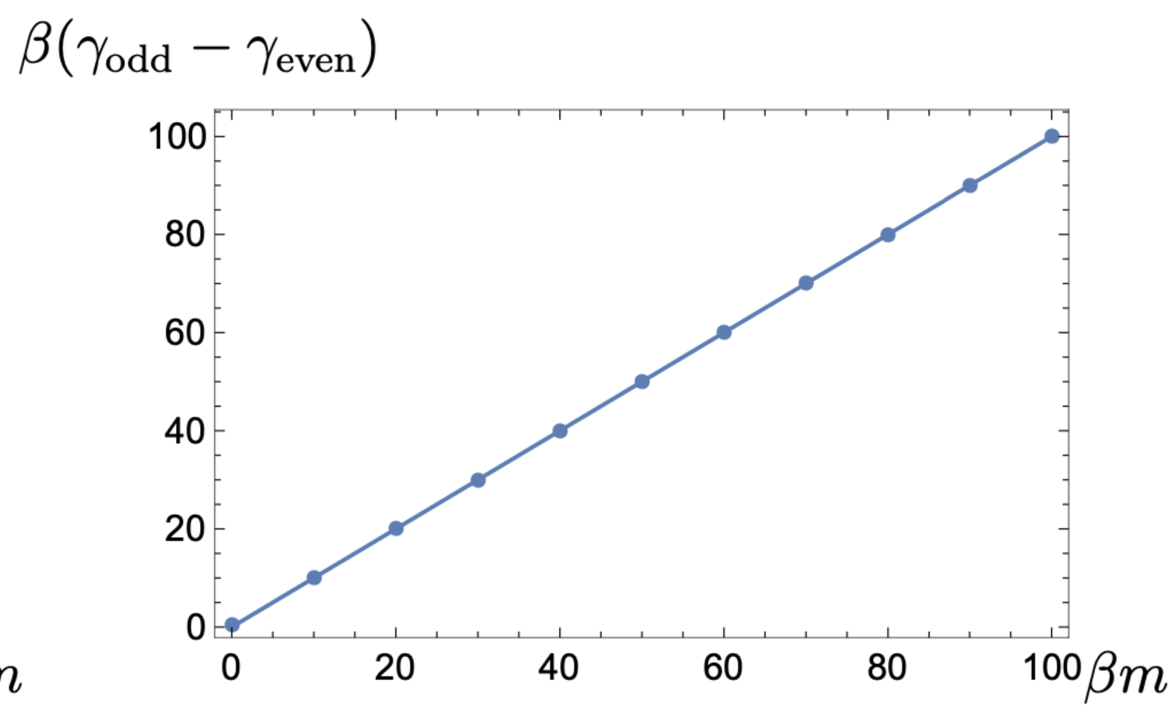


Staggering

$$b_n \sim \alpha_{\text{odd}}\, n + \gamma_{\text{odd}} \quad (\text{odd } n)$$
$$b_n \sim \alpha_{\text{even}}\, n + \gamma_{\text{even}} \quad (\text{even } n)$$

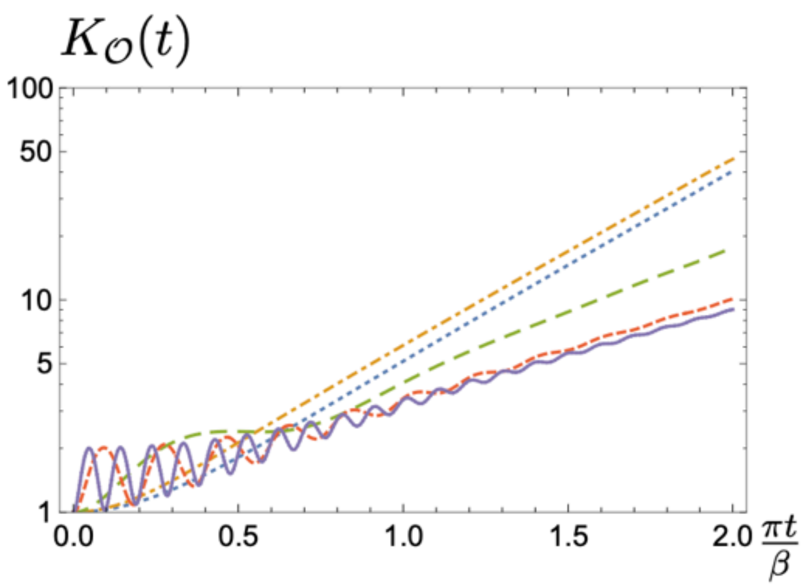


(a) Mass-dependence of α_{odd} and α_{even}

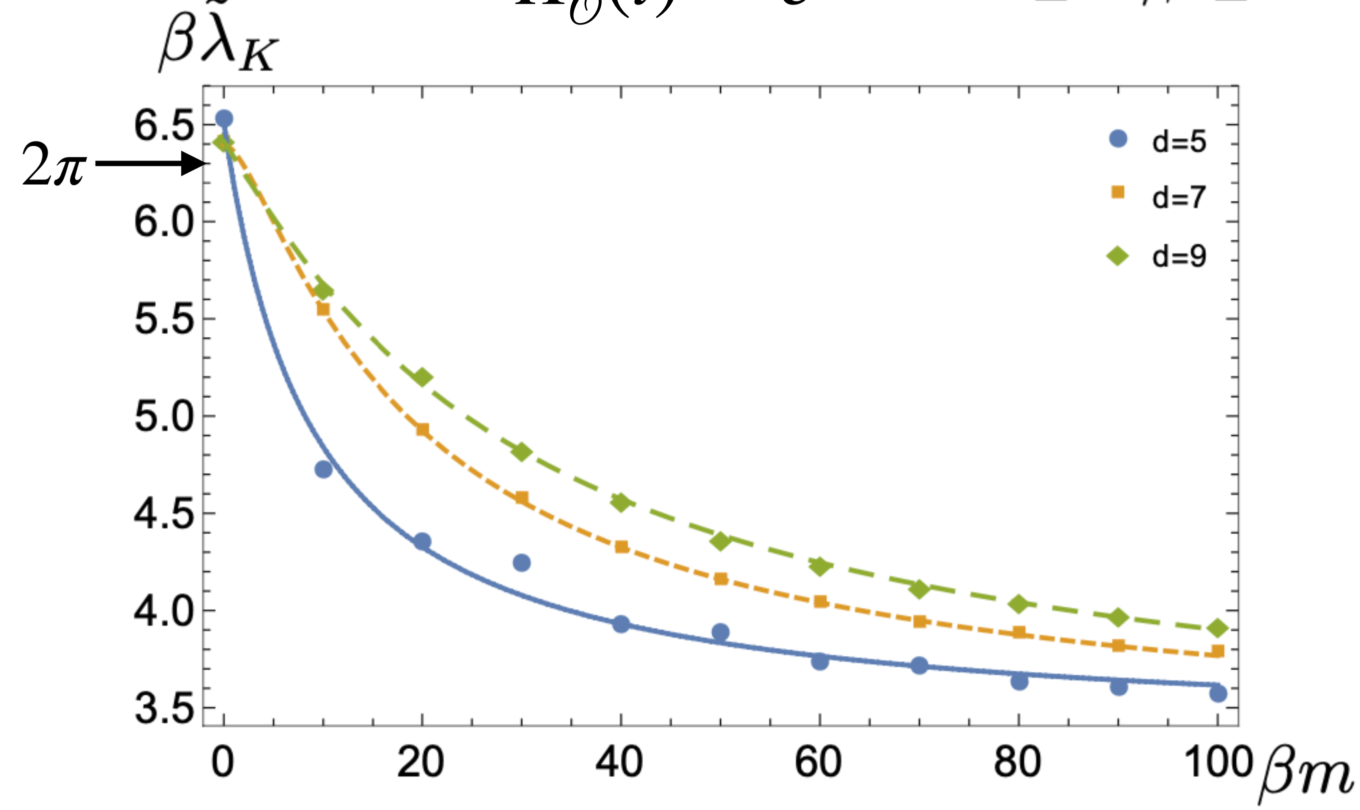


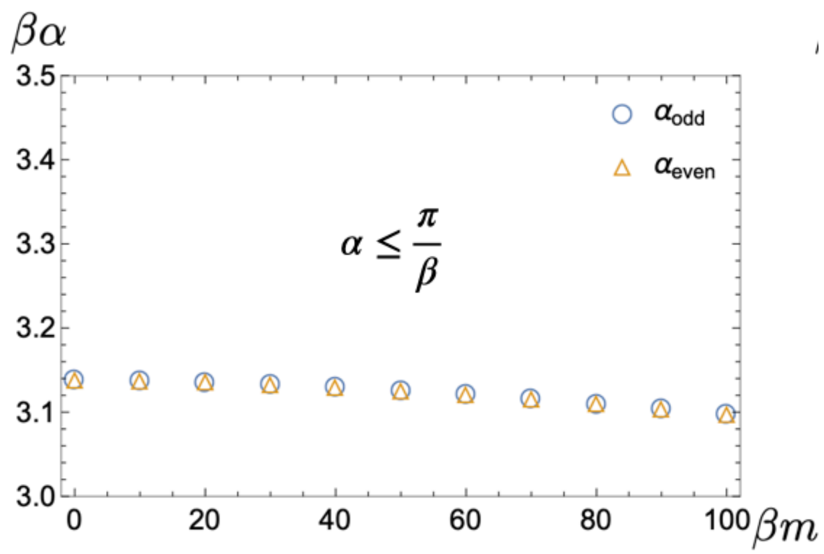
(b) Mass-dependence of $\gamma_{\text{odd}} - \gamma_{\text{even}}$

Summary

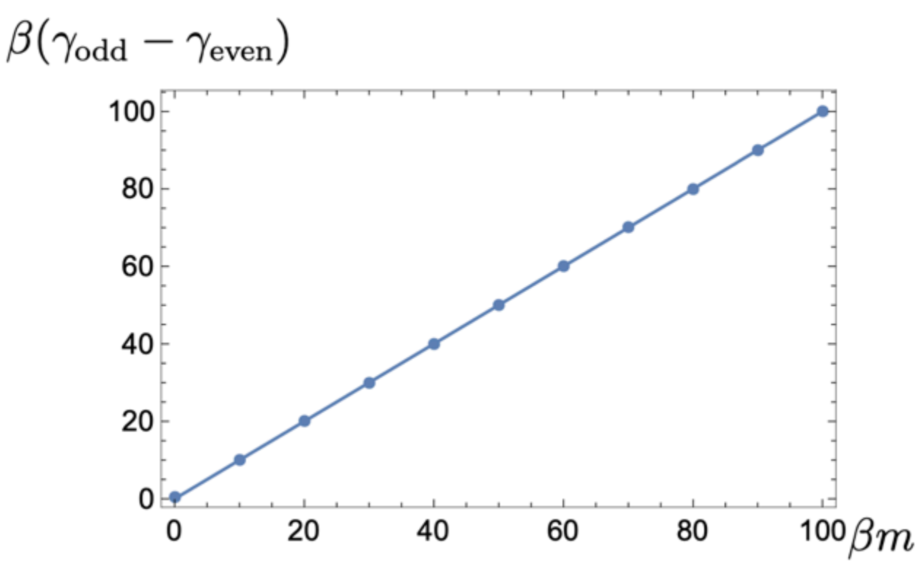


$K_{\mathcal{O}}(t) \sim e^{\tilde{\lambda} t}$, $1.5 \leq \pi t/\beta \leq 2$.





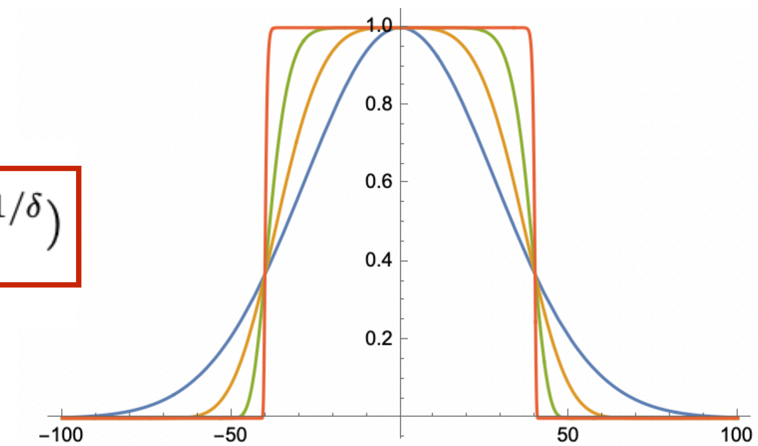
$$\beta\tilde{\lambda}_K^{(d)} = \beta(\alpha_{\text{odd}} + \alpha_{\text{even}}) + k_2^{(d)}\left(\frac{1}{k_3^{(d)} + \beta|\gamma_{\text{odd}} - \gamma_{\text{even}}|} - \frac{1}{k_3^{(d)}}\right) + \\ + k_4^{(d)}\left(\frac{1}{\left(k_3^{(d)} + \beta|\gamma_{\text{odd}} - \gamma_{\text{even}}|\right)^2} - \frac{1}{(k_3^{(d)})^2}\right),$$



Summary

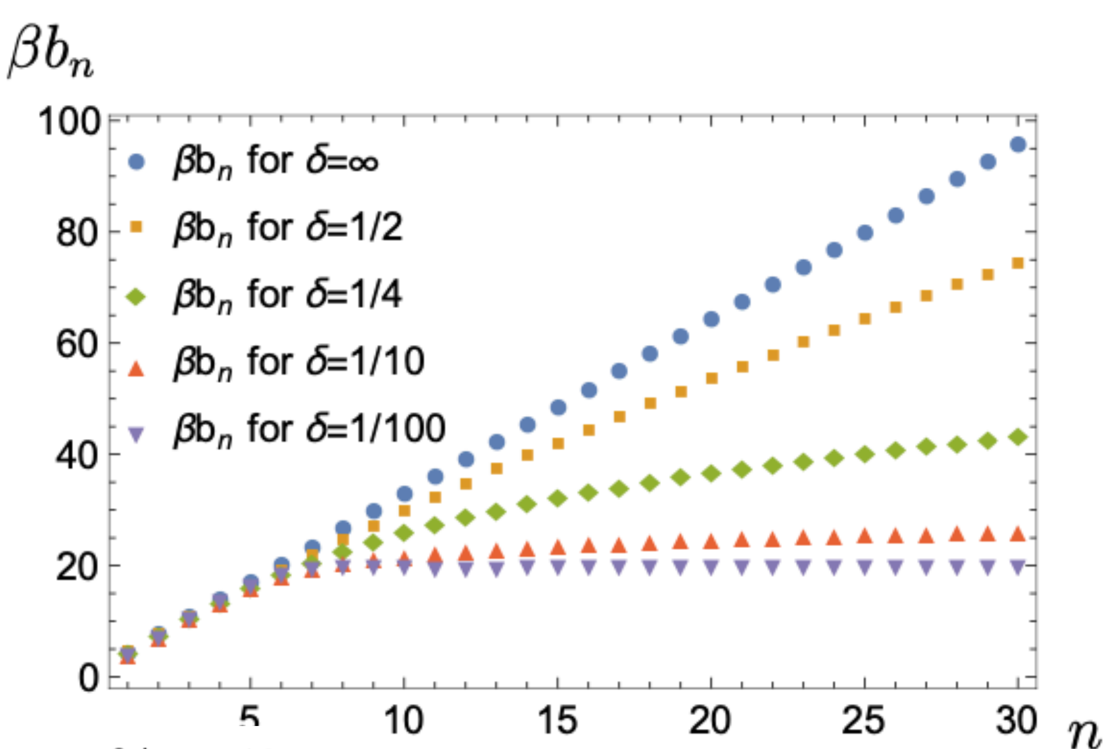
m=0, d=4

$$f^W(\omega) = N(\beta,\Lambda,\delta)\frac{\omega}{\sinh(\frac{\beta\omega}{2})}\exp(-|\omega/\Lambda|^{1/\delta})$$

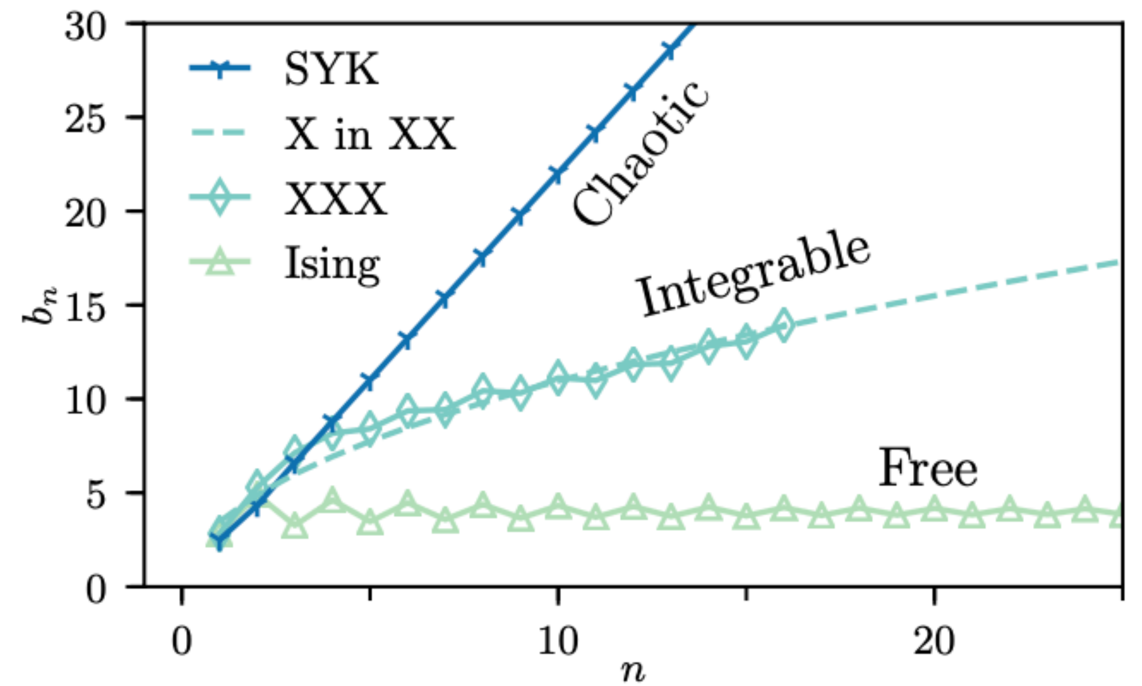


$$b_n \sim n^\delta \Longleftrightarrow f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$

$$\delta \leq 1$$



$\beta\Lambda = 40.$



- Is it possible to extract the chaos-info from a C(t) or the spectral function?
- More scales: compact space, interaction, other spins etc
- Holographic counterpart?
- Observations, conjectures, mathematical justification