

New Dynamical Degrees of Freedom from Invertible Transformations

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6TH INTERNATIONAL CONFERENCE ON HOLOGRAPHY, STRING
THEORY AND SPACETIME IN DA NANG**

Pavel Jiroušek, Keigo Shimada, Alexander Vikman, MY, arXiv: 2208.05951

$$c = \hbar = M_G^2 = 1/(8\pi G) = 1$$

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- **Introduction**

 - How ubiquitous field transformations are ?

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Introduction

Field transformations are ubiquitous in mathematics & physics !!

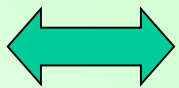
- Gauge (global) transformation of fields
- Bogliubov transformation
- Fourier transformation (series) & Laplace transformation
- Galilean, Lorentz, general coordinate transformations
- ...

It provides better ways to **understand various physical phenomena**, and to **advance calculations**, in particular, to solve more easily differential equations.

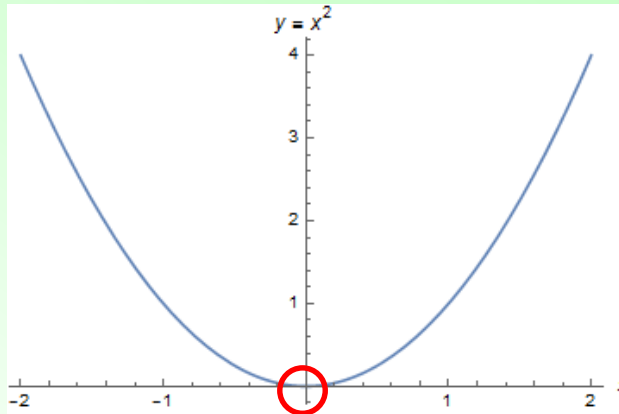
Invertible transformation

Transformation : $y_i = f_i(x_j)$

(Local) invertibility



There is (local) **one-to-one correspondence** between x_j and y_i .



The transformation $y = x^2$ is (locally) invertible except around the origin.

Except around the origin,
one-to-one.

How to judge (local) invertibility ???

Regular or Singular transformation

Transformation : $y_i = f_i(x_j)$

Jacobian matrix :

$$\mathcal{J}_{ij} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

{	Regular	↔	$\det \mathcal{J}_{ij} \neq 0$	↔	No eigenvalue vanishes
	Singular	↔	$\det \mathcal{J}_{ij} = 0$	↔	An eigenvalue vanishes

Inverse function theorem

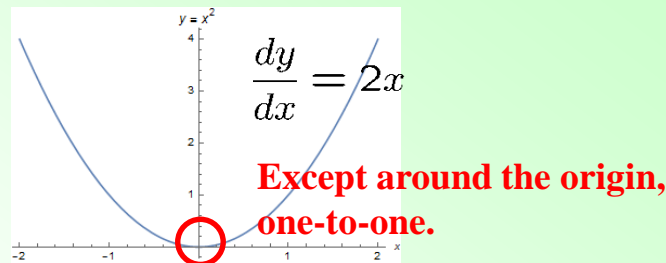
Transformation : $y_i = f_i(x_j)$

Jacobian matrix : $J_{ij} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$

Regular $\longleftrightarrow \det J_{ij} \neq 0 \longleftrightarrow$ No eigenvalue vanishes

Inverse function theorem :

If the transformation is **regular at some point**,
there is **one-to-one correspondence (invertible)** locally around that point.



Consequences of regular (invertible) transformation

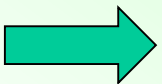
Dynamical system $[y_i]$ characterized by the following action :

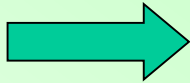
$$S_{\text{Old}}[y_i] = S_{\text{Old}}[y_i, \dot{y}_i, \ddot{y}_i, \dots]$$

Transformation : $y_i = f_i(x_j)$

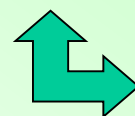
 New dynamical system $[x_j]$ characterized by the following action :

$$S_{\text{New}}[x_i] = S_{\text{Old}}[f_i(x_j)]$$

 Least action principle : $\delta S_{\text{New}} = \int d^4x \frac{\delta S_{\text{Old}}}{\delta y_i} \mathcal{J}_{ij} \delta x_j$

$\mathcal{J}_{ij} \left(= \frac{\delta y_i}{\delta x_j} \right)$: regular (at all points)  $\frac{\delta S_{\text{New}}}{\delta x_j} = 0 \iff \frac{\delta S_{\text{Old}}}{\delta y_i} = 0.$

Both systems are **completely equivalent** (just **re-labeling the dynamical variables**).



One-to-one correspondence

Singular but invertible transformation

N.B. A **regular** and **invertible** transformation gives the **physically same dynamics** (and the **same number of d.o.f.**) because it is nothing but **re-labelling**.

Inverse function theorem II

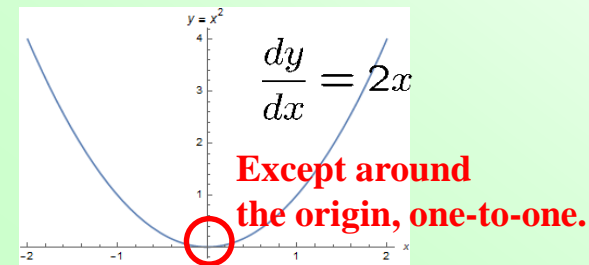
Transformation : $y_i = f_i(x_j)$

Jacobian matrix :
$$\mathcal{J}_{ij} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Regular $\longleftrightarrow \det \mathcal{J}_{ij} \neq 0 \longleftrightarrow$ No eigenvalue vanishes

Inverse function theorem :

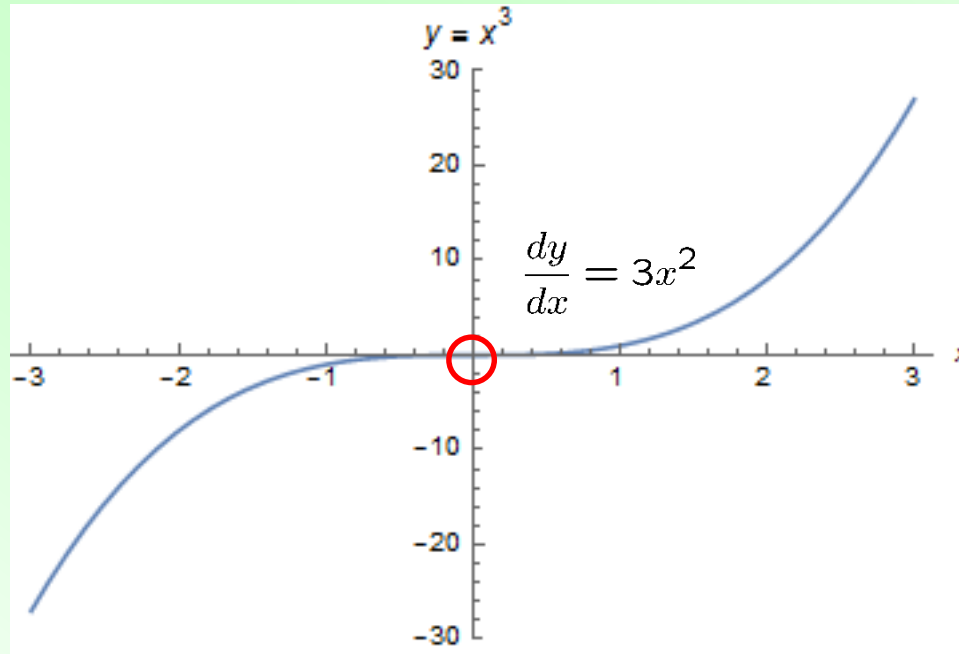
If the transformation is **regular** at some point, there is **one-to-one correspondence (invertible)** locally around that point.



But, **the opposite is not necessarily true**, that is, even if the transformation is **singular** at some point, **it can be (locally) invertible**.

Singular $\longleftrightarrow \det \mathcal{J}_{ij} = 0 \longleftrightarrow$ An eigenvalue vanishes

Example of singular but invertible transformation



This transformation is **singular** at the **origin**.

But, the transformation is **invertible** everywhere!!

Regular $\iff \det \mathcal{J}_{ij} \neq 0 \implies$ (locally) invertible
(Inverse function theorem)

Singular $\iff \det \mathcal{J}_{ij} = 0 \implies$ invertible or non-invertible

Consequences of singular transformation

$$S_{\text{new}}[x_i] = S_{\text{old}}[f_i(x_j)] \quad y_i = f_i(x_j)$$

$$\longrightarrow \delta S_{\text{new}} = \int d^4x \frac{\delta S_{\text{old}}}{\delta y_i} \mathcal{J}_{ij} \delta x_j$$

$$\text{(i)} \quad \mathcal{J}_{ij} \left(= \frac{\delta y_i}{\delta x_j} \right) : \text{regular} \quad \longrightarrow \quad \frac{\delta S_{\text{new}}}{\delta x_j} = 0 \iff \frac{\delta S_{\text{old}}}{\delta y_i} = 0.$$

$$\text{(ii)} \quad \mathcal{J}_{ij} \left(= \frac{\delta y_i}{\delta x_j} \right) : \text{singular (det J = 0 at some point)}$$

$$\longrightarrow \delta S_{\text{new}} = 0 \quad \text{for such a singular point.}$$

$$\longrightarrow \frac{\delta S_{\text{new}}}{\delta x_j} = 0 \quad \not\iff \quad \frac{\delta S_{\text{old}}}{\delta y_i} = 0.$$

New dynamics can appear for **singular** transformation even if it is **invertible** !!

Appearance of “new d.o.f” from singular but invertible transformation

$$S[q] = \frac{1}{2} \int_{t_1}^{t_2} dt \dot{q}^2 \quad \longrightarrow \quad \ddot{q} = 0$$
$$\quad \quad \quad \longrightarrow \quad q(t) = q_0 + v_0 t$$

(2 initial conditions: q_0 (initial position), v_0 (initial velocity) \rightarrow 1 d.o.f.)

(Singular but) invertible transformation of the variable $Q \Leftrightarrow q$:

$$q(Q, \dot{\phi}) = Q^3 + \dot{\phi} \quad \longleftrightarrow \quad Q(q, \dot{\phi}) = \sqrt[3]{q - \dot{\phi}}$$

(one-to-one correspondence $(q, \dot{\phi}) \Leftrightarrow (Q, \dot{\phi})$)

$$J = \frac{\partial q}{\partial Q} = 3Q^2 \quad \text{singular at } Q = 0$$

Appearance of new d.o.f from singular but invertible transformation II

$$q(Q, \dot{\phi}) = Q^3 + \dot{\phi} \quad \left(J = \frac{\partial q}{\partial Q} = 3Q^2 \right)$$

$$S [Q, \phi] = \frac{1}{2} \int_{t_1}^{t_2} dt (\ddot{\phi} + 3Q^2 \dot{Q})^2 \quad \longleftarrow \quad S [q] = \frac{1}{2} \int_{t_1}^{t_2} dt \dot{q}^2$$

$$\longrightarrow \quad \frac{\delta S}{\delta Q} = -\ddot{q} J = 0, \quad \frac{\delta S}{\delta \phi} = \frac{d}{dt} \ddot{q} = 0 .$$

● **Regular** branch with $J \neq 0$ $\longrightarrow \ddot{q} = 0 \longrightarrow q(t) = q_0 + v_0 t$

● **Singular** branch with $J = 0$ at $Q = 0$,

(2 constants, 1 d.o.f.)

$$\longrightarrow \frac{\delta S}{\delta \phi} = \frac{d^4}{dt^4} \phi = 0$$

$$\longrightarrow \phi(t) = c_0 + c_1 t + c_2 \frac{t^2}{2} + c_3 \frac{t^3}{6}$$

$$\longrightarrow q(Q, \dot{\phi})|_{Q=0} = \dot{\phi} = c_1 + c_2 t + c_3 \frac{t^2}{2}$$

4 constants, 2 d.o.f.

New d.o.f. appeared !!

Equivalent Action

$$S_1 [q] = \frac{1}{2} \int_{t_1}^{t_2} dt \dot{q}^2 \quad \longrightarrow \quad S_2 [Q, \phi] = \frac{1}{2} \int_{t_1}^{t_2} dt (\ddot{\phi} + 3Q^2 \dot{Q})^2$$

$$(q(Q, \dot{\phi}) = Q^3 + \dot{\phi})$$

$$\longleftrightarrow S_3 [Q, \phi, \theta, \lambda] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} (\dot{\theta} + 3Q^2 \dot{Q})^2 + \lambda (\dot{\phi} - \theta) \right]$$

$(\frac{\delta S}{\delta \lambda} = \dot{\phi} - \theta = 0)$

gauge symmetry : $\phi \rightarrow \phi + \epsilon, Q^3 \rightarrow Q^3 - \dot{\epsilon}, \theta \rightarrow \theta + \dot{\epsilon}$

\longrightarrow **gauge invariant variable :** $\bar{q} = Q^3 + \theta$

$$\longleftrightarrow S_4 [Q, \phi, \bar{q}, \lambda] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} \dot{\bar{q}}^2 + \lambda (\dot{\phi} + Q^3 - \bar{q}) \right]$$

$$\longrightarrow \frac{\delta S}{\delta Q} = 3\lambda Q^2 = 0 \quad \left\{ \begin{array}{l} \bullet \text{ **Regular** branch with } \lambda = 0 \quad \longrightarrow S_1[\bar{q}] \\ \bullet \text{ **Singular** branch with } Q = 0 \quad \longrightarrow \bar{q} = \dot{\phi} \end{array} \right.$$

Yet another interesting example

N.B. A **regular** and **invertible** transformation gives the **physically same dynamics** (and the **same number of d.o.f.**) because it is nothing but **re-labelling**.

Appearance of new “dynamics” from singular but invertible transformation

$$S[q, \phi] = \frac{1}{2} \int_{t_1}^{t_2} dt (\dot{q}^2 + \dot{\phi}^2) \quad \longrightarrow \quad \ddot{q} = 0, \quad \ddot{\phi} = 0.$$

$$\longrightarrow \quad q(t) = q_0 + v_0 t, \quad \phi(t) = \phi_0 + u_0 t.$$

(4 initial conditions: $q_0, v_0, \phi_0, u_0 \rightarrow$ 2 d.o.f.)

(Singular but) invertible transformation of the variables :

$$\begin{cases} q = Q^3 + \dot{\phi} , \\ \phi = \phi . \end{cases}$$

Appearance of new “dynamics” from singular but invertible transformation II

$$q = Q^3 + \dot{\phi} , \quad \phi = \phi .$$

$$S [Q, \phi] = \frac{1}{2} \int_{t_1}^{t_2} dt \left[(\ddot{\phi} + 3Q^2\dot{Q})^2 + \dot{\phi}^2 \right] \longleftarrow S[q, \phi] = \frac{1}{2} \int_{t_1}^{t_2} dt (\dot{q}^2 + \dot{\phi}^2)$$

$$\longrightarrow \frac{\delta S}{\delta Q} = -3\ddot{q}Q^2 = 0 , \quad \frac{\delta S}{\delta \phi} = \frac{d}{dt} (\ddot{q} - \ddot{\phi}) = 0 .$$

● **Regular** branch with $Q \neq 0$ $\longrightarrow \ddot{q} = 0, \ddot{\phi} = 0.$ (4 constants, 2 d.o.f.)

● **Singular** branch with $Q = 0,$

$$\longrightarrow \frac{\delta S}{\delta \phi} = \frac{d}{dt} (\ddot{\phi} - \phi) = 0$$

$$\longrightarrow \phi(t) = c_0 + c_1 t + c_2 \sinh(t) + c_3 \cosh(t)$$

$$\longrightarrow q(Q, \dot{\phi})|_{Q=0} = \dot{\phi} = c_1 + c_2 \cosh(t) + c_3 \sinh(t)$$

4 constants, 2 d.o.f.

The number of d.o.f. remains unchanged.

But, “new dynamics” appeared!!

Mimetic gravity example

Mimetic gravity (dark matter)

(Chamseddine & Mukhanov 2013)

Seed system : $g_{\mu\nu}$ and matter field (Ψ_M) with a seed action,

$$S_{\text{seed}}[g, \Psi_M]$$

Singular transformation (with conformal (Weyl) invariance)

$$g_{\mu\nu} = g_{\mu\nu}(h_{\sigma\tau}, \phi) \quad \downarrow \quad (g_{\mu\nu} \rightarrow g_{\mu\nu} \text{ for } h_{\mu\nu} \rightarrow \omega^2 h_{\mu\nu})$$

Transformed system : $h_{\mu\nu}$ and matter field (Ψ_M) with
new d.o.f (ϕ) (constrained by conformal inv.)

$$S_{\text{dis}}[h, \phi, \Psi_M] = S_{\text{seed}}[g(h, \phi), \Psi_M]$$

$$\rightarrow \frac{\delta S_{\text{dis}}[h, \phi, \Psi_M]}{\delta h_{\mu\nu}} = 0, \quad \frac{\delta S_{\text{dis}}[h, \phi, \Psi_M]}{\delta \phi} = 0,$$

give gravitational eq. including new d.o.f and its eq.

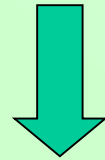
Concrete (original) example of mimetic gravity

(Chamseddine & Mukhanov 2013)

Seed system : $g_{\mu\nu}$ and matter field (Ψ_M) with a seed action,

$$\text{e.g. } S_{\text{seed}}[g, \Psi_M] = \int d^4x \sqrt{-g} R + S_{\text{matter}}$$

**Singular transformation
with conformal invariance**



$$g_{\mu\nu} = \left(h^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right) h_{\mu\nu} \equiv Y h_{\mu\nu}$$

$(g_{\mu\nu} \rightarrow g_{\mu\nu} \text{ for } h_{\mu\nu} \rightarrow \omega^2 h_{\mu\nu})$

(Dis)transformed system : $h_{\mu\nu}$ and matter field (Ψ_M) with
new d.o.f ϕ (constrained by conformal inv.)

$$S_{\text{dis}}[h, \phi, \Psi_M] = S_{\text{seed}}[g(h, \phi), \Psi_M]$$

$$\left\{ \begin{array}{l} \delta S_{\text{seed}}[g, \Psi_M] = \int d^4x \sqrt{-g} (G^{\mu\nu}(g) - T^{\mu\nu}) \delta g_{\mu\nu} \\ \delta g_{\mu\nu} = Y \delta h_{\sigma\rho} \left(\delta_\mu^\sigma \delta_\nu^\rho - g_{\mu\nu} g^{\sigma\alpha} g^{\rho\beta} \partial_\alpha \phi \partial_\beta \phi \right) + 2g_{\mu\nu} g^{\sigma\rho} \partial_\sigma \delta \phi \partial_\rho \phi \end{array} \right.$$



$$\frac{\delta S_{\text{dis}}[h, \phi, \Psi_M]}{\delta h_{\mu\nu}} = 0, \quad \frac{\delta S_{\text{dis}}[h, \phi, \Psi_M]}{\delta \phi} = 0$$

Concrete (original) example of mimetic gravity II

(Chamseddine & Mukhanov 2013)

$$S_{\text{dis}}[h, \phi, \Psi_M] = S_{\text{seed}}[g(h, \phi), \Psi_M] = \int d^4x \sqrt{-g} R + S_{\text{matter}}.$$

$$g_{\mu\nu} = \left(h^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right) h_{\mu\nu} \equiv Y h_{\mu\nu} \quad (g_{\mu\nu} \rightarrow g_{\mu\nu} \text{ for } h_{\mu\nu} \rightarrow \omega^2 h_{\mu\nu})$$

$$\left(\Rightarrow g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \frac{h^{\mu\nu}}{Y} \partial_\mu \phi \partial_\nu \phi = 1 \right) \quad \frac{(G(g) - T)}{\neq 0} (1 - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) = 0.$$

Trace part is trivially satisfied.

➔

{

$$\delta S_{\text{seed}}[g, \Psi_M] = \int d^4x \sqrt{-g} (G^{\mu\nu}(g) - T^{\mu\nu}) \delta g_{\mu\nu}$$

$$\delta g_{\mu\nu} = Y \delta h_{\sigma\rho} (\delta_\mu^\sigma \delta_\nu^\rho - g_{\mu\nu} g^{\sigma\alpha} g^{\rho\beta} \partial_\alpha \phi \partial_\beta \phi) + 2g_{\mu\nu} g^{\sigma\rho} \partial_\sigma \delta \phi \partial_\rho \phi$$

↑

➔

{

$$\frac{\delta S_{\text{dis}}[h, \phi, \Psi_M]}{\delta h_{\mu\nu}} = 0 \quad \Rightarrow \quad G^{\mu\nu}(g) - T^{\mu\nu} - (G(g) - T) g^{\mu\sigma} g^{\nu\rho} \partial_\sigma \phi \partial_\rho \phi = 0$$

$$\Leftrightarrow \quad G^{\mu\nu}(g) = T^{\mu\nu} + \tilde{T}^{\mu\nu} \quad \text{with } \nabla_\mu^{(g)} \tilde{T}^\mu_\nu = 0$$

↖

➔

{

$$\frac{\delta S_{\text{dis}}[h, \phi, \Psi_M]}{\delta \phi} = 0 \quad \Rightarrow \quad \nabla_\alpha^{(g)} ((G(g) - T) \partial^\alpha \phi) = 0$$

↗

Fix ϵ

$$\tilde{T}^{\mu\nu} \equiv (\epsilon + p) u^\mu u^\nu - p g^{\mu\nu}, \quad \epsilon = G(g) - T, \quad p = 0, \quad u^\mu = g^{\mu\alpha} \partial_\alpha \phi \quad (\text{dust})$$

$$(u^\mu u_\mu = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1)$$

What's the essence of mimetic gravity ?

(Pavel Jiroušek, Keigo Shimada, Alexander Vikman, MY, arXiv: 2207.12611)

Singular (disformal) transformation

Disformal transformation

(Bekenstein 1992)

$$g_{\mu\nu} = C(Y, \phi)h_{\mu\nu} + D(Y, \phi)\partial_\mu\phi\partial_\nu\phi, \quad Y = h^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi$$

$$\left(g^{\mu\nu} = \frac{1}{C} \left(h^{\mu\nu} - \frac{D}{C + DY} \partial^\mu\phi\partial^\nu\phi \right) \right)$$

To check the invertibility, we consider **Jacobian**, its **eigenvalues** and **eigenvectors**.

(Zumalacárregui & García-Bellido 2014)

$$\mathcal{J}_{\sigma\rho}^{\mu\nu} = \frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}} = C\delta_\sigma^\mu\delta_\rho^\nu - C_Y h_{\sigma\rho}\partial^\mu\phi\partial^\nu\phi - D_Y\partial_\sigma\phi\partial_\rho\phi\partial^\mu\phi\partial^\nu\phi$$

$$\mathcal{J}_{\sigma\rho}^{\mu\nu}\xi_{\mu\nu}^a = \lambda_a\xi_{\sigma\rho}^a, \quad \zeta_a^{\sigma\rho}\mathcal{J}_{\sigma\rho}^{\mu\nu} = \lambda_a\zeta_a^{\mu\nu}; \quad \left(C_Y \equiv \frac{\partial C}{\partial Y}, \quad D_Y \equiv \frac{\partial D}{\partial Y} \right)$$

- (9) **eigenvalues, eigenvectors, dual-eigenvectors** :

$$\lambda_C = C, \quad \xi_{\mu\nu}^C = \phi_{\mu\nu}^\perp, \quad \zeta_C^{\mu\nu} = \phi_{\top}^{\mu\nu} \quad \left(\phi_{\mu\nu}^\perp\partial^\mu\phi\partial^\nu\phi = 0, \quad \phi_{\top}^{\mu\nu}\xi_{\mu\nu}^D = 0 \right)$$

- (1) **eigenvalue, eigenvector, dual-eigenvector** :

$$\lambda_D = C - C_Y Y - D_Y Y^2, \quad \xi_{\mu\nu}^D = C_Y h_{\mu\nu} + D_Y \partial_\mu\phi\partial_\nu\phi, \quad \zeta_D^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi$$

$\lambda_C = 0$ and/or $\lambda_D = 0 \iff$ **Singular transformation**

Consequences of singular transformation

$$S_{\text{dis}}[h, \phi, \Psi_M] = S_{\text{seed}}[g(h, \phi), \Psi_M]$$

$$\longrightarrow \delta S_{\text{dis}} = \int d^4x \frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} \mathcal{J}_{\sigma\rho}^{\mu\nu} \delta h_{\mu\nu}$$

$$\text{(i)} \quad \mathcal{J}_{\sigma\rho}^{\mu\nu} \left(= \frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}} \right) : \text{regular} \quad \longrightarrow \quad \frac{\delta S_{\text{dis}}}{\delta h_{\mu\nu}} = 0 \iff \frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} = 0.$$

$$\text{(ii)} \quad \mathcal{J}_{\sigma\rho}^{\mu\nu} \left(= \frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}} \right) : \text{singular } (\lambda_a = 0)$$

$$\longrightarrow \delta S_{\text{dis}} = \int d^4x \left(\frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} - \rho \zeta_a^{\sigma\rho} \right) \mathcal{J}_{\sigma\rho}^{\mu\nu} \delta h_{\mu\nu}$$

$$\left(\frac{\delta S_{\text{dis}}}{\delta h_{\mu\nu}} = 0 \right)$$

$$\longrightarrow \frac{\delta S_{\text{seed}}}{\delta g_{\sigma\rho}} = \rho \zeta_a^{\sigma\rho} \quad \left(\zeta_a^{\sigma\rho} \mathcal{J}_{\sigma\rho}^{\mu\nu} = \lambda_a \zeta_a^{\mu\nu}, \lambda_a = 0 \right)$$

In original case ($C=Y \neq 0, D=0$)

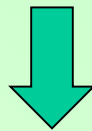
$$\longrightarrow G_{\mu\nu} - T_{\mu\nu} = \tilde{\rho} \partial_\mu \phi \partial_\nu \phi = \tilde{T}_{\mu\nu} \quad \left(\tilde{\rho} = \frac{(C + DY)^2}{\sqrt{-g}} \rho \right)$$

$$(\lambda_D = C - C_Y Y - D_Y Y^2 = 0)$$

\parallel
($\mathbf{G} - \mathbf{T}$)

The important message :

The property of mimetic matter is determined by the (dual) eigenvector with zero eigenvalue of a “singular” transformation.



It has been supposed that **non-invertibility** of a transformation is **inevitable** for mimetic gravity.
But, **this is not the case**, as we will show.

Singular behavior of disformal transformation

$$g_{\mu\nu} = C(Y, \phi)h_{\mu\nu} + D(Y, \phi)\partial_\mu\phi\partial_\nu\phi, \quad Y = h^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi$$

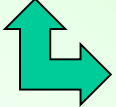
Jacobian matrix, its **eigenvalues** and **eigenvectors**.

$$\mathcal{J}_{\sigma\rho}^{\mu\nu} = \frac{\delta g_{\sigma\rho}}{\delta h_{\mu\nu}} = C\delta_\sigma^\mu\delta_\rho^\nu - C_Y h_{\sigma\rho}\partial^\mu\phi\partial^\nu\phi - D_Y\partial_\sigma\phi\partial_\rho\phi\partial^\mu\phi\partial^\nu\phi, \quad \mathcal{J}_{\sigma\rho}^{\mu\nu}\xi_{\mu\nu}^a = \lambda_a\xi_{\sigma\rho}^a,$$

(1) **eigenvalue**, **eigenvector**:

$$\lambda_D = C - C_Y Y - D_Y Y^2 = -Y^2 \partial_Y \left(\frac{C}{Y} + D \right), \quad \xi_{\mu\nu}^D = C_Y h_{\mu\nu} + D_Y \partial_\mu\phi\partial_\nu\phi,$$

● $\lambda_D = 0$ as a function (for all configurations of ϕ) (Deruelle & Rua 2014)

 $D(Y, \phi) = -\frac{C(Y, \phi)}{Y} + c(\phi)$ ← arbitrary function

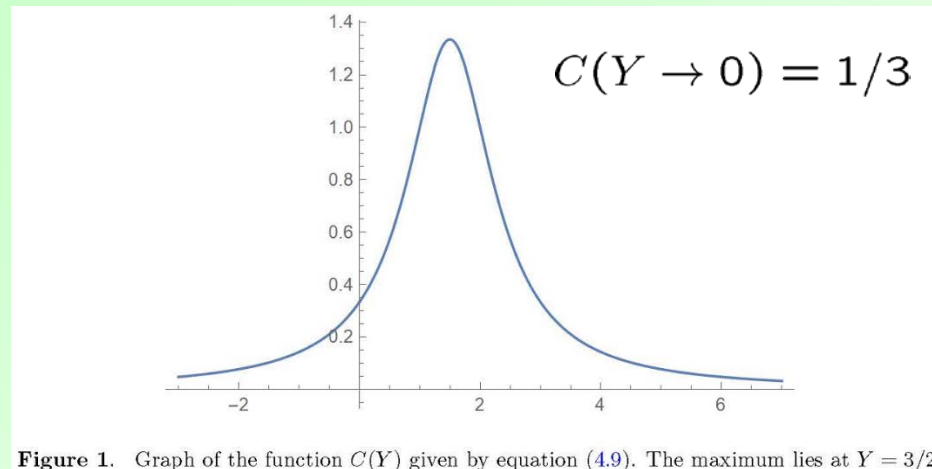
● $\lambda_D = 0$ for **some configuration** for ϕ ← **this talk**

 $C = C_Y Y + D_Y Y^2$

interpreted as a **non-trivial equation of motion**
which may be used to determine the behavior of ϕ .

Concrete example of mimetic gravity with “invertible” transformation

$$g_{\mu\nu} = C(Y, \phi)h_{\mu\nu}, \quad C(Y, \phi) = \frac{Y}{(Y-1)^3 + 1} \geq 0, \quad (Y = h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)$$



Singular at $Y = 1$:
$$C - C_Y Y = -\frac{3Y^2}{((Y-1)^3 + 1)^2} (Y-1)^2$$

But, invertible :
$$h_{\mu\nu} = \frac{X}{\sqrt[3]{X-1} + 1} g_{\mu\nu}$$

Summary

- Transformation is ubiquitous.
- Among them, **invertible** transformation is **special** because there is **one-to-one correspondence** between old and new variables.
- For **regular** transformation with its **Jacobian being non-vanishing**, **the inverse function theorem** guarantees **its (local) invertibility**.
- Two dynamical systems are **completely equivalent** if two systems are connected through **regular (invertible)** transformation.
- **Singular transformation** can be **invertible** or **non-invertible**.
- **Singular but invertible** can **change d.o.f** as well as **change dynamics**.
- A **concrete mimetic example** with a **singular but invertible transformation** is given.