IS HBAR UNIQUE?

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A little bit complicated . . .

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COLLABORATIONS

Based on a series of works with colleagues in Singapore, China, Poland and Germany

1. **Lake, M. J.**, Miller, M., Ganardi, R. F., Liu, Z., Liang, S. D., and Paterek, T. (2019), Generalised uncertainty relations from superpositions of geometries, *Class. Quantum Grav. 36 155012*.
2. **Lake, M. J.** (2019), A solution to the soccer ball problem for generalised uncertainty relations, Proceedings of the 11th Bolyai- Gauss-Lobachevsky Conference, Kiev, Ukraine, May 2019, *Ukr. J. Phys. 64 (11), 1036*.
3. **Lake, M. J.**, Miller, M., and Liang S. D. (2020), Generalised uncertainty relations for angular momentum and spin in quantum gravity, *Universe 6 (4), 56*.
4. **Lake, M. J.** (2021), Why space must be quantised on a different scale to matter, to appear in Proceedings of the 4th International Conference on Holography, Hanoi, Vietnam, August 2020, *SciPost Phys. Proc. 4, 014 (2021)*
5. **Lake, M. J.** (2021), A New Approach to Generalised Uncertainty Relations, to appear in Touring the Planck Scale: Antonio Aurilia Memorial Volume, Piero Nicolini, ed., *Fundamental Theories of Physics*, Springer, ISSN: 0168-1222 [arXiv:2008.1383v2 [gr-qc]].
6. **Lake, M. J.** (2021), How does the Planck scale affect qubits?, invited contribution to the special issue Relevance of Information Geometry for Quantum Information Science, C. Cafaro, ed., *Quantum Rep.* 2021, 3, 196-22
7. **Lake, M. J.** (2022), Generalised Uncertainty Relations and the Problem of Dark Energy, proceedings of the Viewpoints in Astrophysics conference, Cluj-Napoca, Romania, October 2021, *Romanian Astronomical Journal*
8. **among others . . more to come . . . (in preparation)**

OVERVIEW

Part I - Generalised Uncertainty Relations (GURs) and why we need them?

- Heisenberg Uncertainty Principle (HUP)
- Motivation for GURs: minimum length and/or minimum momentum
- Types of GUR proposed in the literature: GUP, EUP and EGUP

Part II - Existing approaches to GURs (circa ~1994)
and what's wrong with them

- GURs from modified commutation relations - an unnecessary headache?
- Violation of the Equivalence Principle
- Violation of Lorentz Invariance in the relativistic limit
- The "soccer ball' problem for multi-particle states
- The reference frame dependence of the "minimum" length
- Other inconsistencies?

Part III - A new approach (circa ~2019)
and how it solves everything *if hbar is not unique!*

Part IV - Space-time as a Quantum Reference Frame (QFR)?

WHAT ARE GUR?

A **Generalised Uncertainty Relation (GUR)** contains extra terms, in addition to those present in the standard Heisenberg Uncertainty Principle (HUP), which are intended to capture some aspect of **quantum gravity phenomenology**

$$\Delta x \gtrsim \hbar/2\Delta p + \dots$$

Key features of GURs

- Introduce minimum length- and / or momentum-scales

$$(\Delta x)_{\rm min} \simeq \sqrt{\hbar G/c^3} \simeq 10^{-33}\,{\rm cm} \qquad (\Delta p)_{\rm min} \simeq ?$$

- "Derived" from model-independent (heuristic) arguments

Key problems

- Implementation in a generalised quantum formalism?
- Existing approaches (~30 years circa 1994) - but are they **consistent**?

TYPES OF GUR: GUP, EUP AND EGUP

The **Generalised Uncertainty Principle (GUP)** can be derived by repeating Heisenberg's thought experiment, including the gravitational interaction between the massive particle and the probing photon

$$\Delta x \gtrsim \frac{\hbar}{2\Delta p} + \alpha \frac{G}{c^3} \Delta p$$

$$\Delta p \gtrsim \frac{\hbar}{2\Delta x} + \eta \hbar \Lambda \Delta x$$

The **Extended Uncertainty Principle (EUP)** can be derived by repeating the thought experiment assuming the presence of **dark energy** (Λ ≃ 10^{-56} cm^{-2}).

$$\alpha,\, \eta \sim \mathcal{O}(1)$$

Using either GUP or EUP alone breaks position-momentum symmetry in the uncertainty relations, but both may be derived as **separate limits** of the **EGUP:**

$$\Delta x \Delta p \gtrsim \frac{\hbar}{2} + \tilde{\alpha}(\Delta p)^2 + \tilde{\eta}(\Delta x)^2$$

GUP, EUP and EGUP are implied by **model-independent** arguments

KEY FEATURES (AND PROBLEMS) OF GUR I

The **GUP** implies the existence of a **minimum length-scale**, of the order of the Planck length, whereas the **EUP** implements a **minimum momentum-scale**, of the order of the de Sitter momentum:

$$(\Delta x)_{\rm min} \simeq l_{\rm Pl} := \sqrt{\hbar G/c^3}$$

$$(\Delta p)_{\rm min} \simeq m_{\rm dS}c := \hbar\sqrt{\Lambda/3}$$

$$l_{\rm Pl} \simeq 10^{-33}{\rm cm}, \ m_{\rm dS} \simeq 10^{-66}{\rm g}$$

Many quantum gravity thought experiments predict a minimum length-scale in nature, of the order of the Planck length. This is an **expected** feature of QG models, but the status of minimum-momentum and / or dark energy on small scales is less clear.

But how can the **GUP** and **EUP** be implemented in the **quantum formalism?**

How can we replace the heuristic Δx and Δp with well-defined standard deviations?
What are the underlying probability distributions that give rise to the GUP and EUP?
=> problem for Quantum Gravity (with dark energy)

Existing approaches in the literature (>2000 papers) rely on **modified commutation relations**, via the standard Schrodinger-Roberston relation, e.g.:

$$[\hat{x}, \hat{p}] = i\hbar(\hat{I} + \tilde{\alpha}\hat{x}^2 + \tilde{\eta}\hat{p}^2)$$

$$\Delta_\psi O_1 \Delta_\psi O_2 \geq \frac{1}{2}\langle[\hat{O}_1, \hat{O}_2]\rangle_\psi$$

KEY FEATURES (AND PROBLEMS) OF GUR II

Among other problems, modified commutation relations lead to:

- **Violation of the Equivalence Principle**

- **Violation of Lorentz invariance in the relativistic limit**

- **Inability to construct sensible multi particle states, known as the "soccer ball problem"**

- **Reference frame-dependence of the "minimum" length**

+ others . . .

KEY FEATURES (AND PROBLEMS) OF GUR III

Violation of the Equivalence Principle

Modified commutators lead to mass-dependent accelerations via the Heisenberg equation:

$$\frac{d}{dt}\hat{O}(t) = \frac{i}{\hbar}[\hat{H},\hat{O}] + \left(\frac{\partial \hat{O}}{\partial t}\right)_H$$

For example …

$[\hat{X}^i, \hat{P}_j] = i\hbar \delta^i{}_j \hat{G}(P)$ leads to the force-mass relation

$$\frac{\hat{F}^i}{m} = \frac{1}{m}\frac{d\hat{P}^i}{dt} = \frac{1}{\hat{G}(P)}\left[\frac{d^2\hat{X}^i}{dt^2} - \frac{\hat{P}^i}{m}\frac{d\hat{G}(P)}{dt}\right]$$

and similar arguments apply to modifications of the form G(X) or G(X,P)

KEY FEATURES (AND PROBLEMS) OF GUR IV

Violation of Lorentz invariance in the relativistic limit

- The standard Heisenberg commutator [x,p] = ih is an (h-scaled) representation of the shift-isometry algebra of Euclidean space
- Modifying it breaks the shift-isometry subgroup of the Poincare group in the relativistic regime
- Breaking the shift-isometry subgroup also breaks the Lorentz boost part (skip the details)

=> Lorentz invariance is broken
(not necessarily replaced with another symmetry)

KEY FEATURES (AND PROBLEMS) OF GUR V

The "soccer ball" problem for multi-particle states

- If the minimum length is enforced via modified commutation relations, single particle states cannot exceed the Planck energy
- Unfortunately, neither can the sum of momenta for multi-particle states (skip technical details, see book chapter)
- The Planck energy is a reasonable limit for sub-atomic particles but is trivially exceeded by, e.g., a soccer ball travelling at standard soccer match velocities

=> No sensible macroscopic limit
(Poisson brackets as h->0 limit of the commutator also problematic)

KEY FEATURES (AND PROBLEMS) OF GUR VI

Reference frame dependence of the "minimum" length

$$(\Delta_\psi x)^2 = \langle \hat{x}^2 \rangle_\psi - \langle \hat{x} \rangle_\psi^2 \qquad (\Delta_\psi p)^2 = \langle \hat{p}^2 \rangle_\psi - \langle \hat{p} \rangle_\psi^2$$

Substituting these relations into the expectation value of a modified commutator leads to p- or x-dependent terms on the right-hand side, e.g. those proportional to $\langle \hat{p} \rangle_\psi^2, \ \langle \hat{x} \rangle_\psi^2$

These are usually dealt with in the literature by restricting the analysis to so-called "symmetric states", e.g. states for which $\langle \hat{p} \rangle_\psi = 0, \ \langle \hat{x} \rangle_\psi = 0$

In short: There are no "symmetric states". This is frame-dependent.
=> no frame-independent minimum length

KEY FEATURES (AND PROBLEMS) OF GUR VII

Are modified commutators self consistent?

The seminal paper on GURs from modified commutator relations is by Kempf, Mangano and Mann [Phys. Rev. D 1995, *52*, 1108] **(28 years ago, >1800 citations)**.

Short Summary:

- Modified commutators require modified operators
- Hilbert space (spectral) representation of the these operators requires a modified phase space volume (integral measure) in the spectral expansions, e.g.

$$[\hat{X}^i, \hat{P}_j] = i\hbar\delta^i{}_j(1+\alpha\hat{\mathbf{P}}^2)\hat{\mathbb{I}}$$

yields the GUP. It is given by the modified P-space vol.:

$$\hat{P}_i := \int P_i|\mathbf{P}\rangle\langle\mathbf{P}|\frac{\mathrm{d}^3\mathrm{P}}{(1+\alpha\mathbf{P}^2)}$$

and the X-space representation is **not well defined**

KEY FEATURES (AND PROBLEMS) OF GUR VIII

Problems with this picture:

- Based on a canonical-type quantisation procedure that maps classical Poisson brackets to commutators and classical Hamiltonians to quantum Hamiltonians
- The phase space of classical Hamiltonian mechanics is a **symplectic manifold** => no notion of distance but volumes are defined via the sympletic 2-form (skip details)
- Sympletic geometry is very "loose" compared to Riemannian or pseudo-Riemannian geometry of standard physics, so changing volume elements is fine, **formally**
- But take a closer look at the "**P**" in the integrand … physically, what is its interpretation?

KEY FEATURES (AND PROBLEMS) OF GUR IX

The standard interpretation in the literature is:

$$\mathbf{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k} \qquad \hat{\mathbf{P}} = \hat{P}_x\hat{i} + \hat{P}_y\hat{j} + \hat{P}_z\hat{k}$$

where {x, y, z} are interpreted as **global Cartesian coordinates**

Global Cartesians only exist in Euclidean space and the associated volume element is **not** flexible: it is **dxdydz**

Must assume either

(a) that {x,y,z} refer to **Global Cartesians** => inconsistency, or

(b) that {x,y,z} do not refer to global Cartesians, but then, what do they refer to?

The latter implies that modified commutators cannot be used to to make **any** physical predictions.

ALTERNATIVES? (WITH FEWER HEADACHES) I

The problems outlined above strongly motivate the search for **alternative mathematical structures** to describe the GUP, EUP and/or EGUP.

But is there a **natural** alternative to modified commutators?

$$\Delta x \gtrsim \frac{\hbar}{2\Delta p} + \alpha \frac{G}{c^3}\Delta p \qquad \Delta p \gtrsim \frac{\hbar}{2\Delta x} + \eta \hbar \Lambda \Delta x$$

Rigorous versions of these relations can also be obtained from first order Taylor expansions of the much more "natural" expressions:

$$(\Delta_\Psi X)^2 = (\Delta_\psi X)^2 + \sigma^2 \quad (\Delta_\Psi P)^2 = (\Delta_\psi P)^2 + \tilde{\sigma}^2$$

where ψ is the canonical quantum wave function for matter, which still obeys the HUP, and $\sigma \simeq l_{\rm Pl}$, $\tilde{\sigma} \simeq m_{\rm dS} c$

So what, then, is Ψ ? (Matter + Geometry)

ALTERNATIVES? II

Basic idea idea of our model:

- Implement minimum length (and momentum) scales by introducing a form of **quantum nonlocal geometry**
- Take each point **'x'** in the classical space and associate it with a basis vector in a rigged Hilbert space, i.e., a ket, **|x>**. Then **'smear' the point** over a volume comparable to the Planck volume by constructing a superposition of these states.

In this model, there are **two position-type variables**: The first represents the quantum state of the geometric 'point' and not the position of a material particle in a fixed classical space! The second represents the measured position of the material particle in the 'fuzzy' space, i.e., new quantum degrees of freedom are introduced to describe Planck-scale fluctuations of the background (c.f. mod. comm. approach).

These fluctuations depend on the 'smearing function' **g(x'-x)** (e.g. a Planck-width 3D Gaussian) and not on the canonical quantum wave function. The variances associated with each source of position uncertainty add linearly (independent random variables) giving a **Generalised Uncertainty Relation** that Taylor expands, to first order, to the GUP:

$$\Delta_\Psi X = \sqrt{(\Delta_\psi X)^2 + (\Delta_g X)^2}$$

ALTERNATIVES? III

Details: The Fundamental Map, from which **all** predictions of the model follow is:

$$S : |\mathbf{x}\rangle \mapsto |\mathbf{x}\rangle \otimes |\mathbf{g_x}\rangle$$

where $$|g_{\mathbf{x}}\rangle = \int g(\mathbf{x}' - \mathbf{x})|\mathbf{x}'\rangle \mathrm{d}^{\mathbf{3}}\mathrm{x}' \quad \langle g_{\mathbf{x}}|g_{\mathbf{x}}\rangle = 1$$

Here, |**x**> is the canonical position eigenvector and |g**x**> represents a ‘smeared’, i.e., quantum mechanically delocalised spatial ‘point’, in the background geometry.

If g(**x’**-**x**) is taken to be a 3D Gaussian then the delocalised point is centred on its classical value, but quantum fluctuations within a few Planck volumes remain relatively likely. This is interpreted as the **probability amplitude** for the coherent transition **x <—> x’**. Note that, in this formalism, **x’** represents the measured value of position!

$$\hat{X}^i = \int x'^i |\mathbf{x}, \mathbf{x}'\rangle\langle \mathbf{x}, \mathbf{x}'| \mathrm{d}^{\mathbf{3}}\mathrm{x}\mathrm{d}^{\mathbf{3}}\mathrm{x}'$$

ALTERNATIVES? IV

The map **S** immediately induces a transformation of the canonical quantum wave vector, such that

$$S : |\psi\rangle \mapsto |\Psi\rangle$$

$$|\Psi\rangle = \int\int \psi(\mathbf{x})\mathbf{g}(\mathbf{x}' - \mathbf{x})|\mathbf{x}, \mathbf{x}'\rangle \mathrm{d}^{\mathbf{3}}\mathrm{x}\mathrm{d}^{\mathbf{3}}\mathrm{x}'$$

The associated probability distribution, incorporating quantum fluctuations in both matter and geometry is

$$|\Psi(\mathbf{x}, \mathbf{x}')|^{\mathbf{2}} = |\psi(\mathbf{x})|^{\mathbf{2}}|\mathbf{g}(\mathbf{x}' - \mathbf{x})|^{\mathbf{2}}$$

The probability of obtaining the outcome "**x**" from a single position measurement is then given by a convolution, which leads to the previous uncertainty relation:

$$\frac{\mathrm{d}^3 P(\mathbf{x}'|\mathbf{\Psi})}{\mathrm{dx}'^3} = \int |\Psi(\mathbf{x}, \mathbf{x}')|^{\mathbf{2}}\mathrm{d}^{\mathbf{3}}\mathrm{x} = (|\psi|^{\mathbf{2}} * |\mathbf{g}|^{\mathbf{2}})(\mathbf{x}')$$

and the canonical HUP holds independently of the GUR (for the matter sector)

ALTERNATIVES? V

Performing the same trick in the **momentum space** picture yields a Generalised Uncertainty Relation that Taylor expands, to first order, to the **EUP**, and combining both representations leads directly to the **EGUP**.

Remarkably, the whole formalism is equivalent to imposing the modified de Broglie relation:

$$\mathbf{p}' = \hbar\mathbf{k} + \beta(\mathbf{k}' - \mathbf{k})$$

where β is a new quantum of action for **geometry** $\beta := 2\hbar\sqrt{\frac{\rho_\Lambda}{\rho_{\mathrm{Pl}}}}$

$$\rho_\Lambda = \Lambda c^2/(8\pi G) \simeq 10^{-30}\,\mathrm{g\,.\,cm}^{-3}$$

$$\rho_{\mathrm{Pl}} = (3/4\pi)m_{\mathrm{Pl}}/l_{\mathrm{Pl}}^3 \simeq 10^{93}\mathrm{g\,.\,cm}^{-3}$$

The resulting generalised commutation relation is: $[\hat{X}^i, \hat{P}_j] = i(\hbar + \beta)\hat{\mathbb{I}}$

This neatly **avoids all 5 major problems** associated with standard modified commutator models. These problems don't occur in our formalism

In the limit β -> h our formalism reduces to that proposed by Giacomini, Castro-Ruiz and Brukner [Nat. Commun. 2019, 10, 494]. **Quantum Geometry as a QRF!**

CONCLUSIONS

- **MODIFIED COMMUTATION RELATIONS ARE EVIL AND MUST BE STOPPED**
- Instead, we can derive the **GUP**, **EUP** and **EGUP** from an alternative mathematical structure whose physical interpretation corresponds to a quantum formalism describing a **superposition of spatial geometries (aka 'smeared-space')**
- The uncertainties appearing in the GURs are well-defined standard deviations of **generalised probability distributions** (unlike in heuristic thought experiment derivations)
- These **generalised probability distributions** contain both **quantum matter** and **quantum geometry** parts
- However, the **[X,P]** commutator, for generalized measurements in smeared-space, remains proportional to the identity operator, but with coefficient **i(h/2π + β)** where **β** is a function of **h, G, c and Λ**.
- Formalism yields the "expected" quantum gravity phenomenology without violating the **Equivalence Principle** of general relativity. (First model in the literature to achieve this.)
- Important implications include:
  (a) the existence of a **minimum energy density** in nature
  (i.e., the dark energy density) which emerges as a **logical necessity** of the model
  (b) that geometry (i.e. space) is quantized on a totally different scale to matter
- **Lots** of future work to be done. (Please ask if interested.)

"ADVANCED" CONCLUSIONS FOR PARTICLE PHYSICS, ASTROPHYSICS AND COSMOLOGY

- Space-time is quantised on a different scale to matter, which should have profound implications for quantised **gravitational waves**
- Dark energy is an **emergent property**, which emerges directly from the quantum properties of space-time (acting in conjunction with the matter fields)
- Dark energy is **granular** on length-scales of order

$$l_\Lambda = \sqrt{l_{\rm Pl} l_{\rm ds}} \simeq 0.1\,{\rm mm}$$

- Quantum mechanical spin is **really** funky
- Definition of a particle (in relativistic particle physics) needs to be fundamentally re-thought (a la Wigner)