Asymptotically Free Quantum Gravity

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Partly based on collaboration with Kevin Falls and Roberto Percacci, "Towards the determination of the dimension of the critical surface in asymptotically safe gravity," Phys. Lett. B 810 (2020) 135773 [arXiv:2004.04126 [hep-th]].

and work in progress.

1 Introduction

We would like to understand how to formulate Quantum gravity (QG).

The fundamental problem is that the Einstein theory is non-renormalizable perturbatively.

We want to consider a formulation that can deal with such phenomena.

⇒ Quantum gravity within the framework of local field theory.

Known facts

- **Higher-derivative (curvature) terms always appear in QG, e.g. quantized Einstein theory and (low-energy effective theory of) superstring theories!**
- **In 4D, quadratic (higher derivative) theory is renormalizable!**

[K. S. Stelle, Phys. Rev. D16 (1977) 953.]

⇒ Possible UV completion? But it is non-unitary! (on flat backgrounds)

It is natural to consider the higher derivative theory in the formulation.

HDG

$$S_{HDG} = \int d^4x\sqrt{-g}\Big[\mathcal{V} - Z_N R + \frac{1}{2\lambda}{C_{\mu\nu\rho\lambda}}^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E\Big],$$

$${C_{\mu\nu\rho\lambda}}^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3}R^2, \qquad E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2,$$

$$Z_N = \frac{1}{16\pi G_N}\ , \ \mathcal{V} = 2\Lambda Z_N\ ,$$

To fully understand the theory, we need nonperturbative method because there is this non-unitarity problem and other difficulties (strong couplings).

⇒ (Functional or Exact) Renormalization Group!

Here comes the Asymptotic Safety.

2 Asymptotic Safety in a nutshell

We consider effective "average" action obtained by integrating out all fluctuations of the fields with momenta larger than k.

$$e^{W_k(J)} = \int [D\phi] e^{-(S[\phi]+\Delta S_k[\phi])+\int J\phi} \textbf{ where } \Delta S_k[\phi] = \frac{1}{2}\int d^d q \phi(-q) R_k(q^2)\phi(q)$$

$R_A(q)$**: a cutoff which gives suppression of IR modes**

Its role is to remove the IR mode from the action, so that the path integral is carried out only over UV modes $\Rightarrow$ **Legendre transf.** $\Rightarrow \Gamma_k[\phi]$

This is still divergent! But by introducing the cutoff function R_k

$$k\partial_k \Gamma_k(\Phi) = \frac{1}{2}\textbf{tr}\left[\left(\frac{\partial^2 \Gamma_k}{\partial\Phi^A \partial\Phi^B} + R_k\right)^{-1} k\partial_k R_k\right] \quad \Leftarrow \quad \textbf{there is no divergence!}$$

because $k\partial_k R_k$ **has contribution from modes only around** $\sim k$

Functional renormalization group equation (FRGE)!

Important fact

We look at the dependence of the effective average action on k, which gives the RG flow, free from any divergence and can be used to define quantum theory.

How?: FRGE gives flow of the effective action in the theory (coupling) space defined by suitable bases $\mathcal{O}_i$.

$$\Gamma_k = \sum_i g_i(k)\mathcal{O}_i \quad \Rightarrow \quad \frac{d\Gamma_k}{dt} = \sum_i \beta_i \mathcal{O}_i, \quad \beta_i = \frac{dg_i}{dt}$$

$$t \equiv \ln k$$

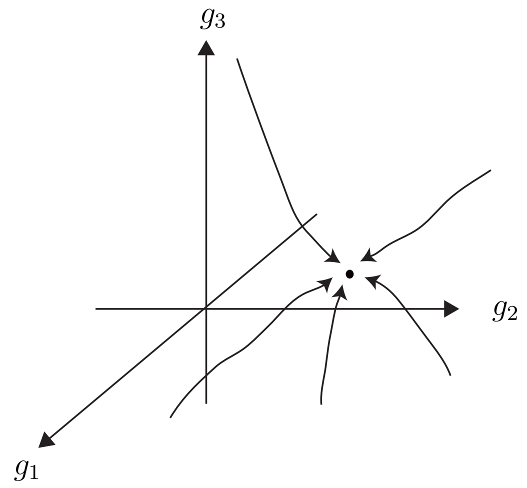


Figure 1: RG flow

We set initial conditions at some point and then flow to $k \to \infty$.

The flows may stop at FPs where $\beta = 0$.

If all couplings go to finite FPs at UV, physical quantities are well defined, giving the UV finite theory $\Rightarrow$ Asymptotic safety

+ There are finite number of the couplings $\Rightarrow$ Predictability

This defines nonperturbative renormalizability.
When integrated to $k = 0$, we get the standard effective action $\Gamma_{k=0}[\phi]$.

An important consequence of the FRGE is that the gravitational couplings depend on the energy scale k.
Note:
The couplings with positive mass dimensions go to infinity as dictated by their dimensions.

Thoes operators whose couplings go to FPs in the infinite energy are called relevant operators, and repell irrelevant operators and others marginal.

Scale invariance is realized in the large energy limit!
$\Rightarrow$ Possible connection to string theory!

The Important problem

How many relevant operators we need? $\cdots$ "Nonperturbatively Renormalizable theory" or Predictability

3 Beta functions

To formulate the theory, we need truncation (keep finite no. of operators). (We cannot deal with infinite no. of couplings.)

Conside up to quadratic curvature terms.

Gauss-Bonnet term is topological, and its coupling does not contribute.

Other beta functions from dim. reg.

J. Julve, M. Tonin, Nuovo Cim. 46B, 137 (1978).

E.S. Fradkin, A.A. Tseytlin, Phys. Lett. 104 B, 377 (1981).

I.G. Avramidi, A.O. Barvinski, Phys. Lett. 159 B, 269 (1985).

Theory is AF $(\lambda \to 0\ ,\ \omega(\equiv -\frac{3\lambda}{\xi}) \to -0.0228\ ,\ \theta \to 0.327)$
only if $\xi > 0$ **(scalar tachyon) and** $\rho > 0$**.**

No nontrivial coupling for λ and ξ was found.

A. Codello and R. Percacci, Phys. Rev. Lett. 97 (2006) 22.

M. Niedermaier, Nucl. Phys. B 833 (2010) 226.

N. O. and R. Percacci, Class. Quant. Grav. 31 (2014) 015024 [arXiv:1308.3398]

K. Groh, S. Rechenberger, F. Saueressig and O. Zanusso, arXiv:1111.1743 [hep-th].

All calculations found Gaussian FPs for dimensionless couplings.

Earlier result found that there are 3 relevant operators (Λ, R, R^2) within the power series of scalar curvature (up to R^{70} order!).

K. Falls, D. Litim, K. Nikolakopulos and C. Rahmede, "A bootstrap towards asymptotic safety," Phys. Rev. D 93 (2016) 104022 [arXiv:1410.4815 [hep-th]].

What about other tensor structure?

Perturbative renormalizability suggests that $R_{\mu\nu}^2$ is also needed.

Namely the number of relevant directions is 4.

$\Lambda, R, R^2, R_{\mu\nu}^2 \cdots$ **on dimensional grounds.**

A surprise:

D. Benedetti, P. F. Machado and F. Saueressig, Nucl. Phys. B 824 (2010) 168 [arXiv:0902.4630 [hep-th]]
studied the problem on Einstein space, keeping R^2 and extra term ${R_{\mu\nu\rho\lambda}}^2$.

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

⇒ Claims that there are only 3 relevant operators, in contrast to perturbation theory which requires $R_{\mu\nu}^2$.

However, Einstein background is not enough.

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \quad \Rightarrow \quad E = R^2_{\mu\nu\alpha\beta}, \qquad C^2 = R^2_{\mu\nu\alpha\beta} - \frac{1}{6} R^2$$

so $$\mathcal{L}_{HDG} = \left(\frac{1}{\xi} - \frac{1}{12\lambda} \right) R^2 + \left(\frac{1}{2\lambda} - \frac{1}{\rho} \right) E$$

The question: Does the above result persist on general backgrounds?

4 Our results

We have studied this problem with quadratic curvatures on general backgrounds.

K. Falls, N. Ohta and R. Percacci, Phys. Lett. B 810 (2020) 135773 [arXiv:2004.04126 [hep-th]].

We keep terms of order $\tilde{Z}_N$ (dimensionless).

$$\begin{aligned}
\beta_\lambda &= -\frac{133}{160\pi^2}\lambda^2 + \tilde{Z}_N \lambda^3 \frac{251\xi - 58\lambda}{120\pi^2 \xi} \\
\beta_\xi &= -\frac{5(72\lambda^2 - 36\lambda\xi + \xi^2)}{576\pi^2} + \tilde{Z}_N \frac{9720\lambda^3 - 1980\lambda^2\xi + 489\lambda\xi^2 - 14\xi^3}{6480\pi^2} \\
\beta_{\tilde{Z}_N} &= \left(-2 + \frac{(30\lambda - \xi)(4\lambda + \xi)}{192\pi^2\xi}\right)\tilde{Z}_N + \frac{-3168\lambda^2 + 654\lambda\xi + 35\xi^2}{1152\pi^2\xi(6\lambda + \xi)} \\
&\quad - \frac{72\lambda^2 - 84\lambda\xi + 65\xi^2}{192\pi^2(6\lambda + \xi)^2}\log\left(\frac{2}{3} - \frac{2\lambda}{\xi}\right).
\end{aligned}$$

Result

We found some nontrivial FPs and strong evidence that the dimension of critical surface is most probably 3.

Unfortunately, these beta functions cannot interpolate to the low-energy where we expect that the Einstein term dominates (nearly zero Newton constant) and the classical picture is good.

5 Asymptotically free quantum gravity

In order to study the behavior down to low-energy, we have to get beta functions to all order in $\frac{1}{\tilde{Z}_N}$.

Fortunately Benjamin Knorr has a strong code which could do this.

However no nontrivial FPs for the dimensionless couplings were found.

We get beta functions in the standard harmonic gauge and regulator (low derivative), and found some nontrivial FPs.

However the fact that the existence or nonexistence depends on the gauge choice and scheme suggests that these FPs might be fake.

In the meantime, Sen, Yamada and Wetterich published a paper in which they have nontrivial FPs, in different gauge fixing and scheme, and different FPs.

What is sure is the existence of asymptotically free (AF) FPs.

I have studied if the AF FP can be smoothly connected to low-energy where the Newton coupling should tend to zero or small value.

5.1 Fixed points

For small λ, the beta functions to the second order are

$$\beta_\lambda = -\frac{133}{160\pi^2}\lambda^2 + O(\lambda^3), \qquad \beta_\xi = -\frac{5(72\lambda^2 - 36\lambda\xi + \xi^2)}{576\pi^2} + O(\lambda^3, \xi^3).$$

Usual study examined only the stability matrix (first order gradient) and $\lambda = \xi = 0$ would be marginal FP.

Checking next order $\Rightarrow$ $\lambda = 0$ is a relevant FP!

We can check the beta function for ξ by setting $\lambda = 0$, and this gives

$$\beta_\xi = -\frac{5\xi^2}{576\pi^2} + O(\xi^3) \quad \Rightarrow \quad \textbf{relevant FP!}$$

Naively we expect that the number of relevant operators at this AF FP is 4, in agreement with perturbation theory.

However this is too naive in the approach of FRG.

We should check how the FP is approached from low energy.

Setting $\lambda = x\xi$, the beta function for ξ gives

$$\beta_\xi = -\frac{5(1 - 36x + 72x^2)}{576\pi^2}\xi^2.$$

So if our trajectory approaches to this FP in the direction with

$$0.0295 < x < 0.470,$$

this FP is repulsive, and it is irrelevant.

We find the UV FP is $\lambda_* = 0$ and $\omega_* = -\frac{3\lambda_*}{\xi_*} \simeq -0.0229\,(x = 0.0076)$, corresponding to relevant direction, giving 4 relevant directions.

With the FP for λ and ξ zero (but their ratio $\omega_* = -0.0229$):

$$\tilde{\Lambda}_* = 0.389, \qquad g_* = 2.38,$$

in agreement with earlier result ($\tilde{\Lambda} = \Lambda k^{-2}, g = G_N k^2$).

5.2 UV and IR behaviors of gravitational couplings and wave function renormalization

It is well known that the wave function renormalization "constant" is unphysical parameter which does not affect any physical quantities.

This point has not been taken into account in most of the literature on the asymptotic safety until recently.

Here we improve this situation (with Hikaru Kawai).

5.2.1 Einstein gravity

Consder the Einstein theory with the cosmological constant:

$$S = \int d^4x\sqrt{g}\left(2\Lambda - \frac{1}{16\pi G_N}R\right).$$

Under the wave function renormalization

$$g_{\mu\nu} \to g'_{\mu\nu} = Z g_{\mu\nu}$$

we have

$$\sqrt{g} \to Z^2\sqrt{g'}, \qquad \sqrt{g}R \to Z\sqrt{g}R'$$

The cosmological constant Λ and the Newton coupling changes as

$$\Lambda \to Z^2\Lambda', \qquad G_N \to Z^{-1}G'_N \quad \Rightarrow \quad \Lambda G_N^2 \textbf{ is invariant}$$

This modifies the FRGE as

$$2(\dot\Lambda + 2\eta_Z\Lambda) + \frac{\dot G_N - \eta_Z G_N}{G_N^2} = \mathrm{Tr}\left[\frac{\dot R_k + \eta_G R_k}{\Gamma^{(2)} + R_k}\right]. \qquad \eta_G = -\frac{\dot G_N}{G_N}, \quad \eta = \frac{\dot Z}{Z}$$

Define dimensionless couplings by

$$\Lambda = \tilde{\Lambda} k^4, \qquad G_N = \tilde{G} k^{-2}$$

The FRGE reduces to

$$\dot{\tilde{\Lambda}} + 4\tilde{\Lambda} - 2\eta_Z \tilde{\Lambda} = \frac{1}{16\pi}(A_1 + A_2 \eta_G), \qquad \dot{\tilde{G}} - (2 + \eta_Z)\tilde{G} = \frac{1}{16\pi}(B_1 + B_2 \eta_G)$$

$$\Rightarrow \tilde{\Lambda}\tilde{G}^2 = \textbf{ invariant}$$

The usual optimized cutoff

$$R_k = (k^2 - \Delta)\theta(k^2 - \Delta)$$

breaks the invariance under the wave function renormalization. We choose

$$R_k = (\sqrt{\Lambda} - \Delta)\theta(\sqrt{\Lambda} - \Delta)$$

$$\Rightarrow \quad A_1 = \frac{(1 + 128\tilde{G}\sqrt{\tilde{\Lambda}}\,)(\dot{\tilde{\Lambda}} + 4\tilde{\Lambda})}{4\pi(1 - 32\pi\tilde{G}\sqrt{\tilde{\Lambda}}\,)}, \qquad A_2 = \frac{5\tilde{\Lambda}}{6\pi(1 - 32\pi\tilde{G}\sqrt{\tilde{\Lambda}}\,)},$$

$$B_1 = -\frac{(11 - 288\pi\tilde{G}\sqrt{\tilde{\Lambda}} + 7(32\pi\tilde{G}\sqrt{\tilde{\Lambda}}\,)^2(\dot{\tilde{\Lambda}} + 4\tilde{\Lambda})}{12\pi(1 - 32\pi\tilde{G}\sqrt{\tilde{\Lambda}}\,)^2\sqrt{\tilde{\Lambda}}}, \quad B_2 = -\frac{160\pi\tilde{G}\sqrt{\tilde{\Lambda}} + 1}{12\pi(1 - 32\pi\tilde{G}\sqrt{\tilde{\Lambda}}\,)}\sqrt{\tilde{\Lambda}},$$

Solve for $\dot{\tilde{\Lambda}}$ and $\dot{\tilde{G}}$:

$$\dot{\tilde{\Lambda}} = f_1(\tilde{\Lambda}, \tilde{G}, \eta_Z), \qquad \dot{G} = f_2(\tilde{\Lambda}, \tilde{G}, \eta_Z)$$

We use the freedom to fix the cosmological constant to a constant.

$$f_1(\tilde{\Lambda}_0, \tilde{G}, \eta_Z) = 0 \quad \Rightarrow \quad \eta_Z \quad \Rightarrow \quad \beta_G$$

What behaviors in the UV and IR limits:

$\tilde{G} = G_N k^2 \to$ **finite**, $(k \to \infty,$ **asymptotic safety**$)$; $\quad \tilde{G} \to 0 \ (k \to 0)$.

For $\tilde{\Lambda} = \frac{1}{2\pi}$**, we set the boundary condition** $g = 0.001$ **at** $t = 0$**, and integrate.**

The behavior of the Newton coupling constant is as follows:

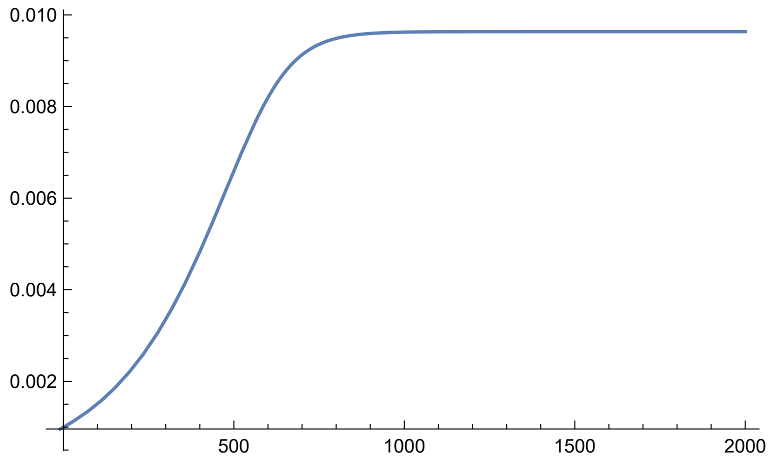
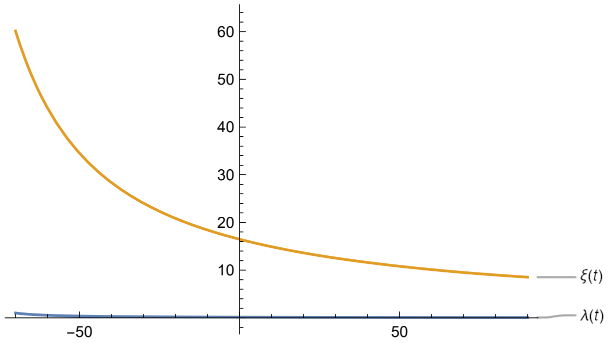


Figure 2: Flow of the Newton coupling G.

The coupling becomes small in IR, and quickly goes to the FP in UV.

Yet to be studied: Higher curvature terms

In the IR limit, we expect that the behavior of $G(= G_N k^2)$ remains the same, whereas λ and ξ becomes large.
In the UV limit, we expect that λ and ξ are asymptotically free.



(a)

Figure 3: Flows of λ and ξ.

If true, both $\frac{1}{\lambda}$ and $\frac{1}{\xi}$ are tending to zero in IR, thus the Einstein term is the dominant term.

In UV, the higher derivative terms dominate.

6 Summary

Expanding to first order in $Z_N \propto \frac{1}{G_N}$ produces three nontrivial FPs in the HDG truncation of the gravitational FRG with three relevant operators.

Our previous analysis involves expansion in $Z_N \propto \frac{1}{G_N}$ so that we could not interpolate to the low-energy region where the Newton coupling is very small.

The beta functions including all order in Z_N indicates that the nontrivial FPs (asymptotic safe points) might be a fake.

However we cannot exculde the possibility of the existence of these FPs, since it depends on the scheme.

We have found an evidence that there are 4 relevant operators in the asymptotic free fixed point, in agreement with perturbation theory.

We expect that it is smoothly connected to the perturbative gravity regime in the low-energy limit.

The redundant wave function renormalization should be taken into account, with higher curvature terms.

Problems to be understood

- **Whether there are nontrivial FP in all order in Z_N.**

  Analysis of higher-loop contributions may be necessary
  $\cdots$ very complicated on general backgrounds

- **Can the unitarity problem really be resolved?**

- **How much do the results depend on matter?**

The real problem is that not much physical consequences are drawn.

- **Physical implications**

  There are certain discussions on implications to particle physics, e.g. Higgs mass, fermion mass, but universal results have not been obtained due to gauge ambiguity and so on.

  E.g. Black hole physics $\cdots$ compatibility with thermodynamics

  Hawking radiation

  Can black hole singularity be resolved? $\cdots$ to some extent

  island idea in Hawking radiation

  Breakdown of global symmetry?