Einstein-Gauss-Bonnet Black holes at large D

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arXiv: 2212.04809 [JHEP 02, 101 (2023)]

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Introduction

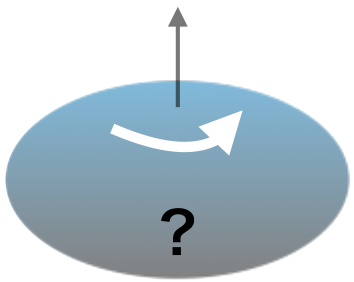
- String Theory predicts (or inspires) Higher Curvature Corrections to GR
  Einstein-Gauss-Bonnet theory is one of the simplest

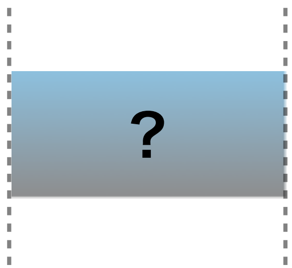
$$S = -\frac{1}{16\pi G}\int (R + \alpha_{\mathrm{GB}}\mathcal{L}_{\mathrm{GB}})d^d x$$

GB correction $\mathcal{L}_{\mathrm{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$

However

- only known exact solution: static spherically symmetric BH
  [Boulware-Deser (1985)]
- No exact solutions like Myers-Perry (Rotating BH) or black string as in GR





BHs in EGB theory

For rotating BHs and black strings,

Numerical or perturbative approaches have been applied

- D=5 equally-rotating BH

  numerical Brihaye-Radu (2008) small $\alpha_{\rm GB}$ approx. Ma-Li-Lu (2021)

- D=5 Singly-rotating BH with small Ω Kim-Cai (2007)

- black strings numerical+small $\alpha_{\rm GB}$ approx. in D=5 Kobayashi-Tanaka (2004)

  D=5…10 Brihaye-Delsate-Radu (2010)

Perturbative sols : only for small GB coupling or small rotation

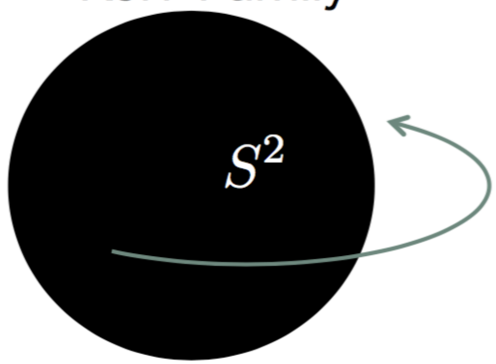
To seek more general analytic sols,
we adopt the large D effective theory approach

Large D limit in GR

Black holes in 4D

Black holes are simple

Kerr Family



"Black holes have no hair"
John Wheeler

Topology theorem $\rightarrow$ horizon topology = S^2

Hawking (1972)

Uniqueness theorem $\rightarrow$ Kerr Family (for vacuum, AF)

Carter (1971), Robinson (1975), Mazur (1982), Bunting (1982)

Perturbation $\rightarrow$ Quasi-local modes

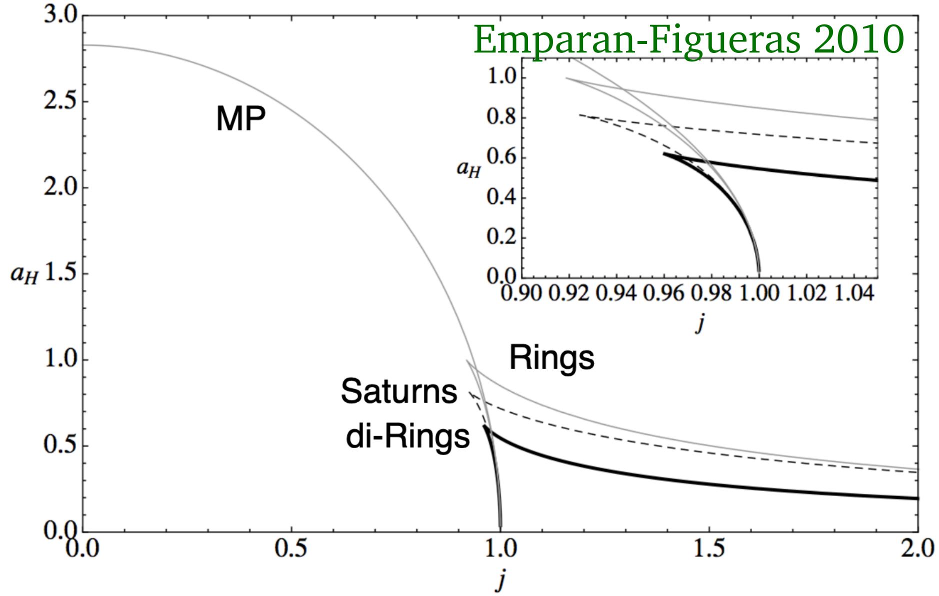
Black holes in D>4

D=5 BHs

Myers-Perry BH | Black Ring

Myers, Perry, '86 — S^3

Emparan, Reall, '02 — $S^2 \times S^1$



- Various horizon topologies : either of $S^3, S^2 \times S^1, S^3/Z_p$ (or connected sum)
  Galloway-Schoen (2005)
- No longer unique for given charges (M,J,Q,...)
- Stationary sols are generated by Inverse-Scattering Method
  Belinsku-Zakharov (1978), Pomeransky (2005)

but only applicable to D=5 (or D=5 $\times T^p$)

D>5 BHs

- No known solution generating technique for general cases

→ Numerical or Analytic approx. (Perturbation, Blackfold, **Large D**)

Large D limit

Asnin-Gorbonos-Hadar-Kol-Levi-Miyamoto(2007)
Emparan-**RS**-Tanabe(2013)

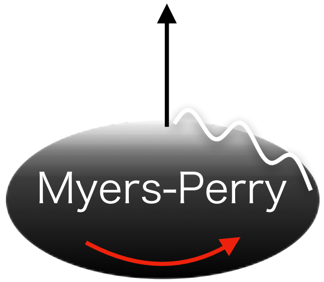
$$S = \frac{1}{16\pi G}\int d^{D}x\sqrt{-g}(R + 2\Lambda + \mathscr{L}_m)$$

Consider the spacetime Dimension is "large"

but how large ?（cf: large N limit → N=3)

→ depends on BH topology

Spherical BHs：qualitatively OK even for D ~ 6,7



spin a

ex) Threshold spin for axisym. instability of Myers-Perry BH

$$a_c(D \to \infty) = \sqrt{3} \approx 1.732, \quad a_{c,D=6} = 1.572, \quad a_{c,D=7} = 1.714$$

RS-Tanabe(2015) Dias-Figueras-Monteiro-Santos(2010)

Black Strings：only reliable up to D ~ 12,13 (critical dimension~13.5)

Einstein eq. is dramatically simplified

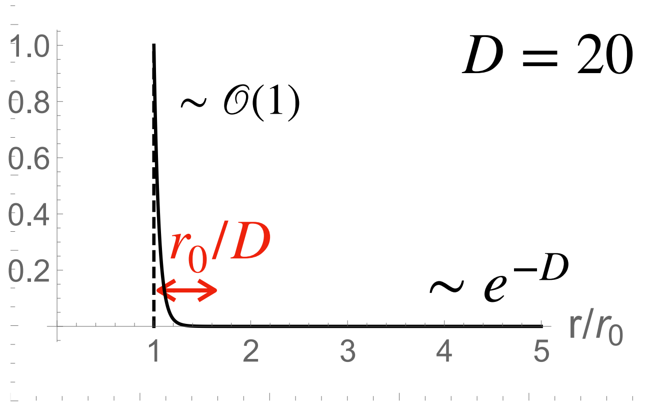
Black hole dynamics → Effective theory@D=∞ ＋ 1/D corrections

Localization of Gravity

Gravity of BH is confined within thin layer of 1/D

new small parameter r_0/D

$$\phi(r) = \left(\frac{r_0}{r}\right)^{D-3}$$

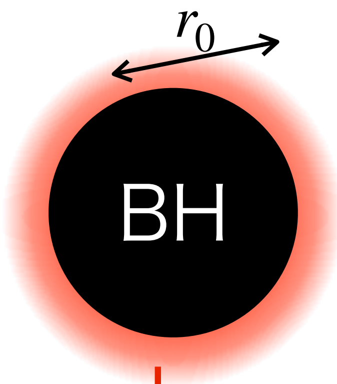


Need new radial coordinate to resolve Near-Horizon region for $D \gg 1$

$$\mathsf{R} := (r/r_0)^{D-3} \longleftrightarrow r = r_0 \mathsf{R}^{\frac{1}{D-3}} \simeq r_0\left(1 + \frac{\ln \mathsf{R}}{D}\right)$$

metric functions are expanded in 1/D as funcs of R

$$g_{\mu\nu}(r) = g_{\mu\nu}^{\text{near},0}(\mathsf{R}) + \frac{1}{D} g_{\mu\nu}^{\text{near},1}(\mathsf{R}) + \frac{1}{D^2} g_{\mu\nu}^{\text{near},2}(\mathsf{R}) + \cdots$$



$$r - r_0 \sim r_0/D$$

Near horizon

Gradient Hierarchy

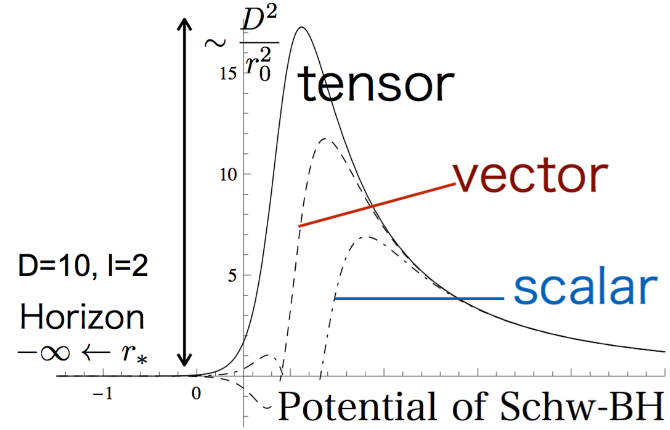
Radial gradient is dominant $\partial_r \sim D \times \mathsf{R}\partial_\mathsf{R} = \mathcal{O}(D) \gg \partial_{||}$

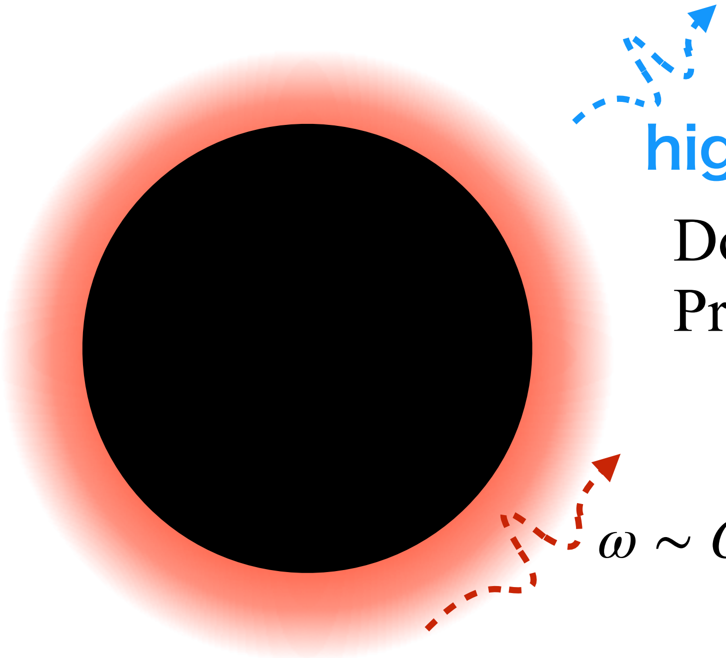
Separation of scales

Potential barrier $\sim O(D^2/r_0^2)$

$\rightarrow$ Dynamics splits into two sectors

Emparan-**RS**-Tanabe(2014)





$\omega \sim O(D/r_0)$

high freq sector

DoF of GW, radiations
Propagate to asym bdry

$\omega \sim O(1/r_0)$

low freq sector

Confined in near-horizon region
DoF of Horizon deformation
$\rightarrow$ reformulated as **Large D Effective theory**

- GW radiation and Horizon dynamics are separated
- **radiating effect is negligible to low freq sector** $\dot{M}/M \propto -D^{-D}$

Andrade-Emparan-Licht-Luna(2019)

Black p-brane@large D

Emparan-Shiromizu-**RS**-Tanabe-Tanaka(2015),
Emparan-**RS**-Tanabe(2015), Minwalla+(2015)

ref) Black p-brane in Eddington Finkelstein coordinate

$$ds^2 = -2dtdr - \left(1 - \frac{r_0^{D-p-3}}{r^{D-p-3}}\right)dt^2 + \frac{1}{D}dz^i dz_i + r^2 d\Omega^2_{D-p-2}$$

rescaled to have $k_{\mathrm{GL}} \sim \sqrt{D}$

$z^i \to z^i/\sqrt{D}$

Large D limit with $\mathsf{R} := (r/r_0)^{D-p-3}$

$$ds^2 = -2dtdr - \left(1 - \frac{m(t,z)}{\mathsf{R}}\right)dt^2 - \frac{2p_i(t,z)}{\mathsf{R}}\frac{dtdz^i}{D} + \frac{1}{D}dz^i dz_i + r_0^2 d\Omega^2_{D-p-2}$$

This is leading order solution with $m(t,z),\ p_i(t,z)$: arbitrary funcs

Large radial gradiant $\partial_r \sim D \times \mathsf{R}\partial_{\mathsf{R}} = \mathcal{O}(D) \gg \partial_{\parallel}$

$x_{\parallel} = (t, z^i)$ dependence is dropped in Leading order

$\rightarrow$ Evolution eq. w.r.t R : **ODE + src term of O(1/D)**

Large D Effective theory

Black p-brane : Leading order metric sol

$$ds^2 = -2dtdr - \left(1 - \frac{m(t,z)}{\mathsf{R}}\right)dt^2 - \frac{2p_i(t,z)}{\mathsf{R}}\frac{dtdz^i}{D} + \frac{1}{D}dz^i dz_i + r_0^2 d\Omega^2_{D-p-2}$$

Constraint eqs

→ **Simple theory of effective fields** $\{m(t,z), p_i(t,z)\}$

$$\begin{cases} \partial_t m - \partial_i \partial^i m = -\partial_i p^i \\ \partial_t p_i - \partial^j \partial_j p_i = \partial_i \left(m\delta_i^j - \dfrac{p_i p^j}{m} \right) \end{cases}$$

Large D Effective Theory (on black p-brane)

Perturbation from Uniform Black String

$$m = m_0 + \delta m(k) e^{\Omega t + ik\cdot z}, \quad p = \delta p(k) e^{\Omega t + ik\cdot z} \rightarrow \Omega(k) = -|k| \pm |k|^2$$

Effective theory captures GL-instability for $0 \leq |k| \leq k_{\rm GL} = 1$

Hydro-elastic complementarity

Emparan-Izumi-Luna-**RS**-Tanabe(2017)

Physical interpretation of Large D effective theory

$(m, p_i) \to (m, v_i)$ Horizon velocity field $v_i(t,x) := (p_i - \partial_i m)/m$

Effective theory in Hydrodynamic form

∇_i : covariant div δ_{ij}

$$\partial_t m + \nabla_i (m v^i) = 0, \quad \partial_t (m v^i) + \nabla_j (m v^i v^j + \tau^{ij}) = 0$$

$$\tau_{ij} := \underbrace{-\, m\delta_{ij}}_{\text{Negative pressure}} \underbrace{-\, 2m\nabla_{(i}v_{j)}}_{\text{Viscosity}} \underbrace{-\, m\nabla_i\nabla_j \log m}_{\text{Young's modulus}}$$

GL instability

Prebend over-bending

Stable Non-uniform phase

Black p-brane@D=∞ = complex of fluid and elastic body

2nd law in Large D ET

Andrade-Emparan-Jansen-Licht-Luna-**RS** (2020)

Black p-brane entropy

$$\text{Entropy} \;=\; \underbrace{\text{const.} \times \text{Mass}}_{\text{conserve}} + \frac{S_1}{D}$$

cf) Schwarzschild

$$S_{BH} \propto M_{BH}^{\frac{D-2}{D-3}} \simeq M_{BH} + O(D^{-1})$$

Entropy conserves at LO in 1/D

difference with Mass up to D^{-1} $\rightarrow$ Entropy functional for $m(t,z), v_i(t,z)$ LO variables

$$\boxed{S_1(t) := \int \left(-\frac{1}{2} m v^2 - \frac{1}{2m} (\partial m)^2 + m \log m \right) d^p x}$$

LO effective equation

2nd Law $\partial_t S_1 := \int 2m \partial_{(i} v_{j)} \partial^{(i} v^{j)} d^p x \geq 0$

Entropy is produced by viscous terms

blob approximation

Andrade+Emparan+Licht (2018)

Effective theory of Black p-brane includes compact BH

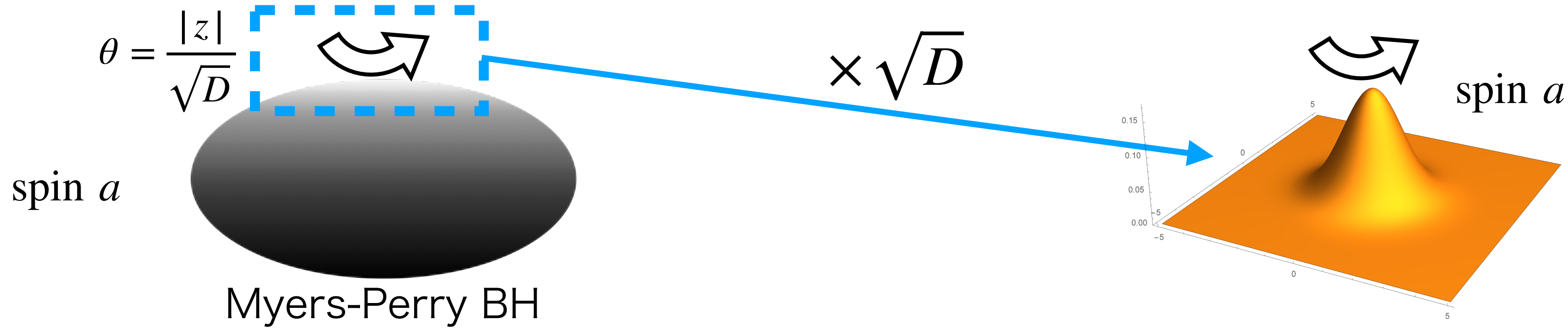
ex) Black 2-brane effective theory $z^i = (x, y)$

$$\partial_t m + \nabla_i (m v^i) = 0, \quad \partial_t (m v^i) + \nabla_j (m v^i v^j + \tau^{ij}) = 0$$

Following spinning Gaussian profile is also a solution

$$m(x, y) = \exp\left(-\frac{x^2 + y^2}{2(1 + a^2)}\right) \qquad v_i(x, y) = \frac{a}{1 + a^2}(y, -x)$$

no translation symm. $m(x, y) \to 0 \quad (|x|, |y| \to \infty)$

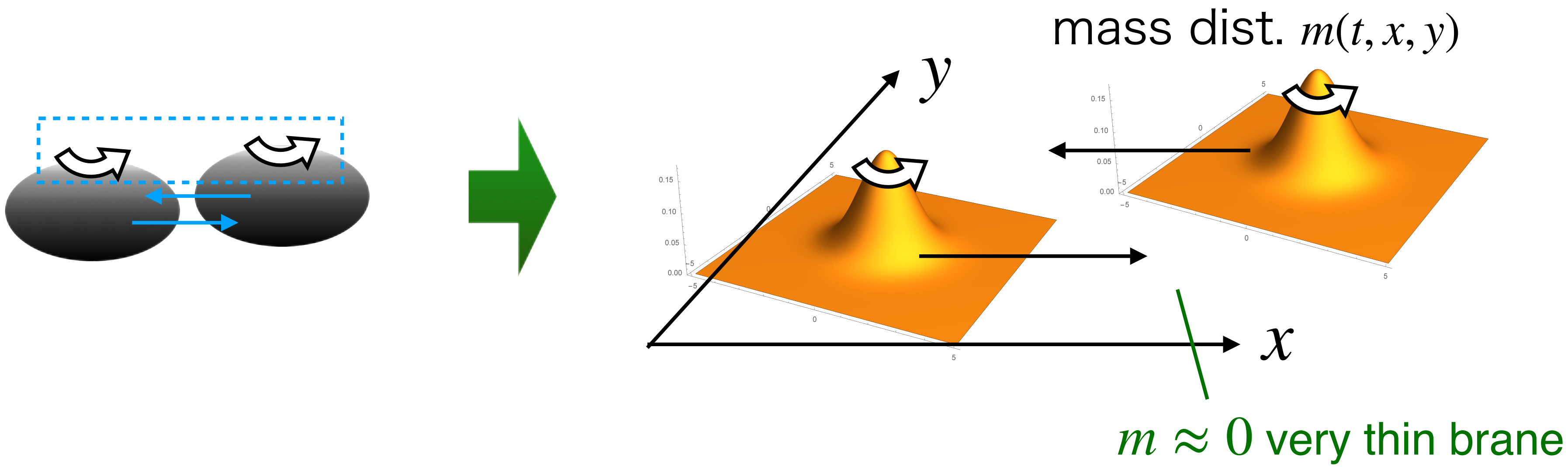


Corresponds to Near-poler region of Myers-Perry BH

2+1 Blob simulation

Andrade-Emparan-Licht-Luna (2018), (2019)
Andrade-Emparan-Jansen-Licht-Luna-**RS** (2020)

dynamics of BHs → motion of “blobs” on black 2-brane



Much easier than full numerical analysis

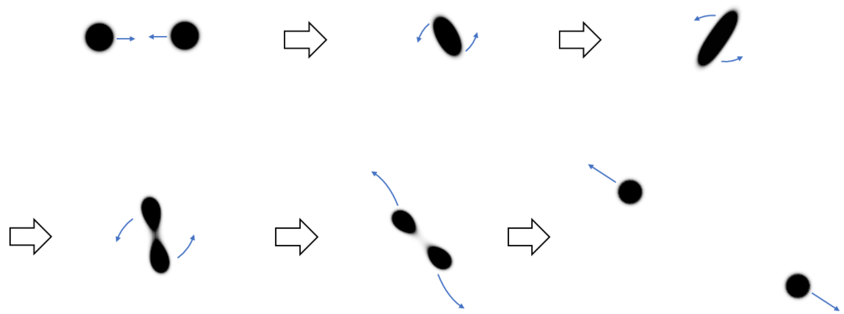
Effective theory→ 2+1 PDE for (m, p_x, p_y)

$$\partial_t m = \ldots, \quad \partial_t p_x = \ldots, \quad \partial_t p_y = \ldots$$

You need just laptop !

2+1 Blob simulation

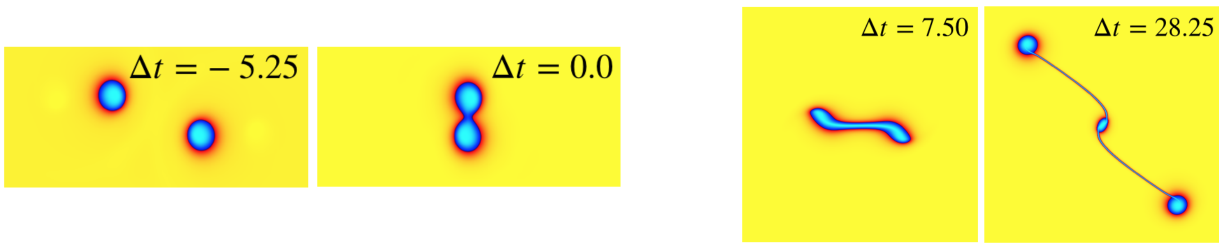
Blob-Blob Collision Andrade-Emparan-Licht-Luna '19 ← **Laptop only !**



$j_c \leq J/M$: unstable Bar $\rightarrow$ split

※only colored $m(t,x,y) \geq \epsilon$, white parts have very thin brane

Consistent with Numerical Relativity in D=6,7 Andrade-Figueras-Sperhake '20



← **super computer**

Holography at Large D

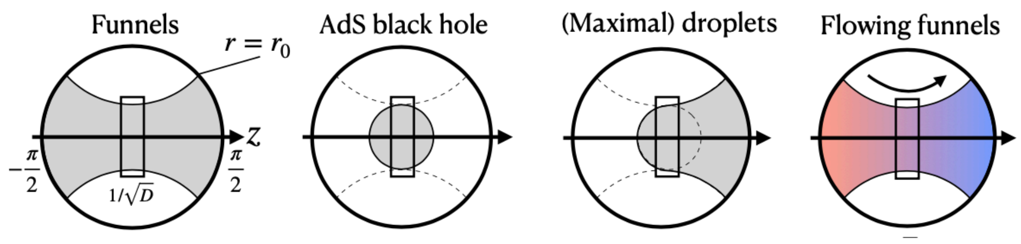
Large D is also useful for holography

- Holographic superconductor@D=∞ Emparan-Tanabe (2014)
- Fluid/Gravity@D=∞ Andrade-Christiana-Pantelidou-Withers (2018)
  C.-Solana-Herzog-Meiring (2018),….
- Various AdS BHs are found in Global AdS Emparan-Licht-Tomasevic-**RS**-Benson (2021)
  <=> thermal bath with or without BHs in S^{n+1} Licht-**RS**-Benson (2022)

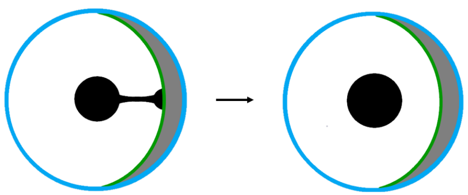
Large D Effective theory
on AdS black funnel

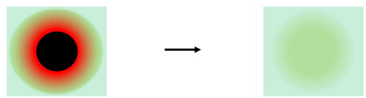
$$\partial_t m + (\partial_x + x)(p - \partial_x m) = 0\,,$$

$$\partial_t p - (\partial_x + x)\left(\partial_x p - \frac{p^2}{m}\right) + p - \left(1 + \frac{1}{r_0^2}\right)\partial_x m = 0\,.$$



- holographic collapse/evapolation, etc…, in Global AdS + KR-branes
  Emparan-Luna-Tomasevic-**RS**-Benson (2023)





EGB gravity at Large D

Overview: EGB-BH@Large D

Large D can be also useful tool beyond GR !

- Simplification at large D comes from localization of gravity around BHs where Near-horizon geometry is simplified (and so is the analysis on it)

  This property is merely about local geometric structure
  $\rightarrow$ Should be applicable regardless of background geom., and theories

- We already knew Large D can be applied to several EGB-BHs:

black string	black ring,	rotating BH (equal ang. mom.)
Chen-Li-Zhang (2017) **RS**-Tomizawa (2022)	Chen-Li-Zhang (2018)	**RS**-Tomizawa (2022)

today's topic

Scaling of $\alpha_{\rm GB}$ at Large D

We must determine how the GB coupling scales at large D

$$R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} + \alpha_{\rm GB}H_{\mu\nu} = 0$$

$$H_{\mu\nu} = -\frac{1}{2}\mathcal{L}_{\rm GB}g_{\mu\nu} + 2RR_{\mu\nu} - 4R_{\mu\alpha}R^{\alpha}{}_{\nu} - 4R_{\mu\alpha\nu\beta}R^{\alpha\beta} + 2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma}.$$

Most interesting regime:

EH term〜GB term@D→∞ → $\alpha_{\rm GB} = O(D^{-2})$

cf) Near Horizon@D→∞ : $R \sim D^2/r_0^2$ (rad: r_0)

NOTE: EH≫GB or EH≪GB cases

are obtained as the parameter limit $D^2\alpha_{\rm GB} \to 0$ or $D^2\alpha_{\rm GB} \to \infty$

Rotating EGB BH at Large D

Myers-Perry BH@Large D

Emparan-Grumiller-Tanabe (2013), Emparan-**RS**-Tanabe (2014)

D=2n+3 Equally-rotating Myers-Perry@Large D $\mathsf{R} := r^{2n}$

$$ds^2_{\rm MP} \simeq -\left(1-\frac{1}{\mathsf{R}}\right)(e^{(0)})^2 + \left(1-\frac{1}{\mathsf{R}}\right)^{-1} dr^2 + \underbrace{(e^{(2)})^2 + \overbrace{d\Sigma^2}^{CP^n}}_{S^{2n+1}}$$

Myers-Perry@D=∞ = static black holes@D=∞

but with $e^{(i)} := \Lambda^i{}_t dt + \Lambda^i{}_\phi(d\phi + \mathscr{A}) \quad \leftrightarrow \quad (dt, d\phi + \mathscr{A})$

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Analysis of Myers-Perry at large D is simplified

- Leading order equation becomes that of static BHs
- Equation decouples to separate ODEs (then Integrable)

ex) perturbative analysis Myers-Perry BH Emparan-**RS**-Tanabe (2014), **RS**-Tanabe (2015)
Charged Myers-Perry Tanabe (2016), Mandlik-Thakur (2018)

Myers-Perry BH@Large D

Emparan-Grumiller-Tanabe (2013), Emparan-**RS**-Tanabe (2014)

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Main Strategy: assume the same property in EGB theory

Setup

For simplicity, we consider equally-rotating BH in D=2n+3.

Background : Minkowski in Eddington-Finkelstein coord.

$$ds^2 = -dt^2 + 2dtdr + \underline{r^2(d\phi + \mathscr{A})^2 + r^2 d\Sigma^2}$$

$S^{2n+1} \to$ a hopf fibration of S^1 on CP^n

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$S^{2n+1} \to$ a hopf fibration of S^1 on CP^n

Assume the ansatz in the boosted frame
(expecting the same simplification as in Myers-Perry)

$$ds^2 = -A(r)(e^{(0)})^2 + 2U(r)e^{(0)}e^{(1)} + 2C(r)e^{(0)}e^{(2)} + H(r)(e^{(2)})^2 + r^2 d\Sigma^2$$

$(dt, dr, d\phi + \mathscr{A})$ $\to$ boosted frame

$$e^{(0)} = \frac{dt - \Omega r(d\phi + \mathcal{A})}{\sqrt{1-\Omega^2}} \qquad e^{(1)} = dr \qquad e^{(2)} = \frac{r(d\phi + \mathcal{A}) - \Omega dt}{\sqrt{1-\Omega^2}}$$

1/D expansion and Assumption

Ansatz

$$ds^2 = -A(r)(e^{(0)})^2 + 2U(r)e^{(0)}e^{(1)} + 2C(r)e^{(0)}e^{(2)} + H(r)(e^{(2)})^2 + r^2 d\Sigma^2$$

1/n-expansion with $\mathsf{R} := r^{2n}$ $(r_H = 1)$ (D=2n+3) $\alpha := (2n)^2 \alpha_{\mathrm{GB}}$

$$A = \sum_{i=0}^{\infty} \frac{1}{n^i} A_i(\mathsf{R}), \quad U = \sum_{i=0}^{\infty} \frac{1}{n^i} U_i(\mathsf{R}), \quad C = \sum_{i=0}^{\infty} \frac{1}{n^i} C_i(\mathsf{R}), \quad H = \sum_{i=0}^{\infty} \frac{1}{n^i} H_i(\mathsf{R}).$$

1/D expansion and Assumption

Ansatz

$$ds^2 = -A(r)(e^{(0)})^2 + 2U(r)e^{(0)}e^{(1)} + \cancel{2C(r)e^{(0)}e^{(2)}} + \underbrace{H(r)(e^{(2)})^2 + r^2 d\Sigma^2}_{S^{2n+1}}$$

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Assumption: LO-metric ≈ static BH (with boosted frame)

$$C(r) = \mathcal{O}(1/n), \quad H(r) = 1 + \mathcal{O}(1/n)$$

1/D expansion and Assumption

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Assumption: LO-metric ≈ static BH (with boosted frame)

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EGB equation decouples to separate ODEs w.r.t R

→ Integrable

Leading order solution

$$A_0 = 1 + \frac{1}{2\alpha} - \frac{1}{2\alpha}\sqrt{1 + \frac{4\alpha(\alpha+1)m}{\mathsf{R}}}$$

$$U_0 = 1, \quad C_0 = 0, \quad H_0 = 1,$$

$$ds^2 \simeq -A_0(e^{(0)})^2 + 2e^{(0)}e^{(1)} + (e^{(2)})^2 + d\Sigma^2 + \mathcal{O}(n^{-1})$$

With $e^{(0)} \leftrightarrow dt, \quad e^{(1)} \leftrightarrow dr, \quad e^{(2)} \leftrightarrow d\phi + \mathcal{A}$

Identical to D=2n+3 Boulware-Deser @large D

(as expected)

Higher order corrections

Higher order corrections are obtained by solving the sourced ODEs w.r.t R

With an auxiliary variable $X := \sqrt{1 + \frac{4\alpha(\alpha+1)m}{\mathrm{R}}}$

Next-to-Leading order sols : A_1, C_1, H_1, U_1

$$A_1 = \frac{\left(X^2-1\right)\Omega^2 \log\left(X^2+1\right)}{16\alpha\left(2\alpha^2+3\alpha+1\right)X\left(\Omega^2-1\right)} + \frac{\left(X^2-1\right)\Omega^2(\arctan X - \arctan(1+2\alpha))}{8\alpha(\alpha+1)X\left(\Omega^2-1\right)} - \frac{(X-1)\left(X+2\Omega^2-1\right)\log(X-1)}{4\alpha X\left(\Omega^2-1\right)}$$
$$- \frac{(X-1)\log(X+1)\left(\alpha\left(4\Omega^2-2\right)+X\left(2\alpha+\Omega^2+2\right)+5\Omega^2-2\right)}{8\alpha(\alpha+1)X\left(\Omega^2-1\right)} + a_0 + a_1 X + \frac{a_2}{X},$$

$$C_1 = \frac{\Omega(X-1)}{4\alpha(1-\Omega^2)}\log\left(\frac{4\alpha(1+\alpha)}{X^2-1}\right) \qquad U_1 = \frac{(X-1)\Omega^2(\alpha(X-1)-1)}{2(\alpha+1)(2\alpha+1)\left(X^2+1\right)\left(\Omega^2-1\right)}$$

$$H_1 = \frac{\Omega^2}{(\alpha+1)(1-\Omega^2)}\left[\log\left(\frac{X+1}{2}\right) - \arctan X + \frac{\pi}{4} - \frac{1}{2(2\alpha+1)}\log\left(\frac{X^2+1}{2}\right)\right],$$

NNLO is also obtained (much more complicated to show)

Ergo Region

Ergo radius is obtained by

$$0 = g_{tt} = (1-\Omega^2)^{-1}(-A - 2\Omega C + \Omega^2 H)$$

Using the leading order solution

$$\mathsf{R}_{\rm ergo} = \frac{(1+\alpha)m}{(1-\Omega^2)(1+\alpha(1-\Omega^2))} + \mathcal{O}(n^{-1})$$

- Ergo region exists for any α
- Size of ergo region monotonically increases with α
  Reach a limit at α→∞

$$\mathsf{R}_{\rm ergo}\Big|_{\alpha=0} = \frac{m}{1-\Omega^2} \Longrightarrow \mathsf{R}_{\rm ergo}\Big|_{\alpha=\infty} = \frac{m}{(1-\Omega^2)^2}$$

Thermodynamics

Metric is solved up to NLO in 1/n-expansion
$\longrightarrow$ Thermodynamic variables are obtained up to the same order

Entropy ← Iyer-Wald formula

The 1st law is checked up to NLO

$$\mathcal{S}=\frac{1}{4G}\int_H (1+2\alpha_{\rm GB}\mathcal{R})\sqrt{h}\,d^{D-2}x$$

scalar curvature of horizon surface

1/n-expansion up to O(1/n) $\alpha_H := \alpha/m^{\frac{1}{2n}}$

$$M=\frac{n\Omega_{2n+1}}{8\pi G}\frac{(1+\alpha_H)m}{1-\Omega^2}\left[1-\frac{1}{8n\left(1-\Omega^2\right)\left(\alpha_H+1\right)\left(2\alpha_H+1\right)}\left(4-8\Omega^2\alpha_H^2+\left(-8\Omega^4+2(\pi-6)\Omega^2+8\right)\alpha_H\right.\right.$$
$$\left.\left.-2\Omega^2\log\left(2\alpha_H^2+2\alpha_H+1\right)+4\Omega^2\left(2\alpha_H+1\right)\left(\log\left(\alpha_H+1\right)-\arctan\left(2\alpha_H+1\right)\right)+\Omega^2(\pi-4)\right)\right],$$

$$J=\frac{n\Omega_{2n+1}}{8\pi G}\frac{(1+\alpha_H)m^{\frac{2n+1}{2n}}\Omega}{1-\Omega^2}\left[1-\frac{1}{8n\left(1-\Omega^2\right)\left(\alpha_H+1\right)\left(2\alpha_H+1\right)}\left(8\left(2\Omega^2-1\right)\alpha_H^2+2\Omega^2\log\left(2\alpha_H^2+2\alpha_H+1\right)\right.\right.$$
$$\left.\left.+4(1+2\alpha_H)\Omega^2(\arctan\left(2\alpha_H+1\right)-\log(\alpha_H+1))-2\alpha_H\left((\pi-16)\Omega^2+10\right)-(\pi-8)\Omega^2-8\right)\right],$$

$$T=\frac{n}{\pi}\frac{1+\alpha_H}{1+2\alpha_H}m^{-\frac{1}{2n}}\sqrt{1-\Omega^2}\left[1-\frac{\left(4\Omega^2+1\right)\alpha_H+4\alpha_H^2+2\Omega^2}{2n\left(1-\Omega^2\right)\left(\alpha_H+1\right)\left(2\alpha_H+1\right)}\right],$$

$$S=\frac{\Omega_{2n+1}}{4G}\frac{(1+2\alpha_H)m^{\frac{2n+1}{2n}}}{\sqrt{1-\Omega^2}}\left[1+\frac{1}{8n(1-\Omega^2)\left(\alpha_H+1\right)\left(2\alpha_H+1\right)}\left(8\left(1-2\Omega^2\right)\alpha_H^2+8\alpha_H\left(1-2\Omega^2\right)\right.\right.$$
$$\left.\left.+4(1+2\alpha_H)\Omega^2\left(\log(1+\alpha_H)-\arctan\left(2\alpha_H+1\right)+\pi/4\right)-2\Omega^2\log\left(2\alpha_H^2+2\alpha_H+1\right)\right)\right].$$

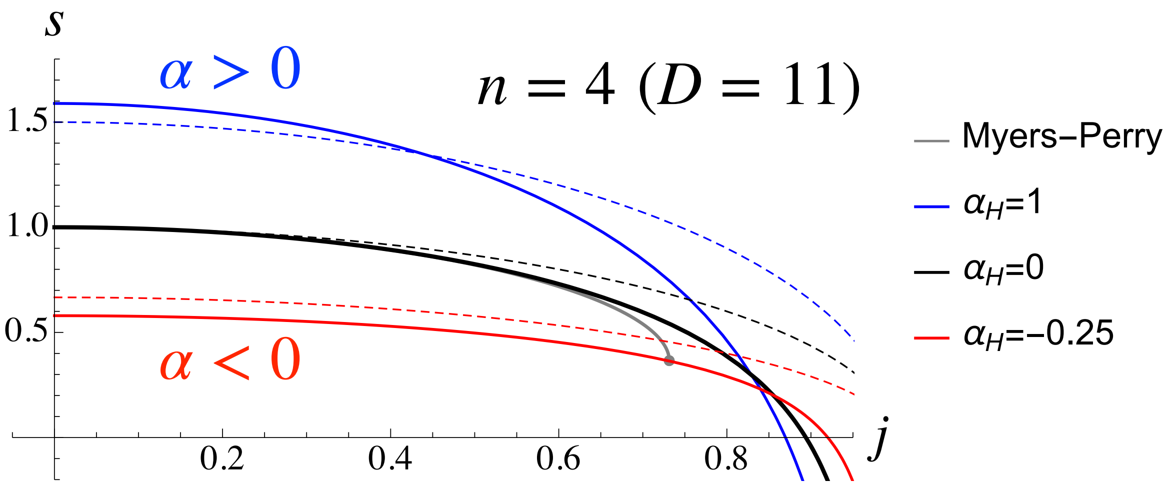
Phase diagram

Mass-Normalized Entropy and Angular momentum

$$j := \frac{8\pi G J}{(n+1)\Omega_{2n+1}}\left(\frac{8\pi G M}{(n+1/2)\Omega_{2n+1}}\right)^{-\frac{2n+1}{2n}} \qquad s := \frac{4GS}{(n+1)\Omega_{2n+1}}\left(\frac{8\pi G M}{(n+1/2)\Omega_{2n+1}}\right)^{-\frac{2n+1}{2n}}$$

Thick：O(1)+O(1/n)

Dotted：O(1)



$$s = \frac{\sqrt{1-j^2}\,(2\alpha_H+1)}{\alpha_H+1} \times\left[1+\frac{1}{2n(1-j^2)}\left(\log\left(\frac{1-j^2}{1+\alpha_H}\right) + \frac{\alpha_H\left(4\left(1-j^2\right)\alpha_H-4j^2+3\right)}{(\alpha_H+1)\,(2\alpha_H+1)}\right)\right]$$

- 1/n expansion is bad around extremal limit (the same is true in GR)
- For any j, Entropy Increase for α>0
  Decrease for α<0

EGB-Black String at Large D

Overview : EGB-BS@Large D

- In Chen-Li-Zhang (2017) , EGB black string is solved at leading order in 1/D
  However, LO results are much similar to GR
  - Threshold of Instability : $k_{\rm GL} = 1 + O(1/D)$
  - Entropy is proportional to the mass@LO

  $$\rightarrow \partial_t S = O(1/D) \quad \text{cf) } S_{\rm Sch} \propto M_{\rm Sch}^{\frac{D-2}{D-3}} \sim M_{\rm Sch}(1 + O(1/D))$$

- 2nd law was proven in Higher curvature theory around stationary BHs with Iyer-Wald(-Wall) formula Wall (2015), Hollands-Kovács-Reall (2022)

  $$S = \int \sqrt{h}(1 + 2\alpha_{GB} R_H) dA$$ Iyer-Wald (1993,1994)

How about with more nonlinear evolution ? (@large D)

This motivate us to study NLO correction to EGB-BS

Ansatz

Ansatz for $D = n + 4$ dynamical black string

$$ds^2 = -Adt^2 + 2Udtdr - \frac{1}{n}Cdtdz + \frac{1}{n}Gdz^2 + r^2 d\Omega_{n+1}^2$$

1/n expansion with $\mathsf{R} := (r/r_0)^n$

$$A = \sum_{i=0} \frac{A_i(t, \mathsf{R}, z)}{n^i}, \quad C = \sum_{i=0} \frac{C_i(t, \mathsf{R}, z)}{n^i} \quad G = 1 + \frac{1}{n}\sum_{i=0} \frac{G_i(t, \mathsf{R}, z)}{n^i}, \quad U = 1 + \frac{1}{n}\sum_{i=0} \frac{U_i(t, \mathsf{R}, z)}{n^i}$$

EGB equation

$$E_{\mu\nu} := R_{\mu\nu} + \alpha_{\mathrm{GB}} \tilde{H}_{\mu\nu} = 0 \qquad \alpha := n^2 \alpha_{\mathrm{GB}} / r_0^2$$

Leading order analysis

S^{n+1}-part $\quad E_{ij} = E_\Omega \gamma_{ij}$ $\quad (E_{\mu\nu} := R_{\mu\nu} + \alpha_{\mathrm{GB}} \tilde{H}_{\mu\nu})$

$$E_\Omega / n = \underbrace{\mathsf{R}(2\alpha A_0 - 2\alpha - 1)\partial_{\mathsf{R}} A_0 + (A_0 - 1)(\alpha A_0 - \alpha - 1)}_{} + \mathcal{O}(1/n)$$

ODE for $A_0(\mathsf{R})$

$$A_0 = 1 + \frac{1}{2\alpha} - \frac{1}{2\alpha}\sqrt{1 + \frac{4\alpha(\alpha+1)m(t,z)}{\mathsf{R}}}$$

C_0

$$C_0 = \frac{p(t,z)}{2\alpha m(t,z)}\left(\sqrt{1 + \frac{4\alpha(\alpha+1)m(t,z)}{\mathsf{R}}} - 1\right),$$

$m(t,z),\ p(t,z)$: arbitrary funcs

Leading order sols

Chen-Li-Zhang (2017)

An auxiliary variable to simplify the expression

$$X := \sqrt{1 + \frac{4\alpha(\alpha+1)m}{\mathsf{R}}} \quad \Rightarrow \quad A_0 = \frac{1+2\alpha - X}{2\alpha}, \quad C_0 = \frac{p}{2\alpha m}(X-1)\,.$$

$$X:\ 1 = X(\mathsf{R}=\infty) \le X \le X(\mathsf{R}=1) = 1+2\alpha$$

We also have G_0, U_0

$$G_0 = \left(\frac{\log\left(X^2+1\right)}{2\alpha+1} - 2\log(X+1) + 2\arctan X\right)\left(\frac{m\partial_z p - p\partial_z m + m^2}{(\alpha+1)m^2}\right)$$
$$-\frac{((2\alpha+1)\pi - 2(4\alpha+1)\log 2)(m\partial_z p - p\partial_z m + m^2)}{2(\alpha+1)(2\alpha+1)m^2} + \frac{p^2}{2\alpha m^2}(X-1)$$

$$U_0 = \frac{p\left(\left(2\alpha^2+3\alpha+1\right)p - 4\alpha^2\partial_z m\right) + 4\alpha^2 m\partial_z p + 4\alpha^2 m^2}{4\alpha(\alpha+1)(2\alpha+1)m^2}$$
$$-\frac{((2\alpha+1)X\partial_z m - 1)\left(-\partial_z m p + m\partial_z p + m^2\right)}{(\alpha+1)(2\alpha+1)\left(X^2+1\right)m^2} - \frac{Xp^2}{4\alpha m^2}\,.$$

Higher order corrections

Consider R-integration in higher order (put aside the constraints)

Higher order corrections A_i, C_i, G_i, U_i are derived by ODEs + sources

$$X := \sqrt{1 + \frac{4\alpha(\alpha+1)m}{\mathsf{R}}}$$

$$\partial_X \left(\frac{X}{1-X^2} A_i \right) = \mathcal{S}_A^{(i)}$$

$$\partial_X^2 C_i = \mathcal{S}_C^{(i)}$$

$$\partial_X [(1+X^{-2})(1+2\alpha - X)\partial_X G_i] = \mathcal{S}_G^{(i)}$$

$$\partial_X \left(U_i - \frac{1-X^2}{4X} \partial_X G_i \right) = \mathcal{S}_U^{(i)}$$

A_1

$$A_1 = \frac{(X^2-1)\left(-p\partial_z m + m\partial_z p + m^2\right)}{4\alpha(\alpha+1)Xm^2}\left(\frac{\log(X^2+1)}{2(2\alpha+1)} + \arctan X\right)$$

$$+\frac{(X-1)\log(X+1)\left((X+1)p\partial_z m + \alpha(X+1)m\partial_z p + (-2\alpha+2\alpha X + X - 3)m^2\right)}{4\alpha(\alpha+1)Xm^2}$$

$$+\frac{(X-1)\log(X-1)\left(2(X-1)m + (X+1)\partial_z p\right)}{4\alpha Xm}$$

$$+\frac{(X-1)\left(p^2 - 2p\partial_z m + 2m\partial_z p + m^2(2\log(4\alpha(\alpha+1)m)+1)\right)}{2\alpha Xm^2}$$

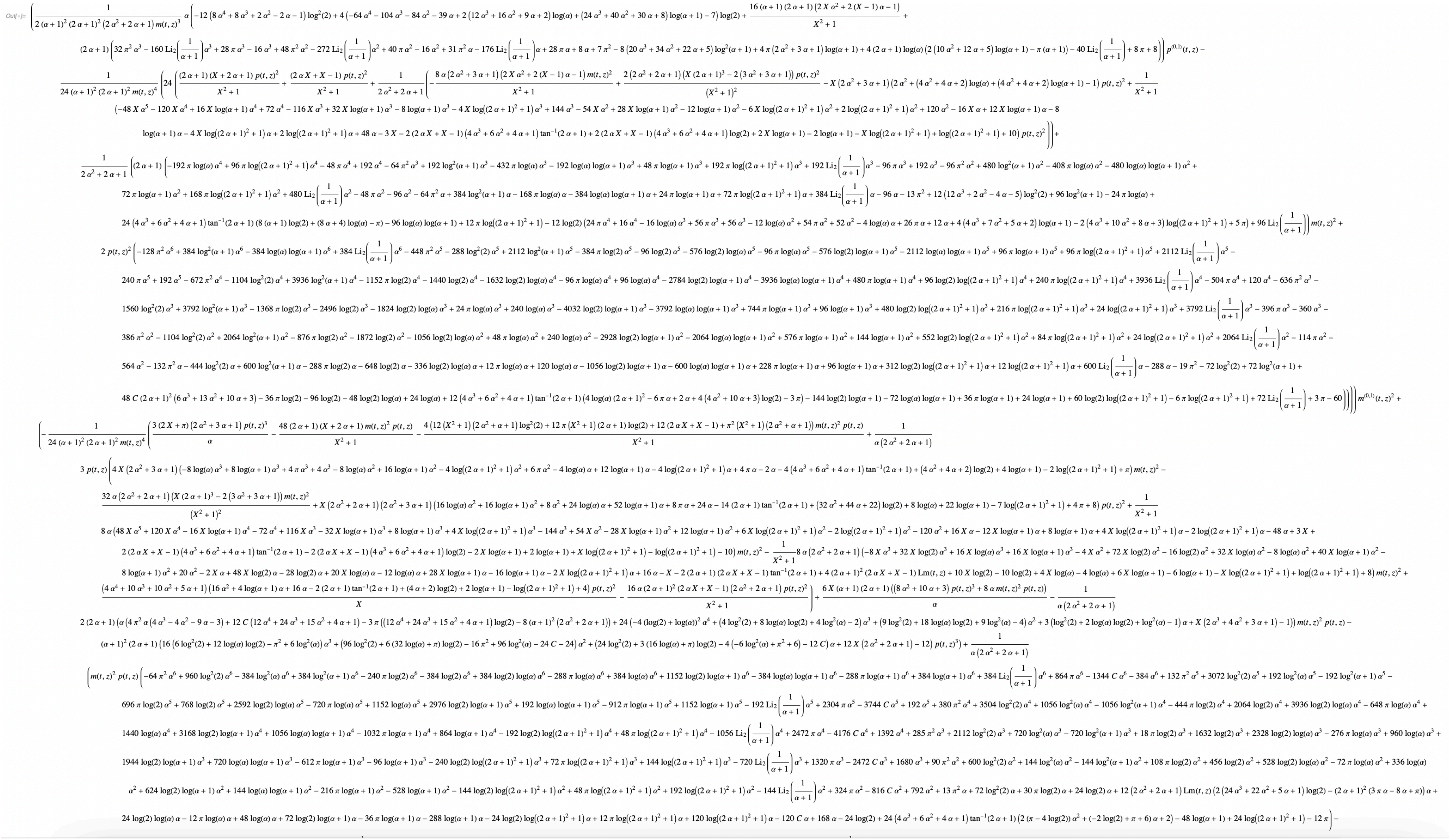
$$+\frac{(1-X^2)(2(2\alpha+1)(2m^2\log m + p^2) + c_1 m^2 + c_2 p\partial_z m + c_3 m\partial_z p)}{8X\alpha(1+\alpha)(2\alpha+1)m^2}$$

c_1, c_2, c_3:constants

G_1

C_1, U_1, H_1 are also integrable but more lengthy form…

$G_1 =$



$\vdots$

Effective equation

From constraint eqs

Effective equation up to NLO is obtained

(LO part is already obtained in Chen-Li-Zhang (2017))

$$\partial_t m - \partial_z^2 m = -\partial_z p + O(1/n)$$

$$\partial_t p - \partial_z^2 p = \partial_z \left(m - \frac{p^2}{m} + \frac{2\alpha}{(\alpha+1)(2\alpha+1)} \left(\frac{p\partial_z m - m\partial_z p - m^2}{m} \right) \right)$$

$$+O(1/n)$$

2nd law

From Iyer-Wald formula $$S = CM + \frac{1}{n} S_1$$

Horizon velocity: $v_i(t,x) := (p_i - \partial_i m)/m$

$$S_1 = \frac{1+\alpha+2\alpha^2}{1+\alpha} \int_0^L \left(-\frac{(\partial_z m)^2}{2m} - \frac{(2\alpha+1)(1+\alpha)}{2(1+\alpha+2\alpha^2)} m v^2 + m \log m \right) dz\,.$$

$$\Rightarrow \quad \partial_t S_1 = \frac{2(1+\alpha+2\alpha^2)}{1+\alpha} \int_0^L m(\partial_z v)^2 dz \geq 0$$

LO effective theory

2nd law holds

Gregory-Laflamme instability

Perturbation of uniform sols

$$m(t,z) = 1 + \varepsilon\, m_1(\Omega, k) e^{\Omega t} \cos(kz), \quad p(t,z) = \varepsilon\, p_1(\Omega, k) e^{\Omega t} \sin(kz),$$

$$\Omega = \Omega_{\mathrm{LO}}(k) + \frac{1}{n}\Omega_{\mathrm{NLO}}(k)$$

Dispersion at LO

$$\Omega_{\mathrm{LO}} = -\frac{1+2\alpha+2\alpha^2}{(2\alpha+1)(\alpha+1)}k^2 \pm \frac{k\sqrt{4\alpha^4+8\alpha^3+4\alpha+\alpha^2\left(k^2+7\right)+1}}{(2\alpha+1)(\alpha+1)},$$

Chen-Li-Zhang 17

$$\Omega_{\mathrm{LO}}(k_{\mathrm{GL}}) = 0 \quad \Rightarrow \quad k_{\mathrm{GL}} = 1$$

threshold is same as GR at LO

Dispersion@NLO

$$\Omega = \Omega_{\mathrm{LO}} + \frac{1}{n}\Omega_{\mathrm{NLO}}$$

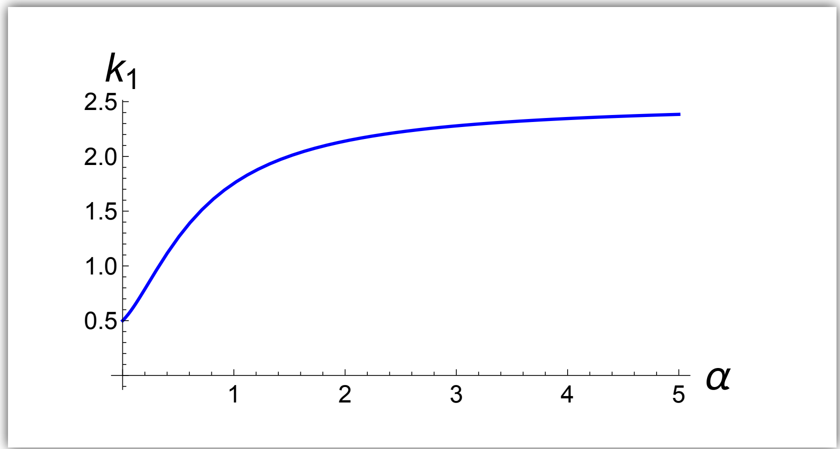
$$\begin{aligned}
\Omega_{\mathrm{NLO}} = &-\frac{2\alpha\left(2\alpha^2+2\alpha+1\right)k^4\arctan(2\alpha+1)}{(\alpha+1)^3(2\alpha+1)^2} - \frac{\alpha\left(2\alpha^2+2\alpha+1\right)k^4\log(2\alpha^2+2\alpha+1)}{(\alpha+1)^3(2\alpha+1)^3} \\
&-\frac{2\alpha\left(2\alpha^2+2\alpha+1\right)k^4\log(1+\alpha)}{(\alpha+1)^3(2\alpha+1)} - \frac{\left(40\alpha^6+104\alpha^5+112\alpha^4+60\alpha^3+20\alpha^2+6\alpha+1\right)k^2}{(\alpha+1)^2(2\alpha+1)^2\left(2\alpha^2+2\alpha+1\right)} \\
&+\frac{\alpha(8\alpha+\pi+4)\left(2\alpha^2+2\alpha+1\right)k^4}{2(\alpha+1)^3(2\alpha+1)^2} \pm \frac{k\sqrt{4\alpha^4+8\alpha^3+7\alpha^2+4\alpha+\alpha^2k^2+1}}{4(\alpha+1)^3(2\alpha+1)^3} \\
&\quad\times\Bigg[\frac{4\left(4\alpha^3+6\alpha^2+4\alpha+1\right)k^3\arctan(2\alpha+1)\left(4\alpha^4+16\alpha^3+8\alpha+\alpha^2\left(19-2k^2\right)+1\right)}{4\alpha^4+8\alpha^3+4\alpha+\alpha^2\left(k^2+7\right)+1} \\
&\qquad+\frac{2\left(2\alpha^2+2\alpha+1\right)k^3\log(2\alpha^2+2\alpha+1)\left(4\alpha^4+16\alpha^3+8\alpha+\alpha^2\left(19-2k^2\right)+1\right)}{4\alpha^4+8\alpha^3+4\alpha+\alpha^2\left(k^2+7\right)+1} \\
&\qquad-\frac{4(2\alpha+1)^2k^3\log(\alpha+1)\left(-8\alpha^5+2\alpha+4\alpha^4\left(k^2-5\right)+2\alpha^3\left(2k^2-9\right)+\alpha^2\left(2k^2-5\right)+1\right)}{4\alpha^4+8\alpha^3+4\alpha+\alpha^2\left(k^2+7\right)+1} \\
&\qquad+\frac{(2\alpha+1)k}{4\alpha^4+8\alpha^3+4\alpha+\alpha^2\left(k^2+7\right)+1}\Big(2(\alpha+1)^2\left(8\alpha^5-28\alpha^4-46\alpha^3-19\alpha^2-4\alpha-1\right) \\
&\qquad+2\alpha^2(8\alpha+\pi+4)\left(2\alpha^2+2\alpha+1\right)k^4-(\alpha+1)\Big(\pi\left(8\alpha^5+32\alpha^4+42\alpha^3+28\alpha^2+9\alpha+1\right) \\
&\qquad\qquad-4\left(16\alpha^6+24\alpha^5+12\alpha^4-9\alpha^3-8\alpha^2+\alpha+1\right)\Big)k^2\Big)\Bigg].
\end{aligned}$$

Correction to kGL

k_{GL} depends on α from NLO

$$\Omega(k_{\mathrm{GL}}) = 0 \to k_{\mathrm{GL}} = 1 - \frac{k_1}{n} \quad \text{or} \quad L_{\mathrm{GL}} = 2\pi\left(1 + \frac{k_1}{n}\right)$$

$$\boxed{k_1 = \frac{20\alpha^3 + 2\pi\alpha^2 + 34\alpha^2 + 2\pi\alpha + 12\alpha + \pi + 2}{4(\alpha+1)(2\alpha^2+\alpha+1)} + \frac{2\log(\alpha+1)}{2(\alpha+1)\left(2\alpha^2+\alpha+1\right)} - \frac{\left(2\alpha^2+2\alpha+1\right)\arctan(2\alpha+1)}{(\alpha+1)\left(2\alpha^2+\alpha+1\right)} - \frac{\left(2\alpha^2+2\alpha+1\right)\log(2\alpha^2+2\alpha+1)}{2(\alpha+1)(2\alpha+1)(2\alpha^2+\alpha+1)}}$$



α: Large, $k_{\mathrm{GL}} \to$ small ($L_{\mathrm{GL}} \to$ Large)

→Uniform phase is stabilized

Weakly Non-Uniform BS

From 0-mode at $k = k_{\rm GL}$, static NUBS appears
NUBS can be obtained by perturbative expansion

$$\lambda := \frac{1}{2}\left((r_{\max}/r_{\min})^n - 1\right)$$

Non-uniformity parameter

$$m(z) = \exp\left(\sum_{i=0}\lambda^i \mu_i \cos\left(\frac{2\pi i z}{L}\right)\right), \quad p(z) = \ldots$$

$$\mu_i = \sum_{j=0,k=0} \mu_{i,j,k}\lambda^j n^{-k}$$

Thermodynamics are obtained in expansion in 1/n and λ

$$\frac{M}{M_{\rm GL}} = 1 + \frac{\lambda^2}{24}\left(1 - \frac{\ell_1 - 12}{n}\right) + \mathcal{O}(\lambda^3)$$

$$\frac{S}{S_{\rm GL}} = 1 + \frac{\lambda^2}{24}\left(1 - \frac{\ell_1 - 12}{n}\right) + \mathcal{O}(\lambda^3)$$

$$\frac{\mathcal{T}}{\mathcal{T}_{\rm GL}} = 1 - \left(1 + \frac{9 - \ell_1}{n}\right)\frac{\lambda^2}{2} + \mathcal{O}(\lambda^3)$$

$$\ell_1 := \frac{48\alpha^3 + 76\alpha^2 + (40+\pi)\alpha + 16}{2(\alpha+1)(2\alpha^2+\alpha+1)} - \frac{\alpha\left(\log\left(2\alpha^2+2\alpha+1\right) + 2(2\alpha+1)^2\log(\alpha+1) + 2(2\alpha+1)\arctan(2\alpha+1)\right)}{(\alpha+1)(2\alpha+1)\left(2\alpha^2+\alpha+1\right)}$$

$M_{\rm GL}(\alpha)$, $S_{\rm GL}(\alpha)$, $\mathcal{T}_{\rm GL}(\alpha)$: Uniform values at GL point

Critical Dimensioin

Critical Dimension in GR $13 < D_* < 14$

NUBS is unstable for $D < D_*$ and stable for $D > D_*$ Sorkin (2004)

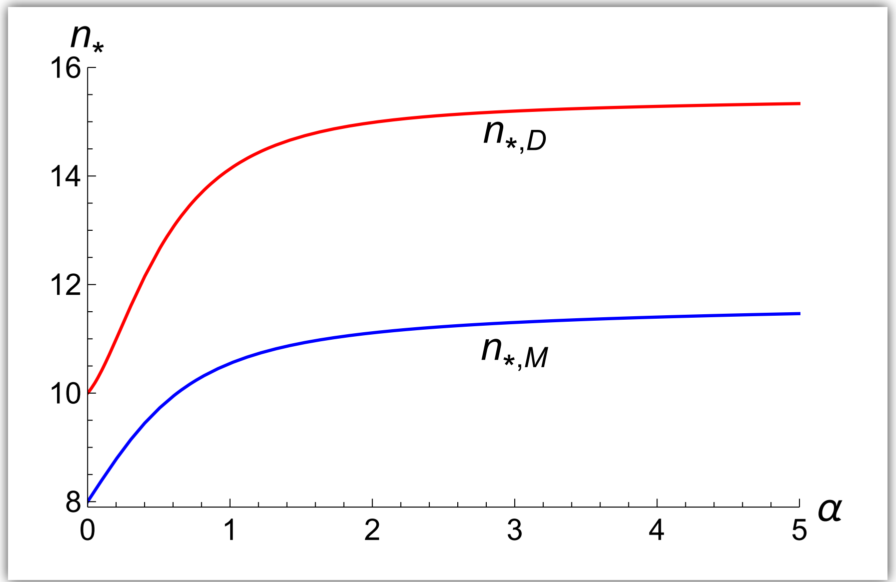
$D_* = n_* + 4$ **depends on α in EGB**

Thermodynamic Critical D

$$\frac{M}{L^{n+1}} \simeq \frac{M_{\rm GL}}{L_{\rm GL}^{n+1}}\left[1+\frac{n\lambda^2}{24}\underbrace{\left(1-\frac{n_{*,M}}{n}\right)}_{=0}\right]$$

Dynamics Critical D

$$\Omega = -\frac{2\alpha^2+\alpha+1}{2\alpha^2+2\alpha+1}\underbrace{\left(1-\frac{n_{*,D}}{n}\right)}_{=0}\frac{\lambda^2}{12}$$



Increasing functions of α

$\rightarrow$ stable NUBS exists in more large D

Note : $D_{*,error}$ = O(1) up to NLO

$$n = n_*(1+O(1/n_*)) = n_* + O(1)$$

Summary

Summary

- Many exact sols of BHs are missing in EGB theory
  (also in Higher curvature theories)

- The difficulty can be circumvented by **the large D limit**
  (at the cost of 1/D expansion)

- Black strings, Rotating BHs are solved in 1/D expansion

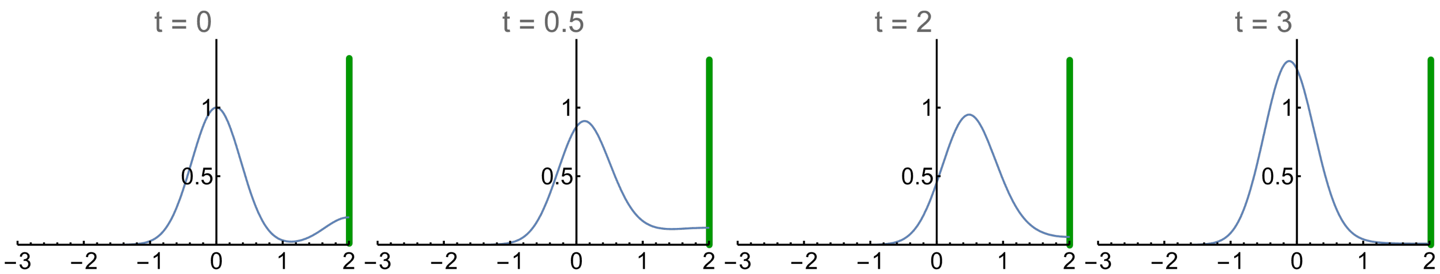
Future Work

- Can large D also allows to solve BHs analytically
  in **Lovelock theory or more generic higher curvature theory** ?

Appendix

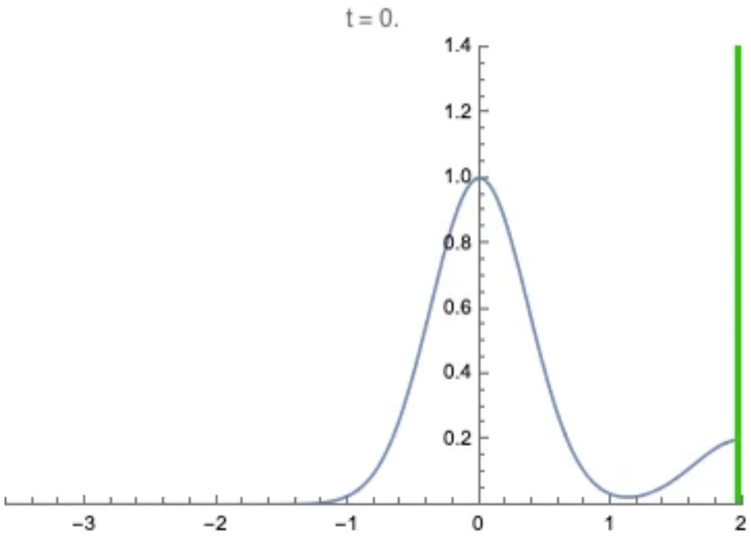
Holographic evaporation

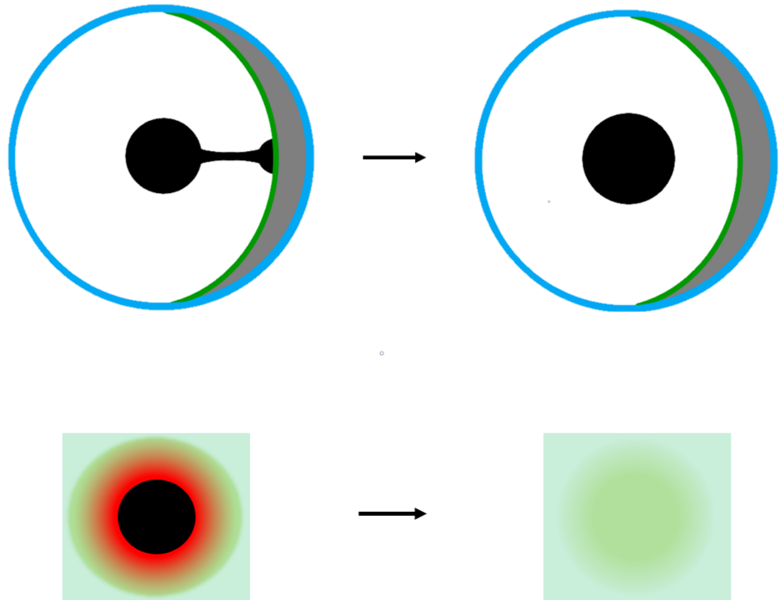
$r_0 = 0.4$ Holographic Evaporation in Emparan et.al. 2002, Tanaka 2002



blob+small droplet

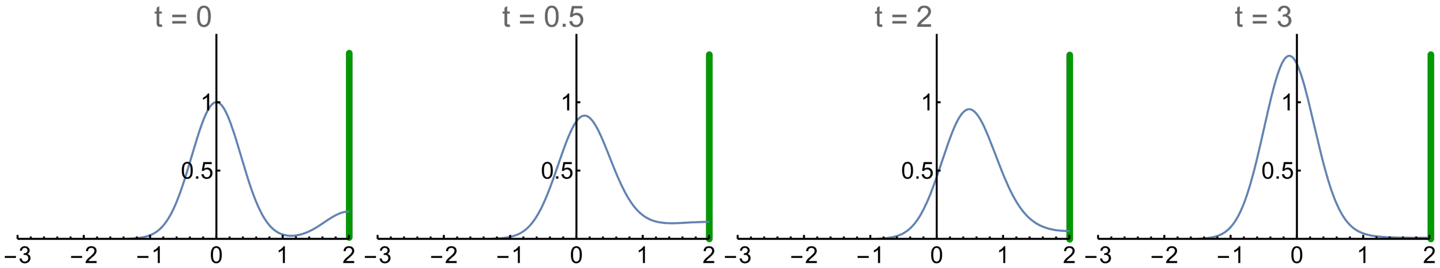
blob





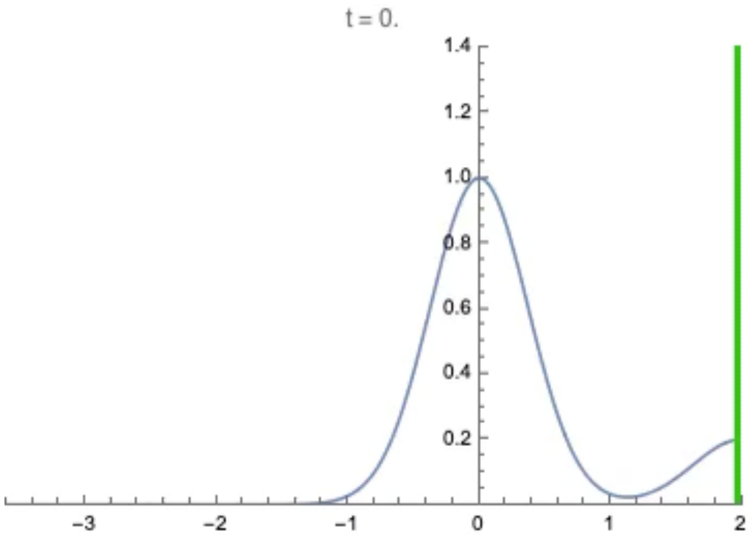
Holographic evaporation

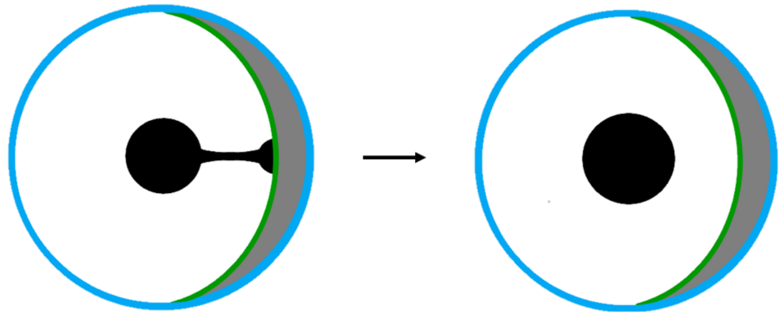
$r_0 = 0.4$ Holographic Evaporation in Emparan et.al. 2002, Tanaka 2002



blob+small droplet

blob





NLO equation

$$\nabla_\nu T^{\mu\nu} = 0$$

$$\begin{aligned}
T^{tt} &= (\alpha+1)m \\
&+ \frac{1}{n}\left[\beta m - \frac{\alpha p^2}{m} + (1+2\alpha-\beta)\frac{p\partial_z m}{m} + (\beta-4\alpha-3)\partial_z p - (2\alpha m + (\alpha+1)\partial_z p)\log m\right]
\end{aligned}$$

$$\begin{aligned}
T^{tz} &= (\alpha+1)(p-\partial_z m) \\
&+ \frac{1}{n}\Bigg[-\left(\beta + \frac{2\alpha^2-3\alpha-1}{2\alpha+1}\right)p - \frac{\alpha p^3}{m^2} + \left(\frac{10\alpha^3+10\alpha^2+5\alpha+1}{(2\alpha+1)\left(2\alpha^2+2\alpha+1\right)} - \beta\right)\frac{p(\partial_z m)^2}{m^2} \\
&- \left(\beta + \frac{\alpha(2\alpha-1)}{2\alpha+1}\right)\frac{p\partial_z p}{m} + \left(\beta + \frac{4\alpha^4-4\alpha^3-6\alpha^2-4\alpha-1}{(2\alpha+1)\left(2\alpha^2+2\alpha+1\right)}\right)\partial_z m \\
&+ \left(\beta + \frac{6\alpha^2-3\alpha-1}{2(2\alpha+1)}\right)\frac{p^2\partial_z m}{m^2} + \left(\frac{4\alpha^4}{(2\alpha+1)\left(2\alpha^2+2\alpha+1\right)} + \beta\right)\frac{\partial_z p \partial_z m}{m} \\
&+ \Bigg(\frac{2\alpha^2+\alpha+1}{1+2\alpha}\partial_z^2 p + (1+\alpha)\frac{p^2\partial_z m - 2mp\partial_z p}{m^2} - 2\alpha p \\
&+ \frac{2\alpha(mp\partial_z^2 + m\partial_z p\partial_z m - p(\partial_z m)^2 + 2\alpha m^2\partial_z m)}{(2\alpha+1)m^2}\Bigg)\log m\Bigg],
\end{aligned}$$

$$\begin{aligned}
T^{zz} &= -\frac{2\alpha p\partial_z m}{(2\alpha+1)m} + (\alpha+1)\partial_t m + \frac{(\alpha+1)p^2}{m} - \frac{\left(2\alpha^2+\alpha+1\right)(m+\partial_z p)}{2\alpha+1} \\
&+ \frac{1}{n}\Bigg[\left(\frac{-4\alpha^4+8\alpha^3+15\alpha^2+6\alpha+1}{(\alpha+1)(2\alpha+1)^2} - \frac{\left(2\alpha^2+\alpha+1\right)\beta}{(2\alpha+1)(\alpha+1)}\right)m + \left(3\beta - \frac{2\alpha^3+9\alpha^2+13\alpha+4}{2\alpha^2+3\alpha+1}\right)\frac{p^2}{m} \\
&- \frac{\alpha p^4}{m^3} + \frac{\gamma_1 p(\partial_z m)^3}{m^3} + \frac{\gamma_2(\partial_z p)^2}{m} + \gamma_3\frac{m\partial_z^2 m - 2(\partial_z m)^2}{m} + \frac{\gamma_4 p^2(\partial_z m)^2}{m^3} + \frac{\gamma_5\partial_z p(\partial_z m)^2}{m^2} \\
&+ \frac{\gamma_6 p^2\partial_z^2 m}{m^2} + \frac{\gamma_7 p\partial_z^2 p}{m} + \gamma_8\frac{m\partial_z^3 p - p\partial_z^3 m}{m} + \left(\gamma_9 + \frac{4\alpha^2+2\alpha+1}{2\alpha+1}\frac{p^2}{m^2}\right)\partial_t m \\
&- \frac{2(4\alpha^2+3\alpha+1)}{2\alpha+1}\frac{p\partial_t p}{m} + \gamma_{10}\partial_z p + \left(\frac{2\alpha^3+10\alpha^2+13\alpha+3}{(\alpha+1)(2\alpha+1)} - 3\beta\right)\frac{p^2\partial_z p}{m^2} + \gamma_{11}\frac{\partial_z p\partial_z^2 m}{m} \\
&+ \gamma_{12}\frac{\partial_t m\partial_z p}{m} + \gamma_{13}\frac{m\partial_t\partial_z p - p\partial_t\partial_z m - \partial_z m\partial_t p}{m} + \gamma_{14}\frac{p\partial_z m}{m} + \left(3\beta - \frac{7\alpha^2+12\alpha+3}{(\alpha+1)(2\alpha+1)}\right)\frac{p^3\partial_z m}{m^3} \\
&+ \gamma_{15}\frac{p\partial_z p\partial_z m}{m^2} + \gamma_{16}\frac{p\partial_z^2 m\partial_z m}{m^2} + \gamma_{17}\frac{\partial_z^2 p\partial_z m}{m} + \gamma_{18}\frac{p\partial_z m\partial_t m}{m^2} \\
&+ \log m\Bigg(\frac{2\alpha\partial_z m}{2\alpha+1}\left(\frac{p\partial_t m}{m^2} - \frac{\partial_t p}{m} - \frac{(2\alpha-1)p}{(2\alpha+1)m}\right) - \partial_t m\left(\frac{(\alpha+1)p^2}{m^2} + \frac{4\alpha^2}{2\alpha+1}\right) \\
&- \frac{2\alpha p\partial_t\partial_z m}{(2\alpha+1)m} + \frac{2(\alpha+1)p\partial_t p}{m} - \frac{2\alpha p^2}{m} + \frac{2\alpha(4\alpha^2+4\alpha-1)m}{(2\alpha+1)^2} \\
&+ \frac{\left(4\alpha^3+4\alpha^2-5\alpha-1\right)\partial_z p}{(2\alpha+1)^2} - \frac{\left(2\alpha^2+\alpha+1\right)\partial_t\partial_z p}{2\alpha+1}\Bigg)\Bigg]
\end{aligned}$$

Thermodynamics of NUBS

Expand in nonuniformity λ

$$\frac{M}{M_{\mathrm{GL}}} = 1 + \frac{\lambda^2}{24}\left(1 - \frac{\ell_1 - 12}{n}\right) - \frac{\lambda^3}{12}\left(1 - \frac{\ell_1 - 12}{n}\right) + \lambda^4\left(\frac{5448 - 443\ell_1}{3456n} + \frac{971}{6912}\right) + \mathcal{O}(\lambda^5)$$

$$\frac{S}{S_{\mathrm{GL}}} = 1 + \frac{\lambda^2}{24}\left(1 - \frac{\ell_1 - 12}{n}\right) - \frac{\lambda^3}{12}\left(1 - \frac{\ell_1 - 12}{n}\right) + \lambda^4\left(\frac{5448 - 443\ell_1}{3456n} + \frac{971}{6912}\right) + \mathcal{O}(\lambda^5)$$

$$\frac{\mathcal{T}}{\mathcal{T}_{\mathrm{GL}}} = 1 - \left(1 + \frac{9 - \ell_1}{n}\right)\frac{\lambda^2}{2} + \left(1 + \frac{9 - \ell_1}{n}\right)\lambda^3 - \left(\frac{121}{72} + \frac{55(9 - \ell_1)}{36n}\right)\lambda^4 + \mathcal{O}(\lambda^5)$$ ← tension

$$M_{\mathrm{GL}} = (1+\alpha)r_0^{n+1}\left(1 + \frac{(\alpha+1)\ell_1 - 7\alpha - 5}{2n(1+\alpha)}\right), \qquad S_{\mathrm{GL}} = (1+2\alpha)r_0^{n+2}\left(1 + \frac{(2\alpha+1)\ell_1 - 10\alpha - 7}{2n(1+2\alpha)}\right)$$

$$\mathcal{T}_{\mathrm{GL}} = \frac{r_0^n}{n}\frac{1+\alpha+2\alpha^2}{1+2\alpha}\left[1 + \frac{1}{n}\left(\frac{\alpha\left(4\alpha^2 - 8\alpha - 3\right)}{(1+2\alpha)\left(1+\alpha+2\alpha^2\right)} + \frac{\arctan(1+2\alpha) - \pi/4 - \log(1+\alpha)}{1+\alpha} + \frac{\log\left(1+2\alpha+2\alpha^2\right)}{2(2\alpha+1)(\alpha+1)}\right)\right]$$

$$T_H = T_{H,\mathrm{GL}} = \frac{1}{r_0}\left(\frac{1+\alpha}{1+2\alpha} - \frac{\alpha(1+4\alpha)}{n(1+2\alpha)^2}\right).$$

Comparison with Large α limit

relative tension/binding energy

$$\tau := \frac{L\mathcal{T}}{M} = \tau_{\rm GL}\left[1 - \left(1 - \frac{\ell_1 - 10}{n}\right)\frac{\lambda^2}{2} + \left(1 - \frac{\ell_1 - 10}{n}\right)\lambda^3 - \left(\frac{481}{288} + \frac{2171 - 217\ell_1}{144n}\right)\lambda^4 + \mathcal{O}(\lambda^5)\right]$$

relative tension for UBS

$$\tau_{\rm GL} = \frac{L_{\rm GL}\mathcal{T}_{\rm GL}}{M_{\rm GL}} = \frac{1}{n}\frac{2\alpha^2 + \alpha + 1}{(2\alpha+1)(\alpha+1)} - \frac{1 + 6\alpha + 15\alpha^2 + 8\alpha^3 - 4\alpha^4}{n^2(1+\alpha)^2(1+2\alpha)^2}$$

$$\tau_{\rm GL} \xrightarrow{\alpha \to 0} n^{-1} - n^{-2} \simeq \frac{1}{n+1} \to \text{GR}$$

$$\tau_{\rm GL} \xrightarrow{\alpha \to \infty} n^{-1} + n^{-2} \simeq \frac{1}{n-1} \to$$ **large α limit** **RS**-Tomizawa 22

Large D and Large α are compatible