Complexity Equals Anything

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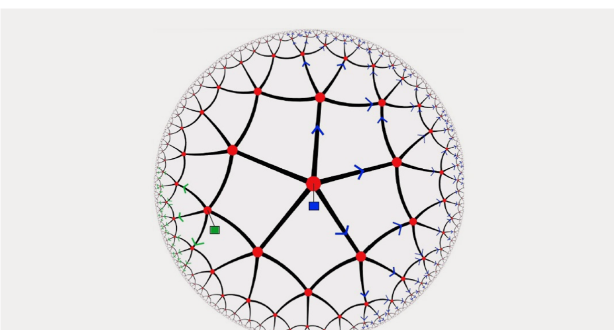
Feb-20-2023 | HOLOGRAPHY, STRING THEORY AND SPACETIME IN DA NANG

arXiv:2111.02429 arXiv:2210.09647 with A. Berlin, R. Myers, G. Sarosi, A. Seperanza

arXiv:2302.XXXXX with E. Jørstad, R. Myers



It From Qubit Collaboration

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

Extreme Universe Collaboration

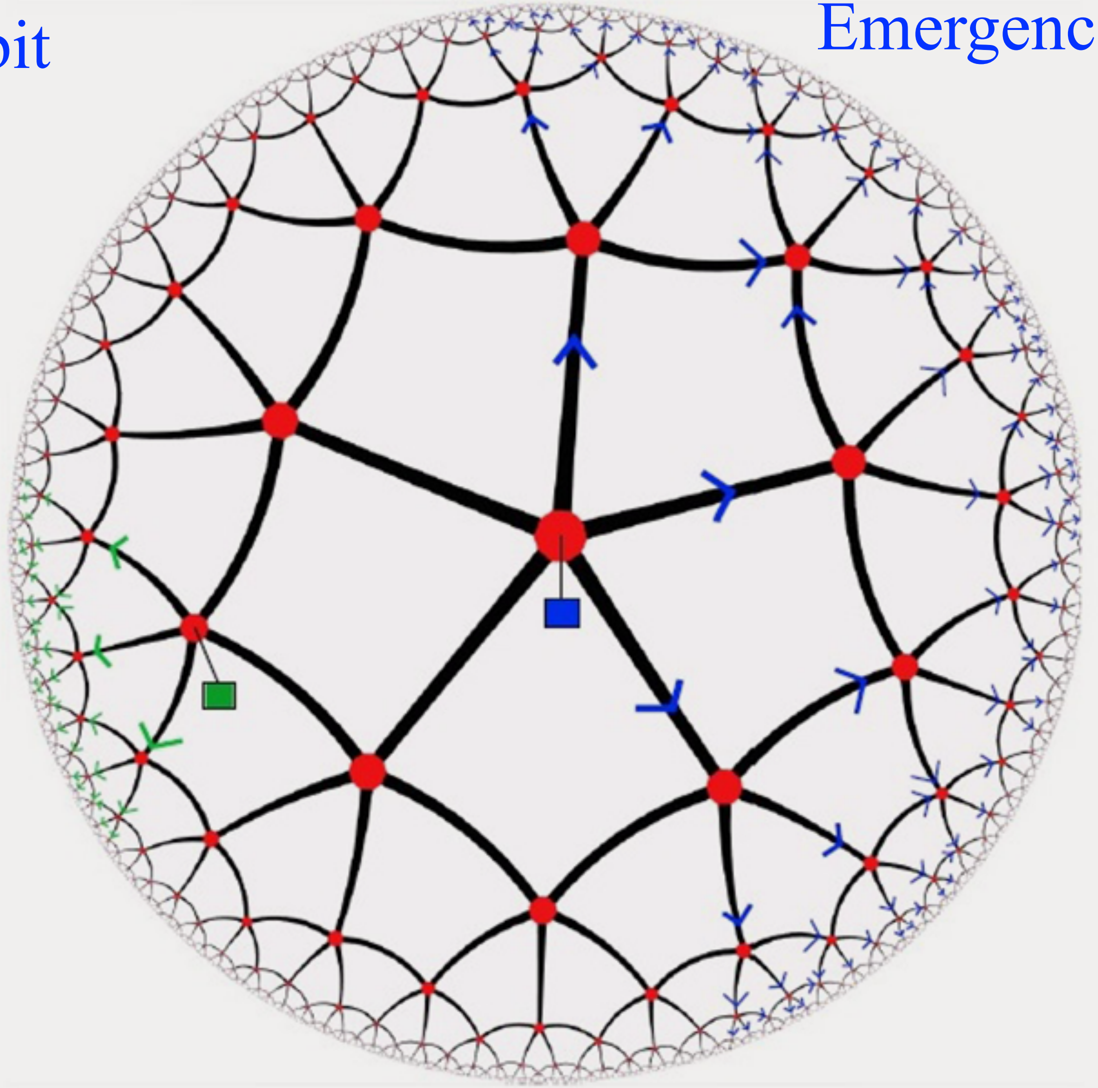
Outline

- Motivations and Background
- Circuit Complexity and Holographic Complexity
- Complexity=Anything
- Singularity Probes

01. Motivation- “Quantum Circuit”

It from Qubit

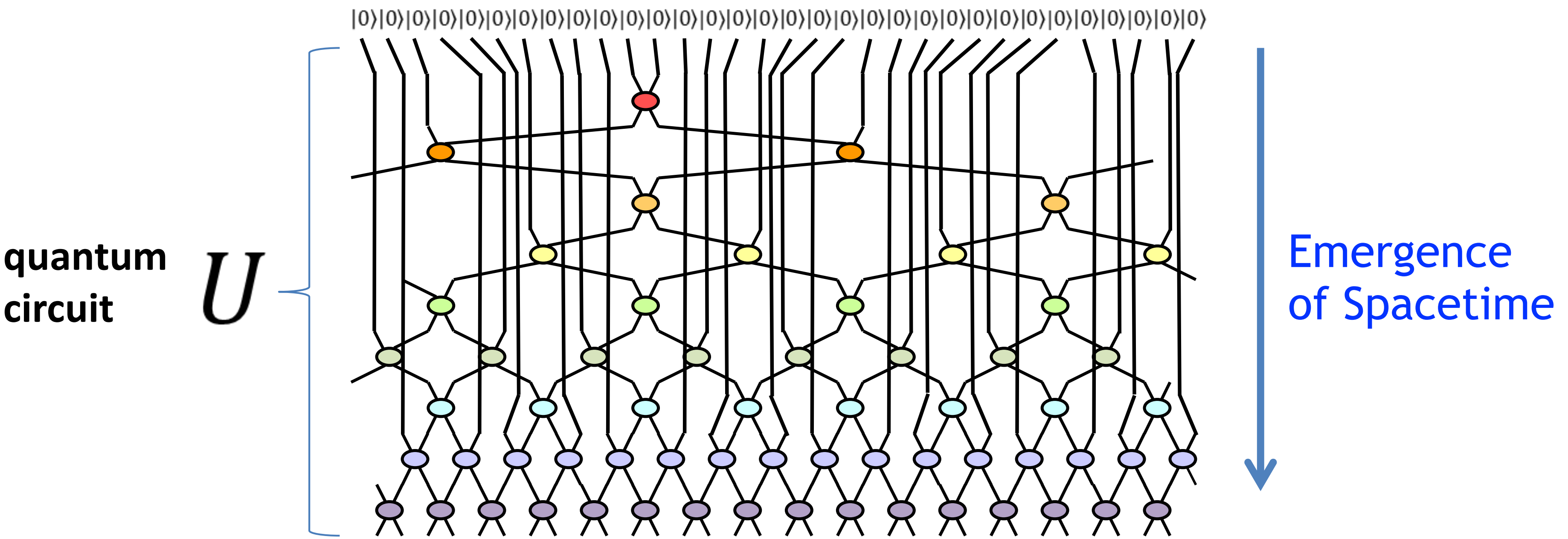
Emergence of Spacetime



01. Motivation- "Quantum Circuit"

AdS/MERA (Multiscale Entanglement Renormalization Ansatz)

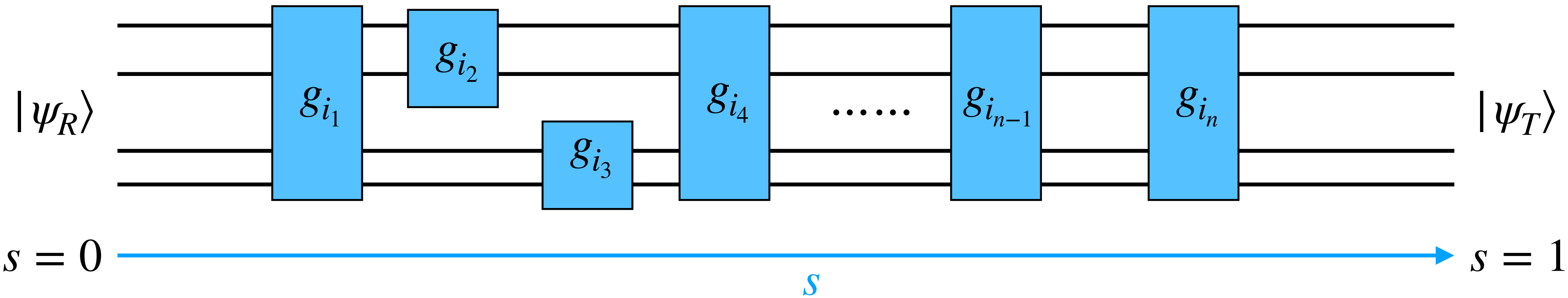
$|\Psi_R\rangle = |0\rangle|0\rangle|0\rangle\cdots|0\rangle$ ⟷ no entanglement



$|\Psi_T\rangle = |\Omega\rangle$ ⟷ (AdS/CFT) AdS spacetime

$|\Psi_T\rangle = |TFD\rangle$ AdS Black hole

Quantum Circuit

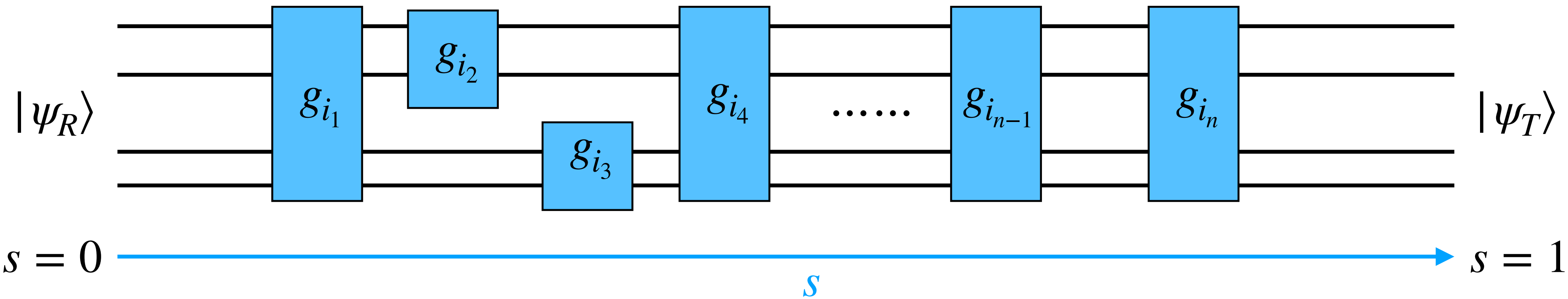


$$|\psi_{\mathrm{T}}\rangle = U_{\mathrm{TR}}\,|\psi_{\mathrm{R}}\rangle = g_{i_n}\cdots g_{i_2}\,g_{i_1}\,|\psi_{\mathrm{R}}\rangle\,,$$

Target state Gates Reference state

What is Circuit Complexity?

Quantum Circuit



$$|\psi_{\rm T}\rangle = U_{\rm TR}\,|\psi_{\rm R}\rangle = g_{i_n}\cdots g_{i_2}\, g_{i_1}\,|\psi_{\rm R}\rangle\,,$$

Target state　　　Gates　　Reference state

Circuit Complexity

The minimal number(cost) of gates in quantum circuits $|\Psi_R\rangle \longrightarrow |\Psi_T\rangle \equiv U_{TR}|\Psi_R\rangle$

02. Circuit Complexity in QFT

Nielsen's Geometric Approach

Nielsen and collaborators, quant-ph/0603161; R.Jefferson, R. Myers, 1707.08570

Quantum Circuit in QFTs $\qquad |\Psi_R\rangle \longrightarrow |\Psi_T\rangle \equiv U_{TR}|\Psi_R\rangle$

A. Continuous construction of the unitary transformations for the target state

$$U(\sigma) = \overleftarrow{\mathcal{P}} \exp\left[-i\int_0^\sigma ds\mathcal{H}(s)\right], \quad \text{where} \quad \mathcal{H}(s) = \sum_I Y^I(s)\mathcal{O}_I$$

B. Find the optimal circuit: minimize the cost function of circuits

"geodesic" in the space of unitaries/states

What is Circuit Complexity?

The minimal number of gates in quantum circuits for $|\Psi_R\rangle \longrightarrow |\Psi_T\rangle \equiv U_{TR}|\Psi_R\rangle$

Circuit Complexity: $\mathcal{C}\left(|\Psi_{\mathrm{T}}\rangle\right) = \min \mathcal{D}(U(\sigma)) = \min \int_0^1 ds\, F\left(U(s), Y^I(s)\right)$

For other approaches toward the complexity in QFT, see eg, :
1707.08582, S. Chapman, M. P. Heller, H. Marrochio and F. Pastawski;
1703.00456, 1706.07056, P. Caputa, N. Kundu, M. Miyaji, T. Takayanagi, K. Watanabe;

Why Circuit Complexity?

Free Choices (Ambiguities)?

References State?	Set of Gates?	Cost Functions?
Unentangled State?	Minimal Set?	?

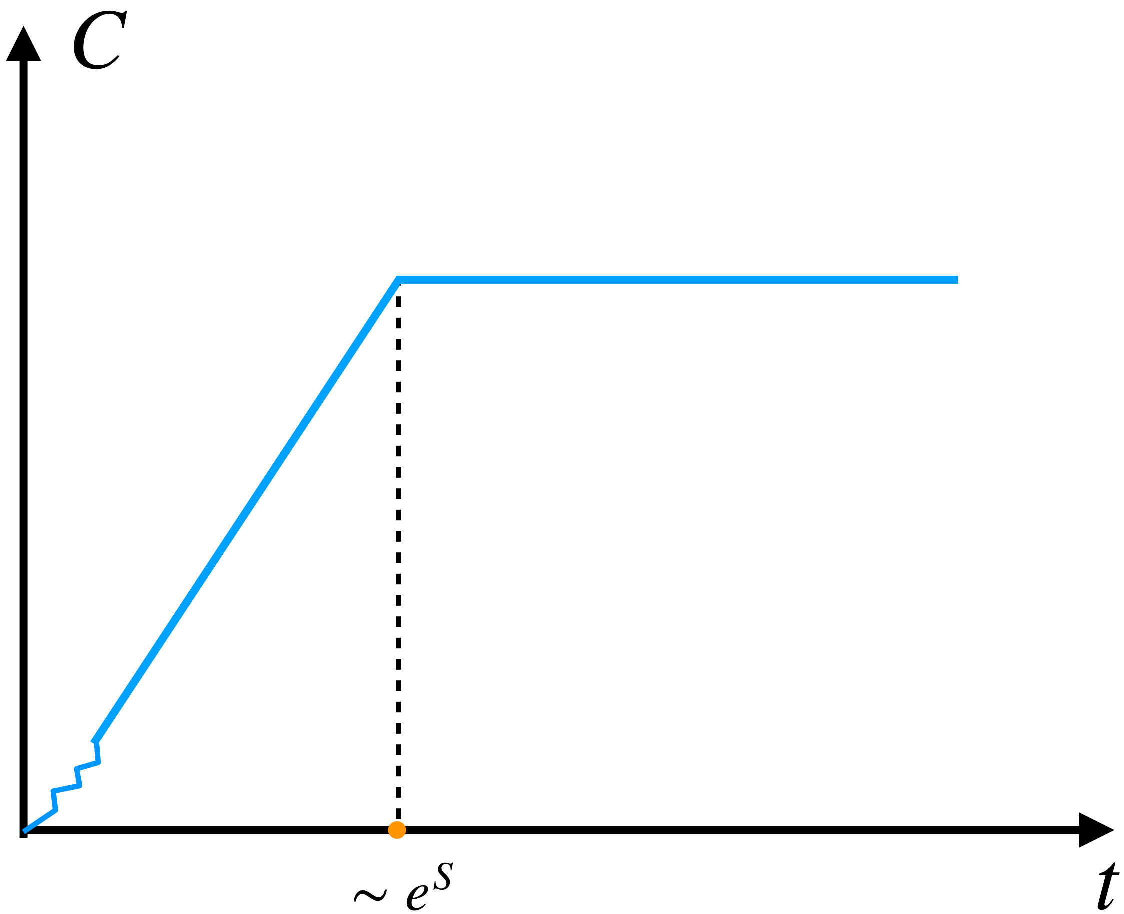
Why Circuit Complexity?

Universalities: Linear growth and Switchback effect

02. Universalities: Linear Growth

A conjecture of the time evolution of complexity

$$|\psi_{\mathrm{T}}(t)\rangle = e^{-iHt}|\psi\rangle$$



02. Universalities: Linear Growth

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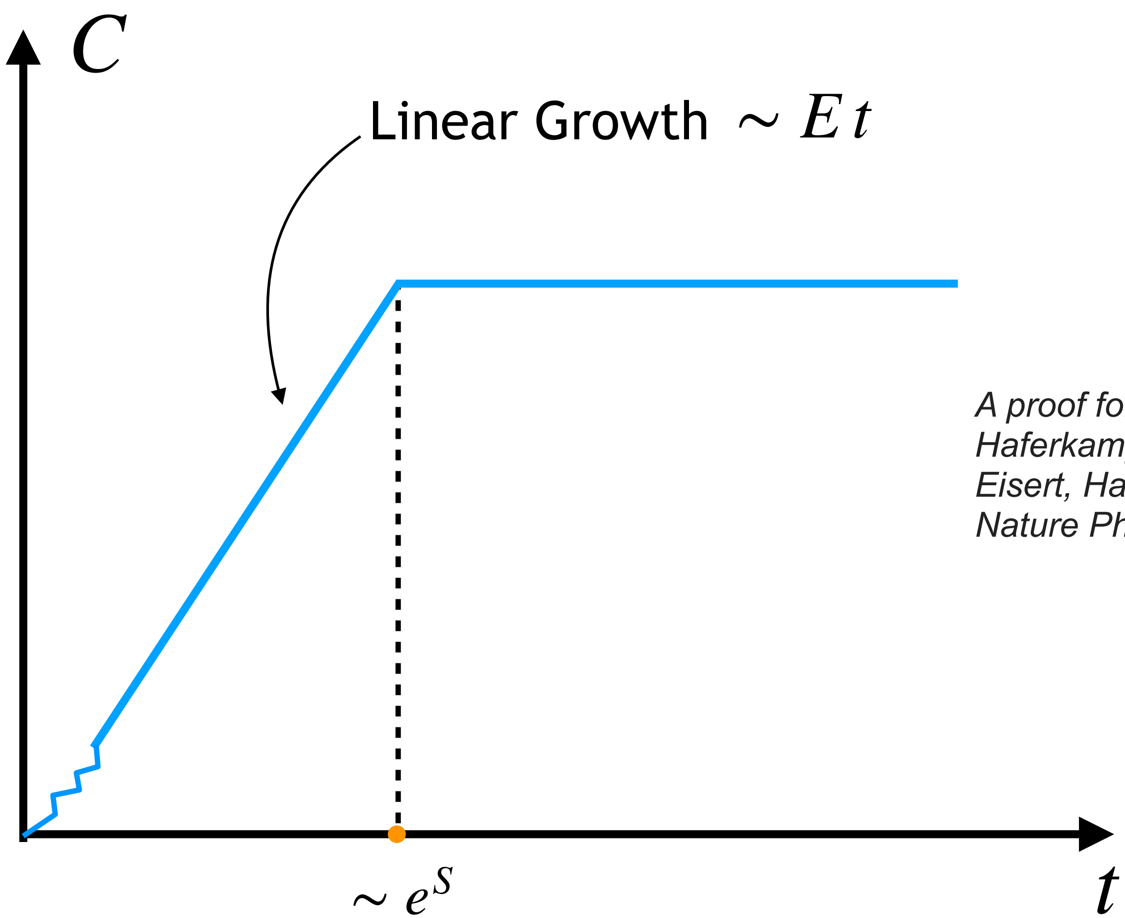

A proof for random circuits of qubits
Haferkamp, Faist, Kothakonda, Eisert, Halpern
Nature Physics, *18*(5), 528-532

02. Universalities: Linear Growth

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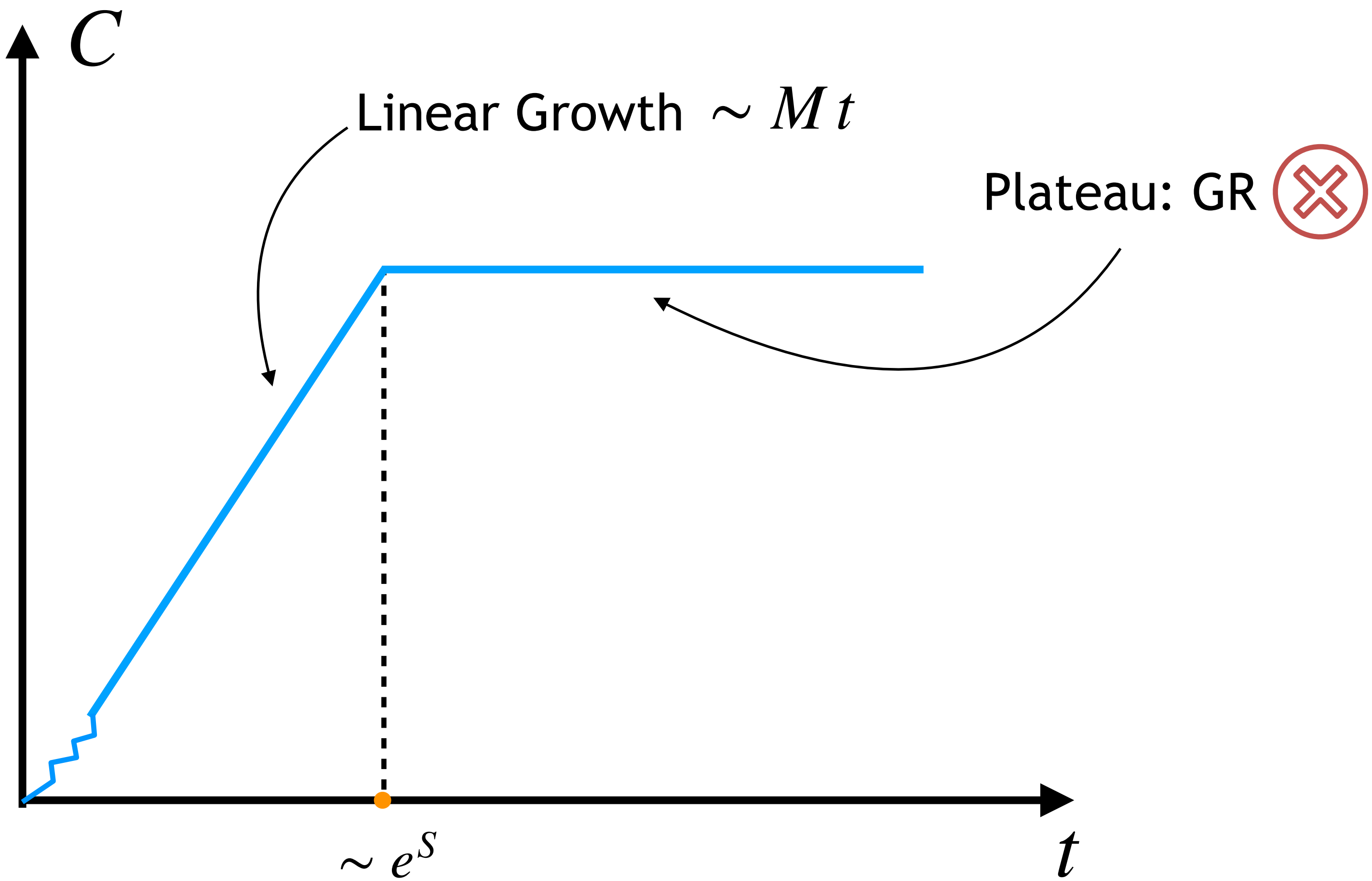
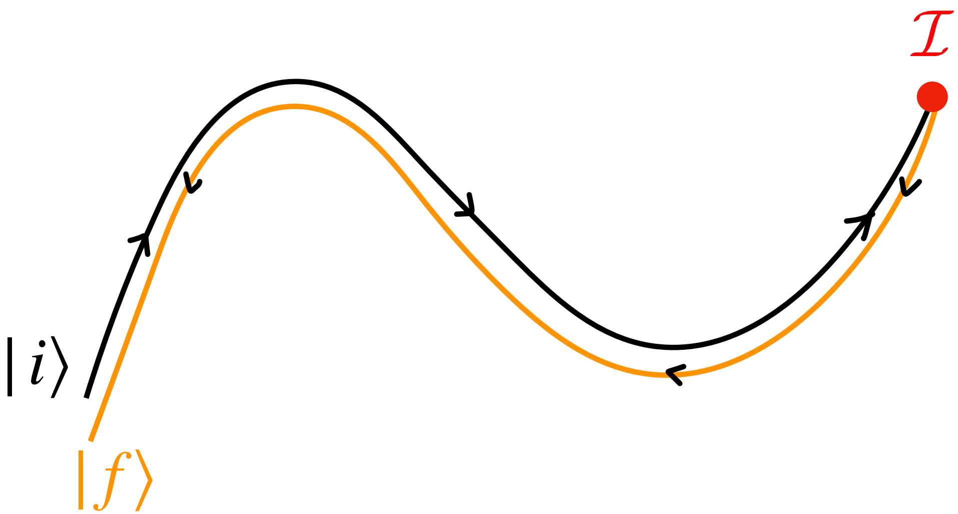


Switchback Effect

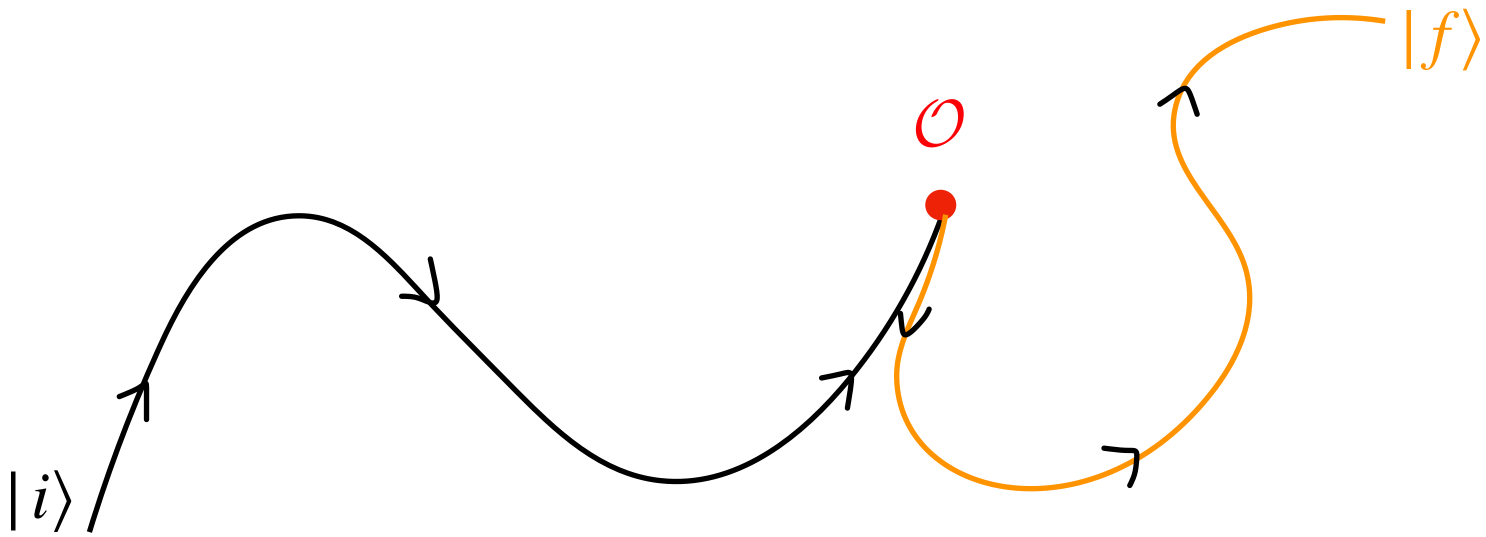
Complexity of a precursor $\mathcal{O}(t) = U(t)\, \mathcal{O} U^{\dagger}(t)\ ,\qquad U(t) = e^{-iHt}$



$$\mathcal{C} \sim 0$$

Switchback Effect

Complexity of a precursor $\mathcal{O}(t) = U(t)\,\mathcal{O}U^{\dagger}(t)$, $U(t) = e^{-iHt}$



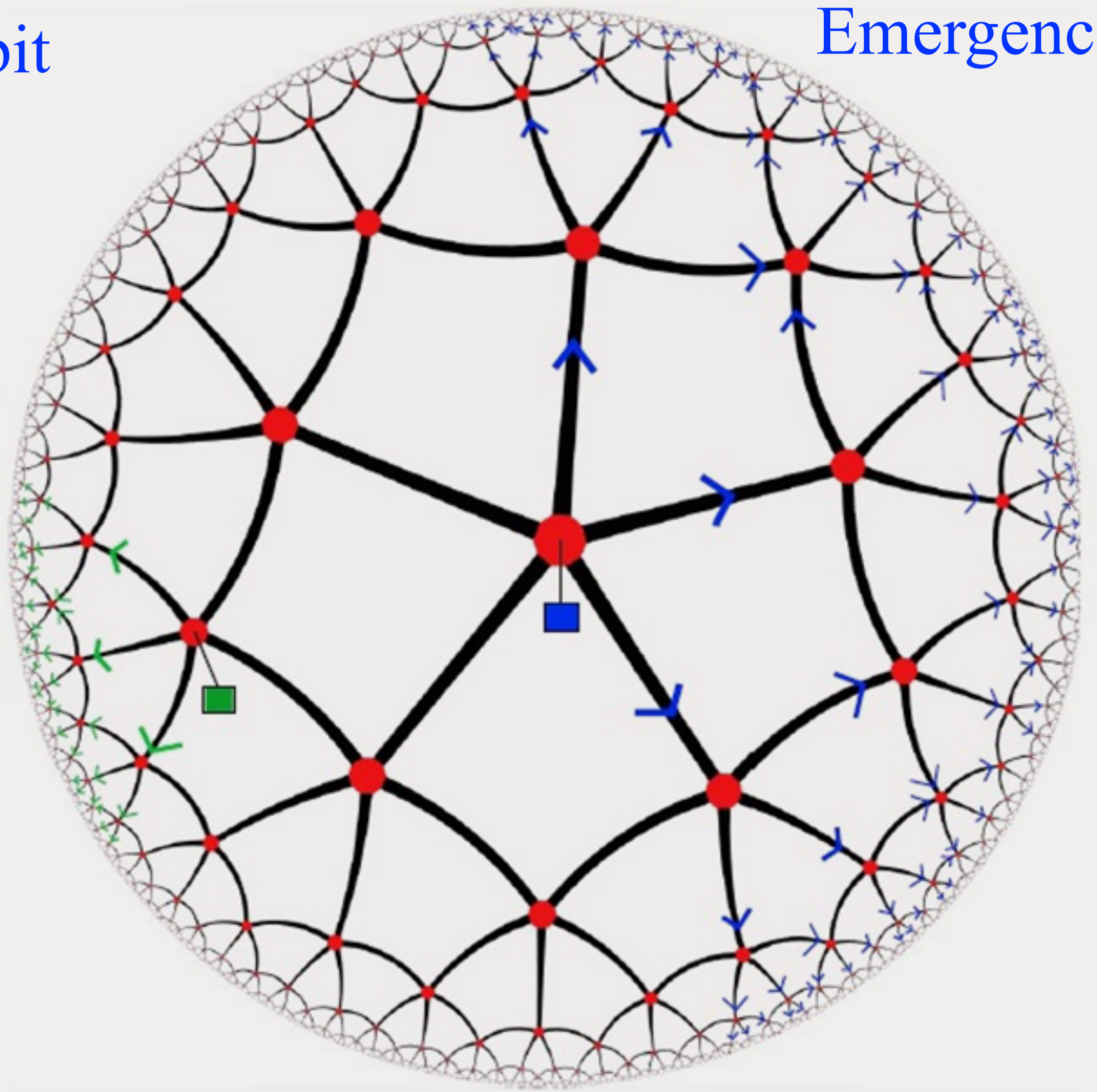
$$\mathcal{C} \sim 0\,, \qquad t < t_{\rm scr}$$

$$\mathcal{C} \sim 2M(t - t_{\rm scr})\,, \qquad t \gg t_{\rm scr}$$

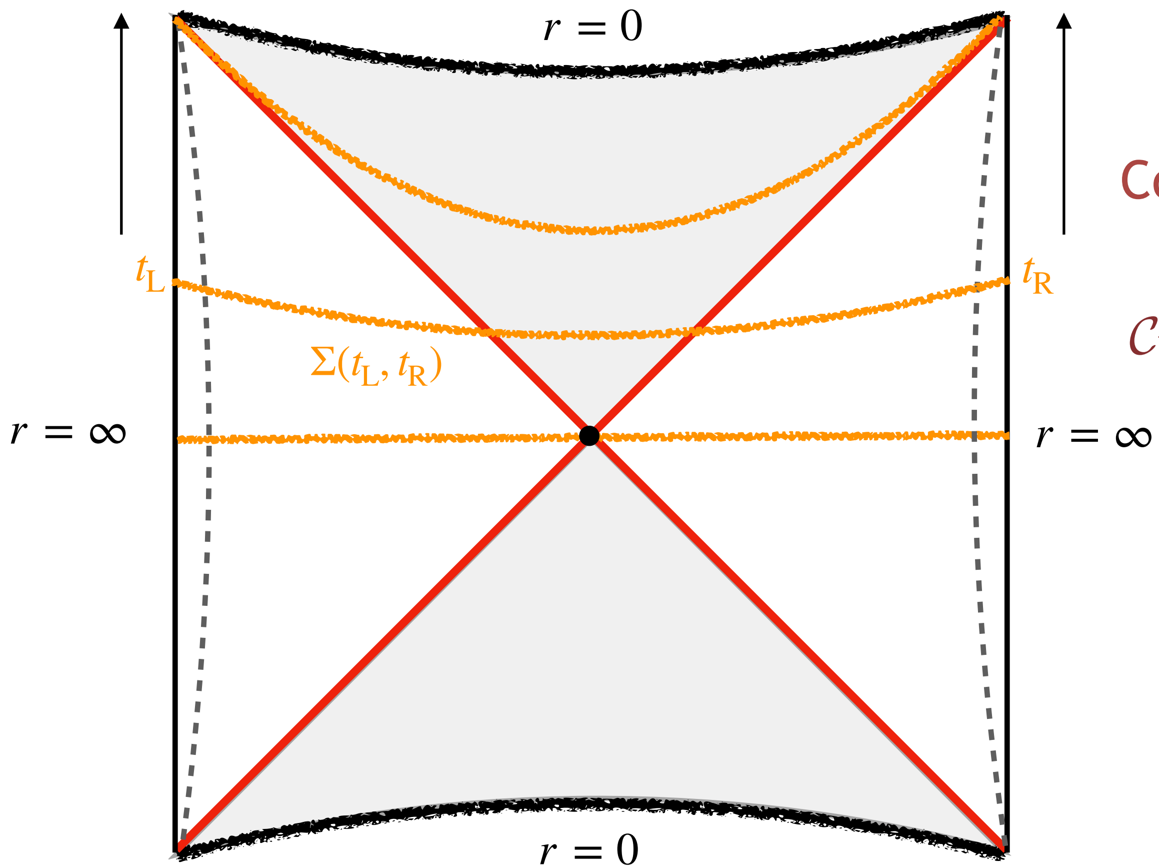
02. "Quantum Circuit"

It from Qubit

Emergence of Spacetime



Susskind, Stanford: arXiv: 1406.2678

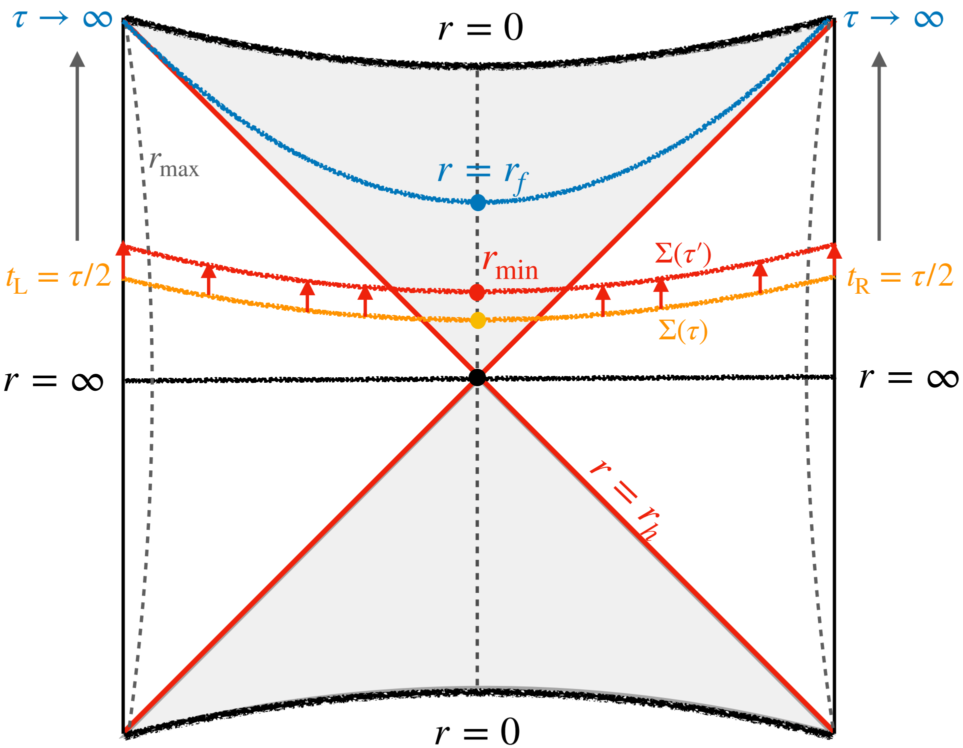


Complexity = Volume Conjecture

$$\mathcal{C}_{\rm V}\left(\Sigma_{\rm CFT}\right) = \max_{\Sigma_{\rm CFT} = \partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_{\rm N}\ell_{\rm bulk}}\right]$$

02. Holographic Complexity

$$|\psi_{\rm TFD}(\tau)\rangle = \sum_{E_n} e^{-\beta E_n/2 - iE_n\tau}|n\rangle_{\rm L}\otimes|n\rangle_{\rm R}$$



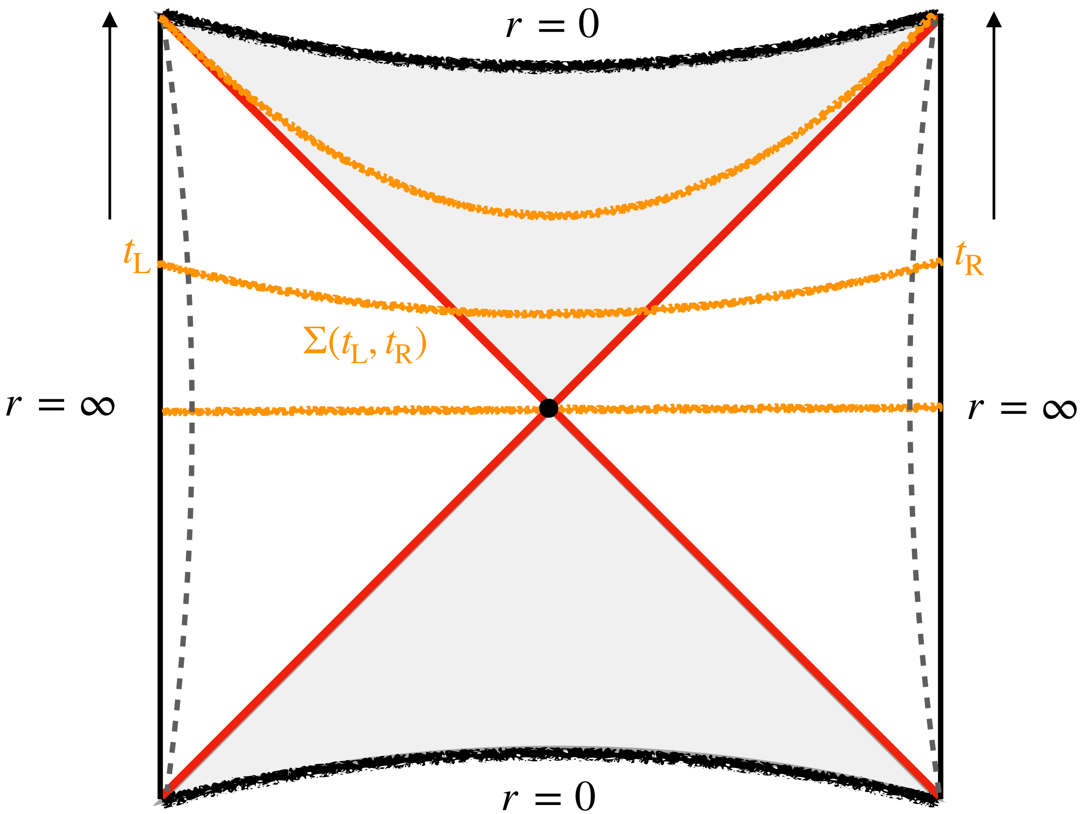
$$\mathcal{C}_{\rm V} = \max\left[\frac{\mathcal{V}}{G_N \ell_{\rm bulk}}\right]$$

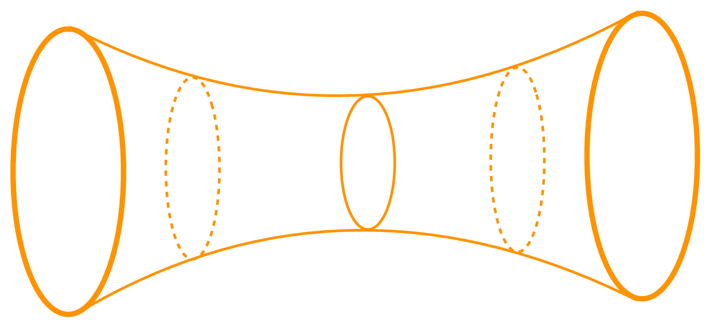
Linear growth at late times

$$\mathcal{C}_{\rm V} \sim M(t_{\rm L}+t_{\rm R}) + \mathcal{O}(1)$$

02. Holographic Complexity

Universality: Linear growth of the size of wormhole





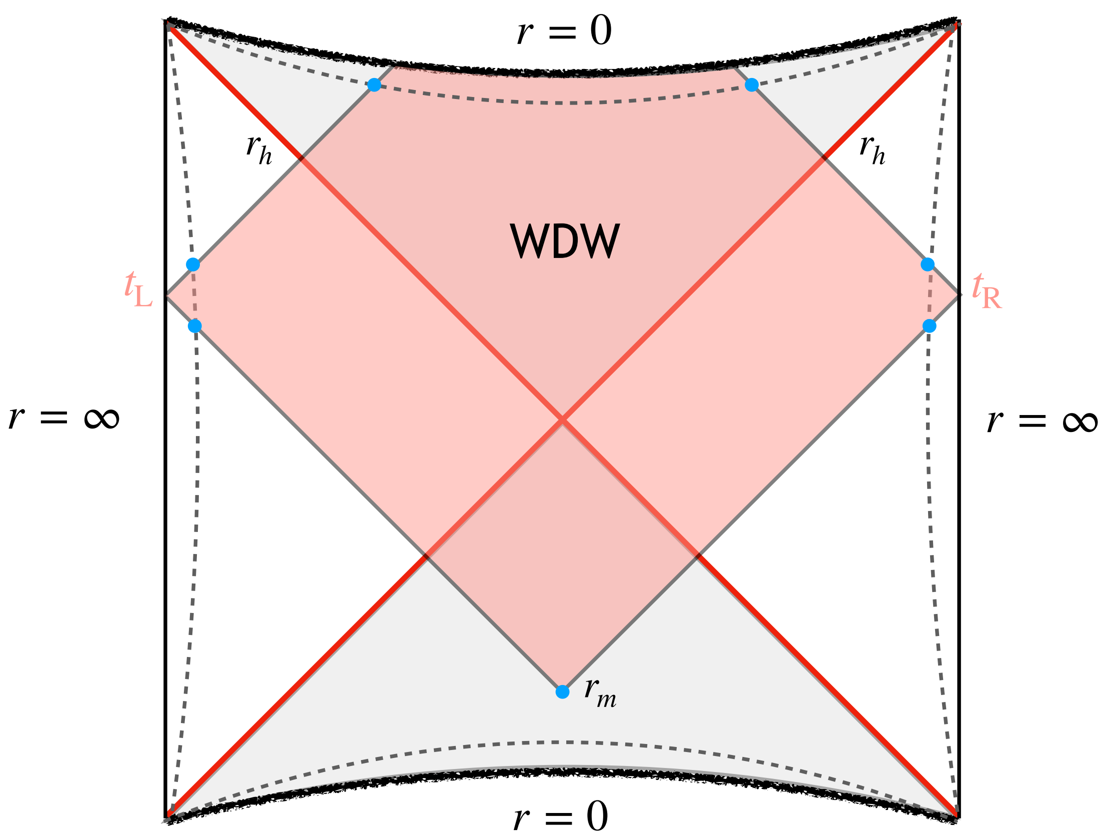
size $\sim t_L + t_R$

Geometries behind the horizon

02. Holographic Complexity

Susskind et al: arXiv:1509.07876; 1512.04993

Wheeler-DeWitt patch



Complexity = Action Conjecture

$$\mathcal{C}_A = \frac{I_{\rm WDW}}{\pi \hbar}$$

Full gravitational action

$$I = I_{\rm bulk} + I_{\rm GHY} + I_{\kappa} + I_{\rm ct} + I_{\rm jt}$$

Lehner, Myers, Possion and Sorkin
arXiv:1609.00207

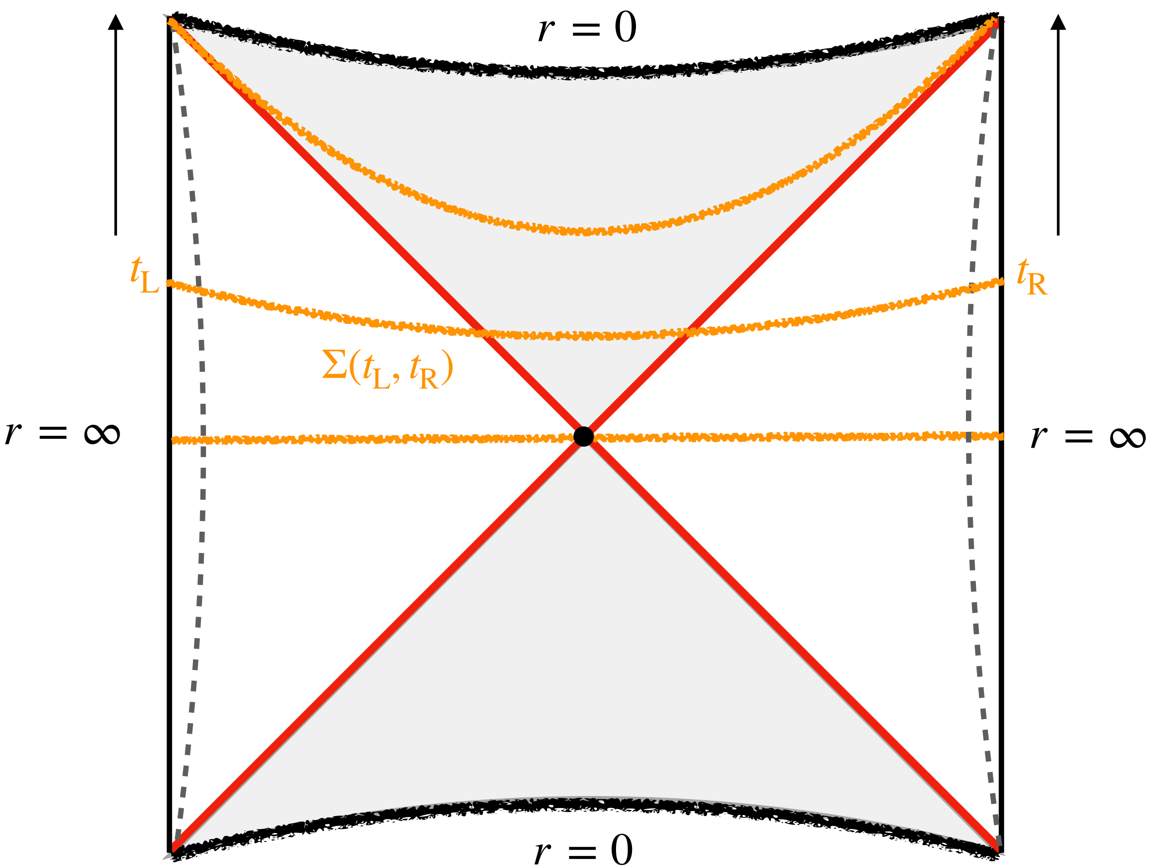
Ambiguities: Complexity Equals Anything?

arXiv:2111.02429 arXiv:2210.09647

A. Berlin, R. Myers, S-M Ruan, G. Sarosi, A. Seperanza

03. Complexity Equals Anything

$$O_{F_1,\Sigma_{F_2}}\left(\Sigma_{\mathrm{CFT}}\right)=\frac{1}{G_{\mathrm{N}}L}\int_{\Sigma_{F_2}} d^d\sigma\sqrt{h}F_1\left(g_{\mu\nu};X^\mu\right)$$



arXiv:2111.02429, A. Berlin, R. Myers, SM.Ruan, G. Sarosi, A. Seperanza

Extremal hypersurface

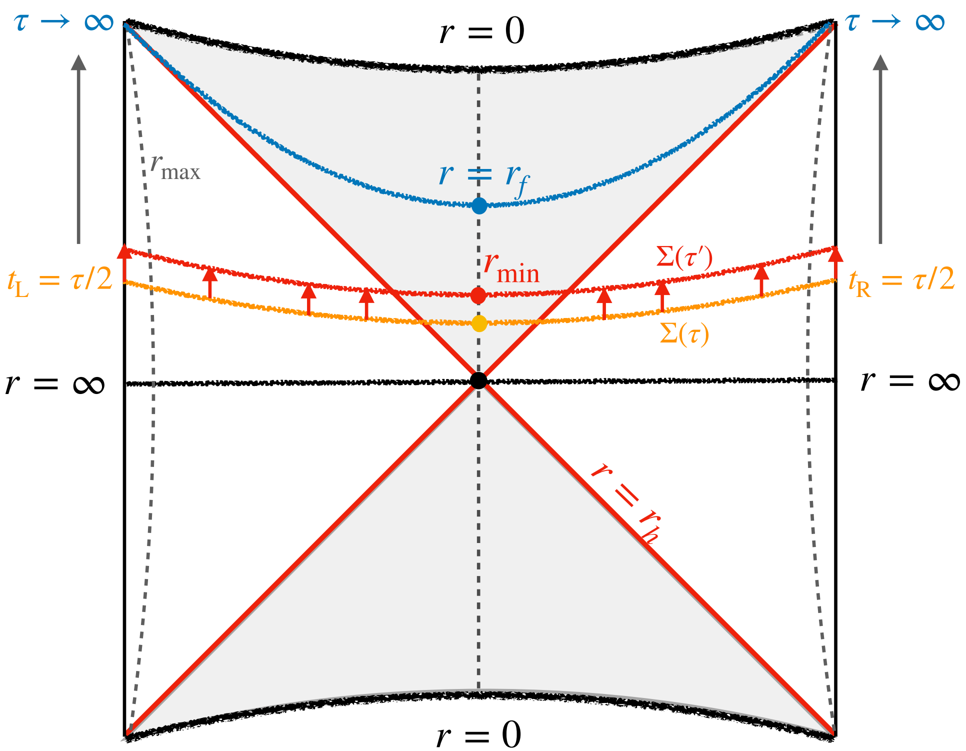
$$\delta_{\mathrm{x}}\left(\int_{\Sigma} d^d\sigma\sqrt{h}F_2\left(g_{\mu\nu};X^\mu\right)\right)=0$$

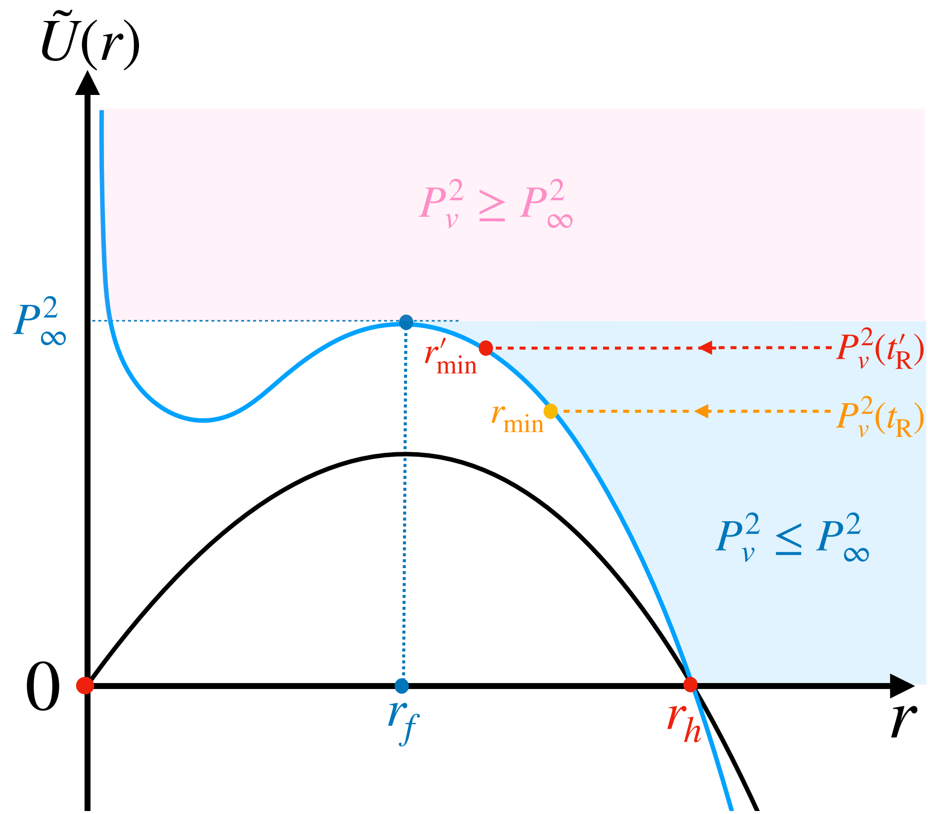
$F_1=F_2=1$

$$\mathcal{C}_{\mathrm{V}}=\max\left[\frac{\mathcal{V}}{G_N\ell_{\mathrm{bulk}}}\right]$$

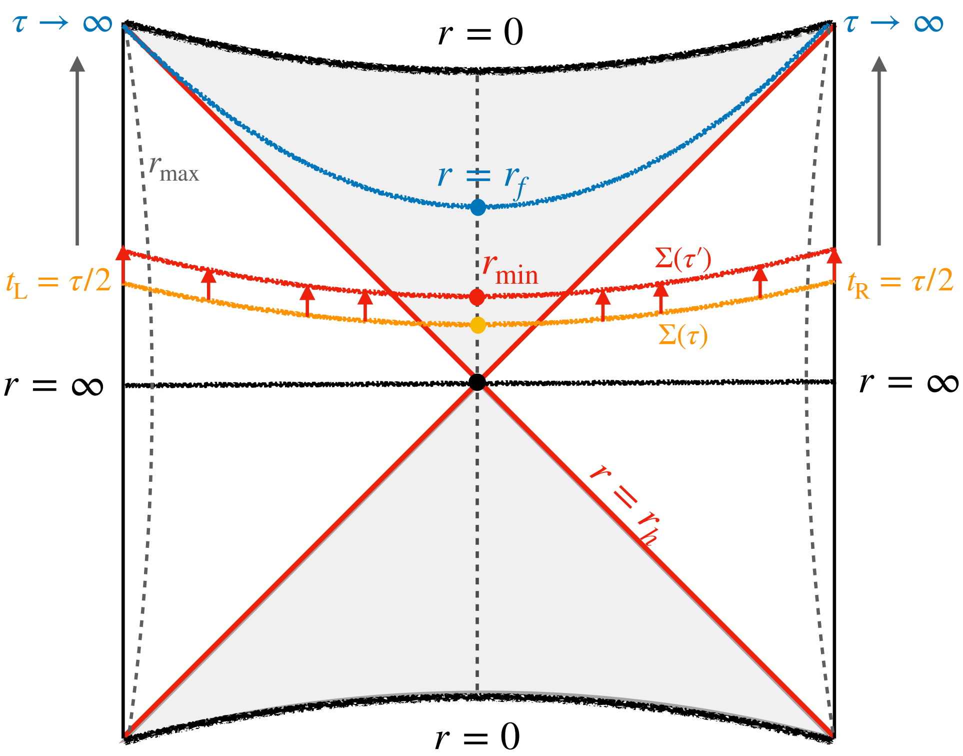
A classical non-relativistic particle with a effective potential

$$\dot{r}^2 + \widetilde{U}(r) = P_v^2 \quad \text{with} \quad \widetilde{U}(r) = -f(r)a^2(r)\left(\frac{r}{L}\right)^{2(d-1)}$$





$F_1 = F_2:$ $$\mathcal{C}_{\rm gen}(\tau) = \max_{\partial\Sigma(\tau)=\Sigma_\tau} \left[\frac{1}{G_N L}\int_\Sigma d^d\sigma \sqrt{h} F_1\left(g_{\mu\nu}; X^\mu(\sigma)\right)\right]$$



Time Evolution:

$$\frac{d\mathcal{C}_{\rm gen}}{d\tau} = \text{boundary term}\Big|_{\partial\Sigma} = \frac{V_x}{G_N L} P_v(\tau)$$

Linear Growth at late times:

$$\lim_{\tau\to\infty} \frac{d\mathcal{C}_{\rm gen}}{d\tau} = \text{constant} = \frac{V_x}{G_N L} P_\infty$$

corrections

$$P_\infty - P_v(\tau) \propto e^{-\kappa\tau},$$

$$\text{with} \quad \kappa = \frac{-f(r_f)\sqrt{-2\tilde{U}''(r_f)}}{P_\infty}$$

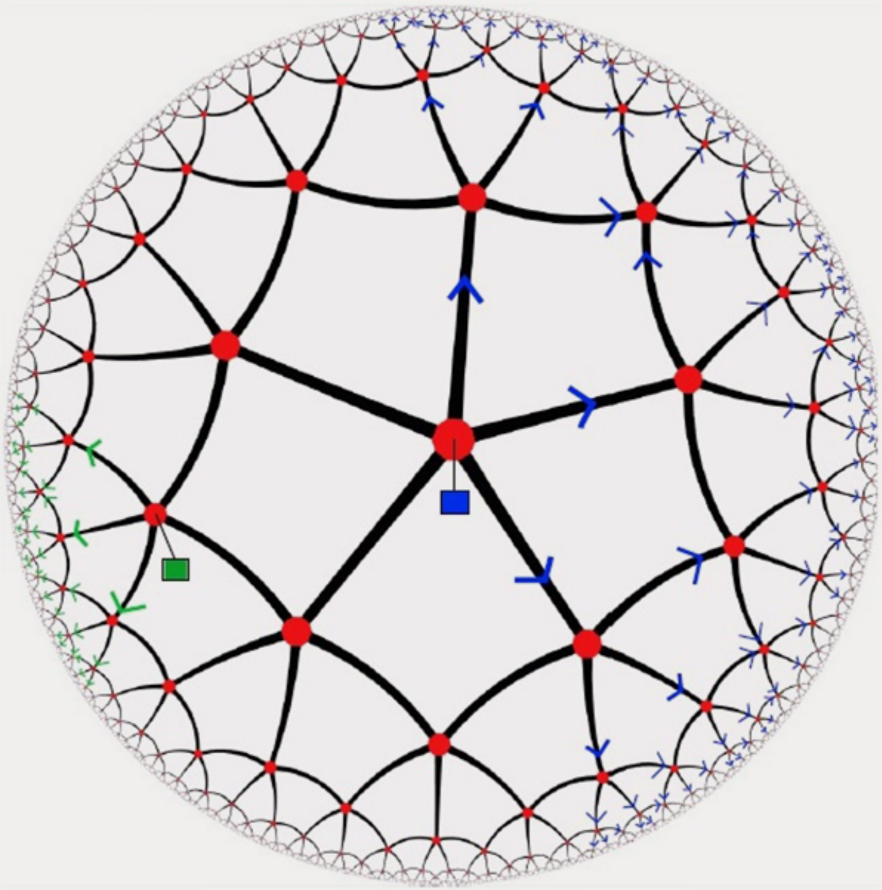
03. Complexity Equals Anything

$$F_1 \neq F_2 \qquad \mathcal{O}_{F_1,\Sigma_{F_2}}(\Sigma_{\mathrm{CFT}}) = \frac{1}{G_{\mathrm{N}} L} \int_{\Sigma_{F_2}} d^d\sigma \sqrt{h} F_1(g_{\mu\nu}; X^\mu)$$

$F_2 = 1$

↓

Extremal Surface



F_1

↓

Complexity measure

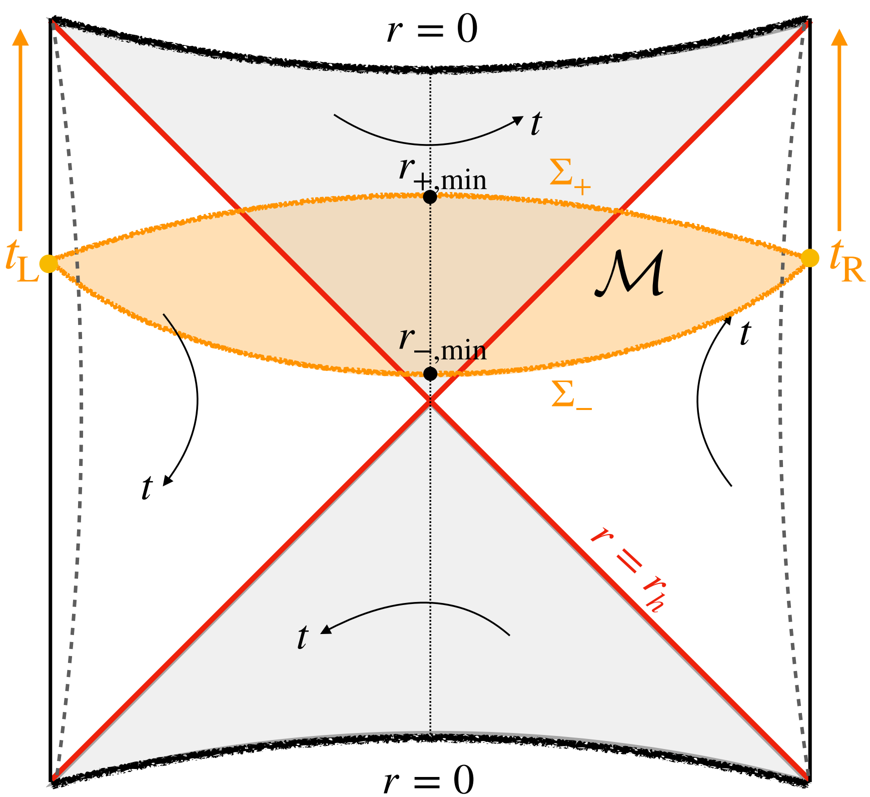
03. Complexity Equals Anything

A generic codimension-zero gravitational observables:

$$O\left[G_1, F_{1,\pm}, \mathcal{M}_{G_2,F_{2,\pm}}\right](\Sigma_{\rm CFT}) = \frac{1}{G_{\rm N}L}\int_{\Sigma_+[G_2,F_{2,+}]} d^d\sigma\sqrt{h}F_{1,+}\left(g_{\mu\nu};X^\mu_+\right)$$
$$+\frac{1}{G_N L}\int_{\Sigma_-[G_2,F_{2,-}]} d^d\sigma\sqrt{h}F_{1,-}\left(g_{\mu\nu};X^\mu_-\right) + \frac{1}{G_N L^2}\int_{\mathcal{M}_{G_2,F_{2,\pm}}} d^{d+1}x\sqrt{g}G_1\left(g_{\mu\nu}\right)$$

Codimension-one

Codimension-zero

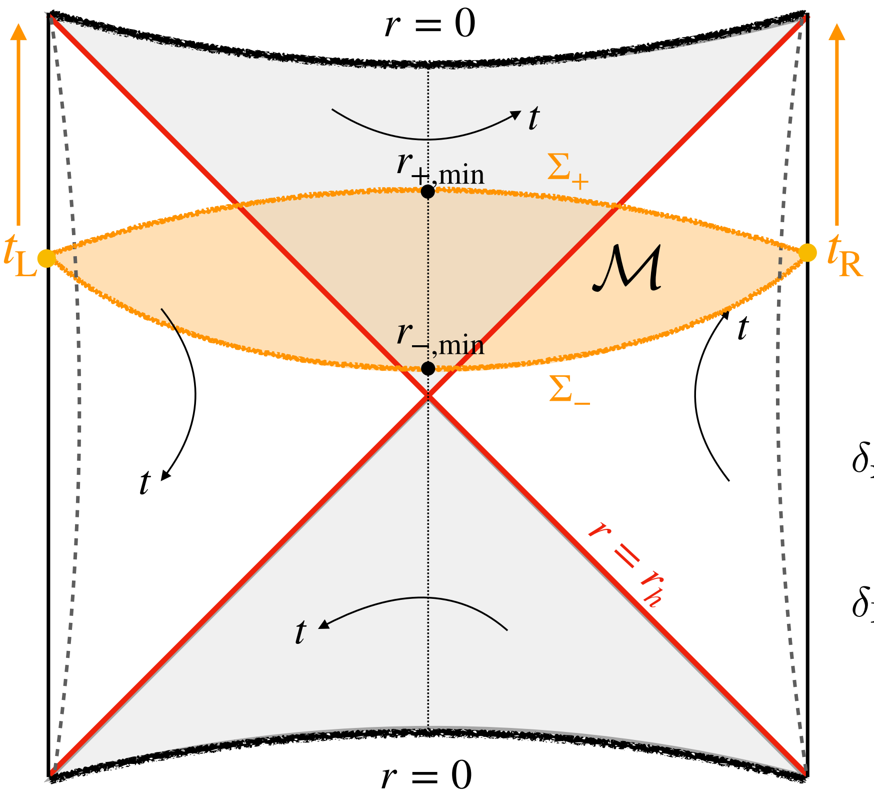


Extremal Codimension-zero Subregion

$$\delta_{{\rm x}_\pm}\left[W_{G_2,F_{2,\pm}}(\mathcal{M})\right] = 0$$

Stokes' theorem
$$\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\,G_2(g_{\mu\nu}) = L\int_{\Sigma_+\cup\Sigma_-=\partial\mathcal{M}} d^d\sigma\sqrt{h}\,\widetilde{G}_2(g_{\mu\nu};X^\mu)$$

Poincaré lemma: every top form on non-compact orientable manifold is always exact.



Extremal codimension-zero subregion:

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Two independent extremal surfaces

$$\delta_{X_+}\left[\int_{\Sigma_+} d^d\sigma\sqrt{h}\left(F_{2,+}(g_{\mu\nu};X^\mu_+)+\widetilde{G}_2(g_{\mu\nu};X^\mu_+)\right)\right]=0,$$
$$\delta_{X_-}\left[\int_{\Sigma_-} d^d\sigma\sqrt{h}\left(F_{2,-}(g_{\mu\nu};X^\mu_-)-\widetilde{G}_2(g_{\mu\nu};X^\mu_-)\right)\right]=0.$$

03. Complexity Equals Anything

Ambiguities: Complexity Equals Anything

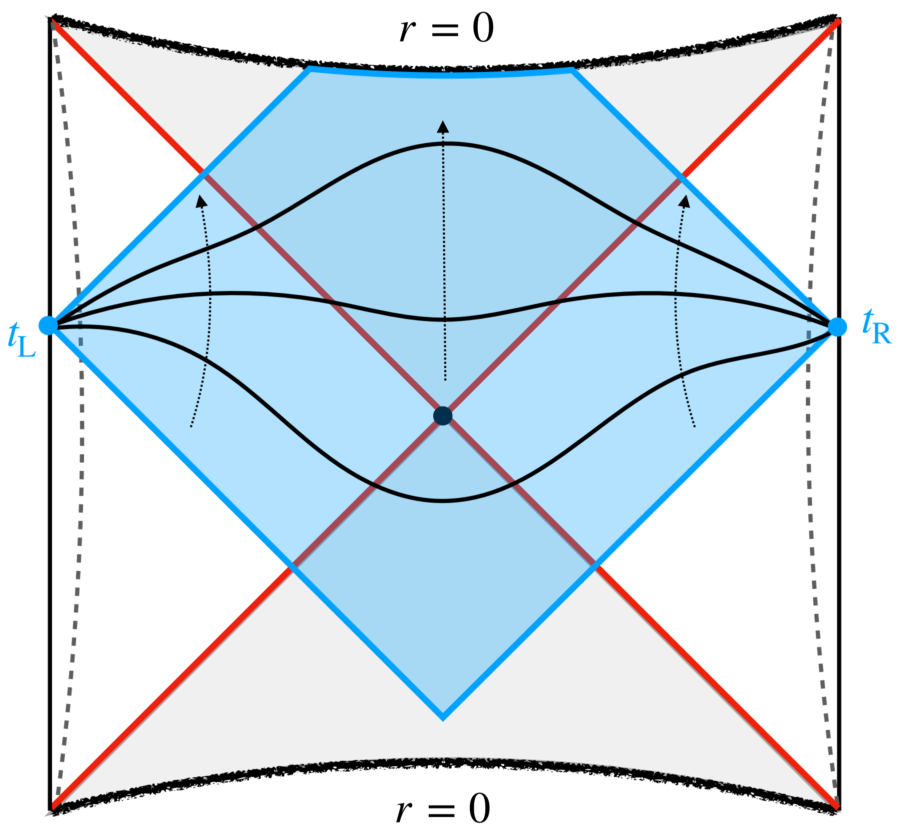
$$O\left[G_1, F_{1,\pm}, \mathcal{M}_{G_2,F_{2,\pm}}\right](\Sigma_{\mathrm{CFT}}) = \frac{1}{G_\mathrm{N} L}\int_{\Sigma_+[G_2,F_{2,+}]} d^d\sigma\sqrt{h}F_{1,+}\left(g_{\mu\nu}; X^\mu_+\right)$$
$$+\frac{1}{G_N L}\int_{\Sigma_-[G_2,F_{2,-}]} d^d\sigma\sqrt{h}F_{1,-}\left(g_{\mu\nu}; X^\mu_-\right) + \frac{1}{G_N L^2}\int_{\mathcal{M}_{G_2,F_{2,\pm}}} d^{d+1}x\sqrt{g}G_1\left(g_{\mu\nu}\right)$$

For two-sided AdS BH: $$\lim_{\tau\to\infty}\frac{dO}{d\tau} \sim M + \mathcal{O}(e^{-\kappa\tau})$$

Universalities: 01. Linear growth at late times
02. Switchback effect in shockwave geometries

04. Singularity Probe

$$\mathcal{C}_{\text{gen}} = \frac{1}{G_N L}\int_\Sigma F_1\left(g_{\mu\nu}, \mathcal{R}_{\mu\nu\rho\sigma}, \nabla_\mu\right)\sqrt{h}d^d y$$
$$= \frac{V_x}{G_N L}\int d\sigma \left(\frac{r}{L}\right)^{d-1}\sqrt{-f(r)\dot{v}^2 + 2\dot{v}\dot{r}}\, a(r)$$



(Maximal) extremal hypersurface

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Singularity

Singularity Probe:

01. Asymptotic geometries near the singularity

02. Distinguish spacelike/timelike singularity

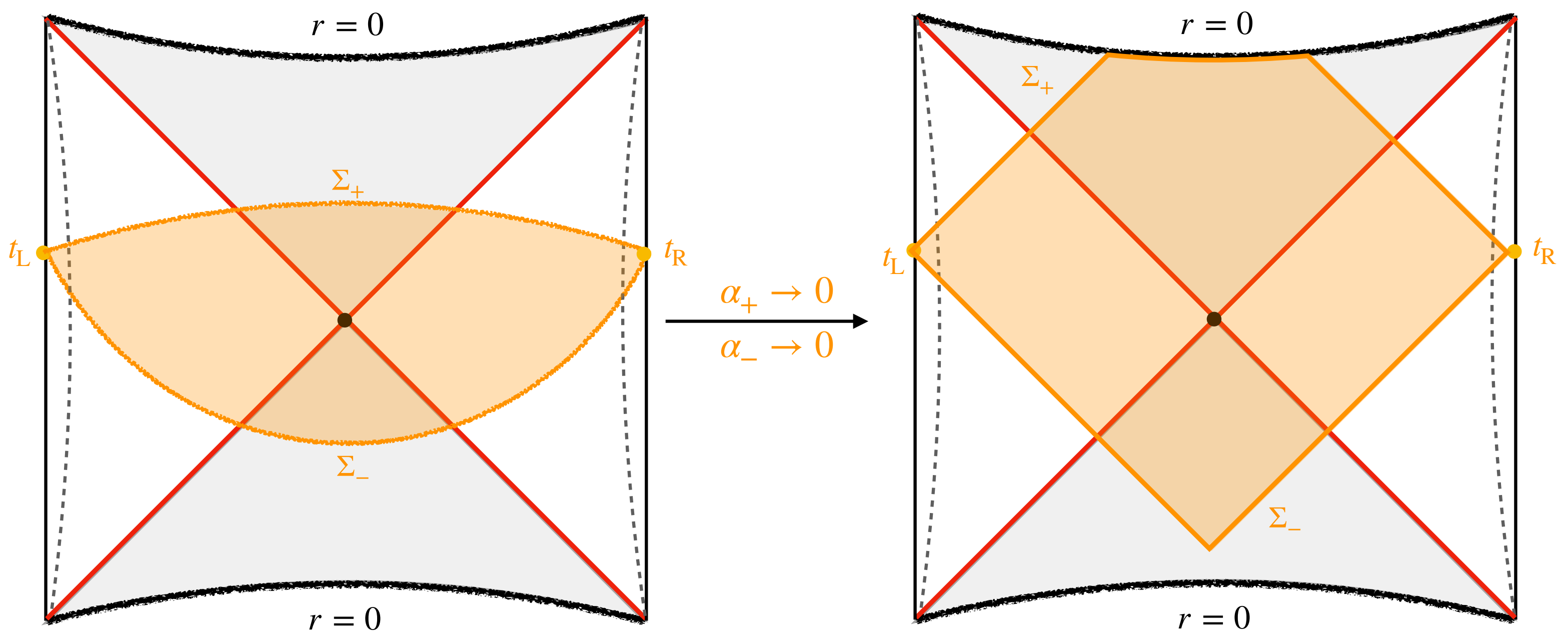
04. Singularity Probes

Simplest Example: $F = G = 1$

Spacetime volume between CMC slices

$$\mathcal{C}_{\text{gen}} = \frac{1}{G_{\text{N}} L} \left[\alpha_+ \int_{\Sigma_+} d^d\sigma \sqrt{h} + \alpha_- \int_{\Sigma_-} d^d\sigma \sqrt{h} + \frac{\alpha_{\text{B}}}{L} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \right],$$

Extremization Conditions: $K_{\Sigma_+} = \frac{\alpha_{\rm B}}{\alpha_+ L}, \quad K_{\Sigma_-} = -\frac{\alpha_{\rm B}}{\alpha_- L}$ constant mean curvature (CMC)



Lessons and Questions

- Infinite Gravitational Observables
- Finding extremal hypersurface —> EoMs of a classical particle
- Linear Growth & Switchback effect
- Two Independent Measures: Extremal Surface + Complexity Measure
- All as candidates for holographic complexity?
- Higher curvature corrections & Quantum corrections C_{bulk}?
- A derivation of holographic complexity?
- ……….

Thanks for your attention!