M2-branes and q-Painlevé equations

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based on: Lett. Math. Phys. 112 (2022) 109 [arXiv:2202.10654 [hep-th]]
with G. Bonelli, N. Kubo, F. Globlek, A. Tanzini

+ work in progress with S. Moriyama

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Introduction

Object: 3d supersymmetric Chern-Simons matter theories on M2-branes

[Hosomichi,Lee,Lee,Lee,Park,'08][Aharony,Bergman,Jafferis,Maldacena,'08]

Original motivations:

- Fundamental objects in M-theory
- Unconventional scaling of free energy $F \sim N^{\frac{3}{2}}$
- Fundamental example of AdS/CFT correspondence

supersymmetric observables ➡ solvable (written in matrix model)

It turned out that M2-matrix models are interesting by themselves

M2 partition function $\longleftrightarrow$ 1d quantum curve

all order 1/*N* pert. = Airy

topological string

q-Painlevé $\longrightarrow$ This talk

Painlevé equations and q-uplift of PIII_3

Original motivation: new special function from non-linear differential equations

Painlevé property: solution should not have an initial value-dependent branch point

example: $\dot{\lambda} = \lambda^{\alpha}$ ⇒ $\lambda = (t - t_0)^{\frac{1}{1-\alpha}}$

$\alpha = 0, 1 \longrightarrow$ regular ○

$\alpha = 2 \longrightarrow$ pole ○

Non-trivial 2nd order ODE with Painlevé property were classified into six:

$$\begin{array}{ccccccccc} \mathrm{PVI} & \longrightarrow & \mathrm{PV} & \longrightarrow & \mathrm{PIII}_1 & \longrightarrow & \mathrm{PIII}_2 & \longrightarrow & \mathrm{PIII}_3 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathrm{PIV} & \longrightarrow & \mathrm{PII} & \longrightarrow & \mathrm{PI} & & \end{array}$$

PIII_3: $$\ddot{\lambda} = \frac{\dot{\lambda}^2}{\lambda} - \frac{\dot{\lambda}}{t} + \frac{2\lambda^2}{t^2} - \frac{2}{t}$$

τ-form of PIII_3 : $\tau_i(\partial^2_{\log t}\tau_1) - (\partial_{\log t}\tau_1)^2 = t^{\frac{1}{2}}\tau_2^2 \qquad (\partial^2_{\log t}\log\tau_1 = \lambda, \quad \partial^2_{\log t}\log\tau_2 = t\lambda^{-1})$

q-uplift $\left(\partial_{\log t} \to \dfrac{\mathfrak{q}^{\partial \log t} - 1}{\mathfrak{q} - 1}\right)$: $\boxed{\tau_1(\mathfrak{q}t)\tau_1(\mathfrak{q}^{-1}t) = \tau_1(t)^2 + t^{\frac{1}{2}}\tau_2(t)^2}$

Higher q-deformed Painlevé equations

Mathematics literatures [Sakai,'01][Tsuda,Masuda,'06]

τ^{qPX} = function on ∞-lattice generated by affine Weyl group $\widehat{W}(E_n)$

$W(E_n)$ + translation

which is a birational representation of $\widehat{W}(E_n)$

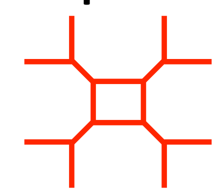
$\tau \to \tau^{-1}(\tau\tau + \tau\tau)$

This talk

τ^{qPX} = function on moduli space of genus 1 curve $C^X(v,w)=0$ with $W(E_n)$ symmetry

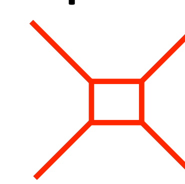
obeying bilinear equation $\tau\tau+\tau\tau+\tau\tau=0$ among translated points

qPVI

$\cdots$ 

$E_5(=D_5)$

qPIII$_3$

$\cdots$ 

$E_1(=A_1)$

qPVI ⇒

$(\cdots)\prod_{\pm}\tau_{i_\pm j_\pm}+(\cdots)\prod_{\pm}\tau_{i_\pm j_\mp}+(\cdots)\prod_{\pm}\tau_{k_{\pm\sigma_1}l_{\pm\sigma_2}m_{\pm\sigma_3}}=0$

(40 equations)

$\tau=\tau(t_1,t_2,t_3,t_4,t_5)$

$\tau_{i_\sigma j_{\sigma'}\cdots}=\tau|_{t_i+\frac{\sigma}{2},t_j+\frac{\sigma'}{2},\cdots}$

*Mathematics literatures tell us the consistency of overdetermining equations, but do not tell us about general solutions.

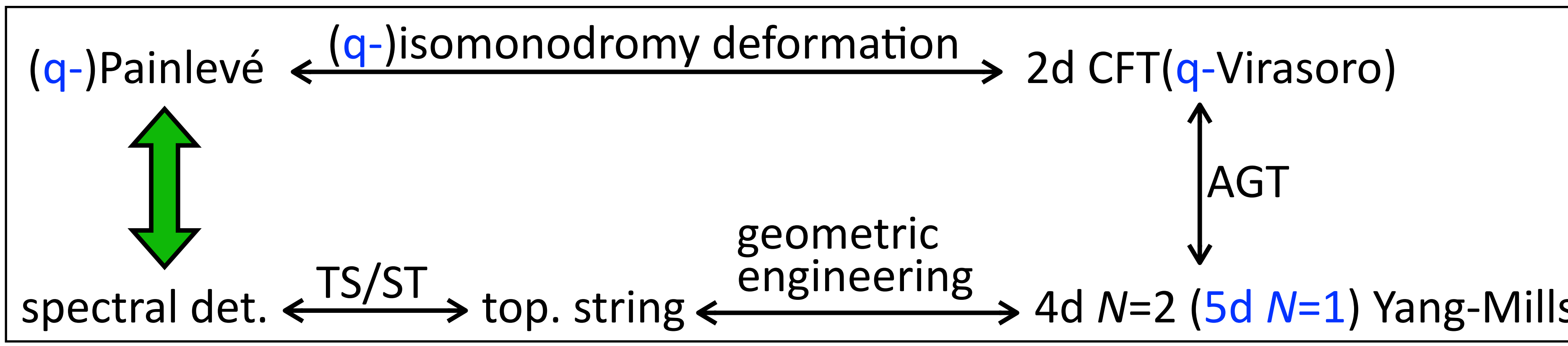
Solution of q-Painlevé equations?

Conjecture: $\tau^{\mathrm{qPX}} = \bigcirc \mathrm{Det}(1+\kappa(C^{\mathrm{X}}(e^{\widehat{x}}, e^{\widehat{p}}))^{-1})$

[Bonelli,Grassi,Tanzini,'17]

κ: arbitrary parameter (initial condition)

Motivation (not this talk):



This talk: M2-brane realization of τ^{qPX}

$$\tau^{\mathrm{qPIII}_3} = \sum_{N=0}^{\infty} \kappa^N Z^{U(N)_k \times U(N+M)_{-k}} = Z^{U(M)_k} \mathrm{Det}(1+\kappa(\widehat{C}^{\mathrm{III}_3})^{-1})$$

$$\tau^{\mathrm{qPVI}} = \sum_{N=0}^{\infty} \kappa^N Z^{U(N+L_1)_k \times U(N+L)_0 \times U(N+L_2)_{-k} \times U(N)_0} = Z^{U(L_1)_k \times U(L)_0 \times U(L_2)_{-k}} \mathrm{Det}(1+\kappa(\widehat{C}^{\mathrm{VI}})^{-1})$$

For qPVI, M2-brane realization is crucial for turning on all moduli

Overall coefficient $Z(N=0)$ is beyond the original motivation

new explanation for q-Painlevé from 3d ?

Plan of talk

1. M2-branes and Fermi gas formalism

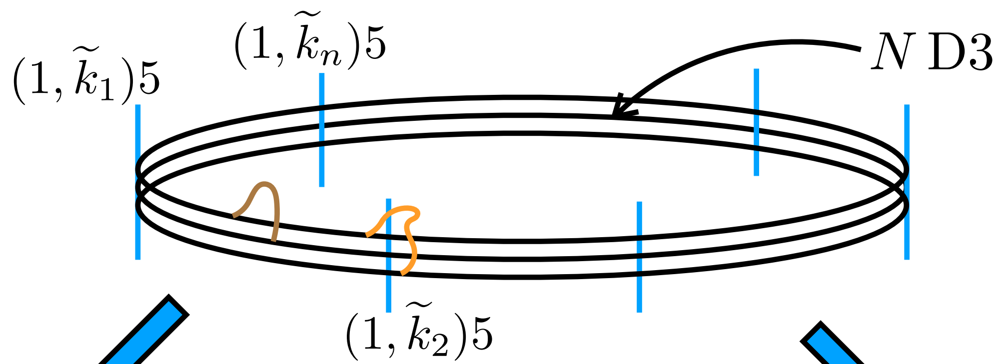
2. qPVI parameters from four node quiver

3. $\tau^{\text{qPVI}} = \sum_{N=0}^{\infty} \kappa^N Z^{\text{four node}}(N)$

4. Summary

M2-branes = Quiver Chern-Simons theories

In type IIB string theory



open string interactions

[Kitao,Ohta,Ohta,'98]
[Bergman,Hanany,Karch,Kol,'99]

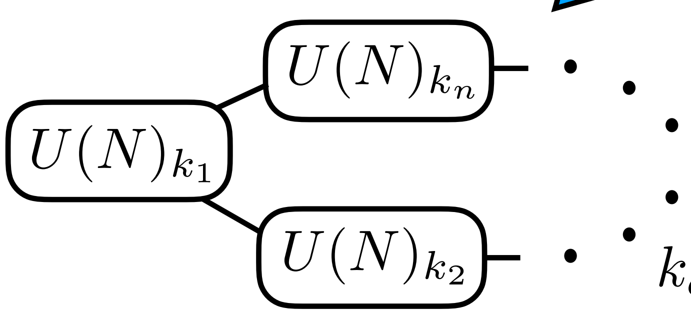
T-dual & IIA→M-theory uplift

D3→D2→M2

D5→D6→orbifold

NS5→orbifold→orbifold

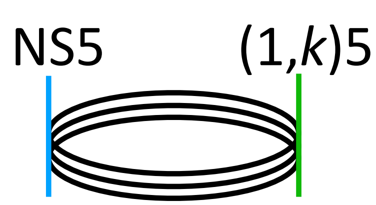
[Aharony,Bergman,Jafferis,Maldacena,'08]
[Imamura,Kimura,'08]



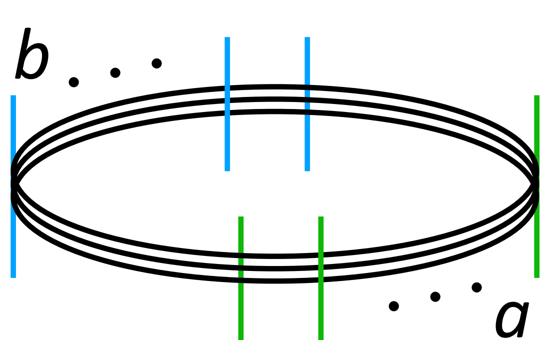
$k_a = \widetilde{k}_a - \widetilde{k}_{a-1}$

↔ M2-branes on orbifold background

For simplicity we consider only $\widetilde{k}_a = 0, k$



→ *N* M2 on $\mathbb{C}^4/\mathbb{Z}_k$



→ *N* M2 on $(\mathbb{C}^2/\mathbb{Z}_a \times \mathbb{C}^2/\mathbb{Z}_b)/\mathbb{Z}_k$

Supersymmetry localization formula

For a general supersymmetric field theory

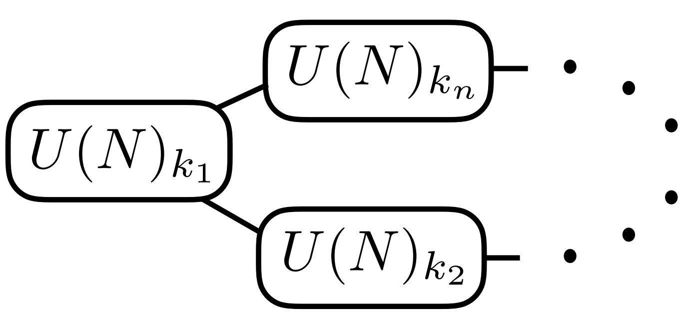
$$Z = \int \mathcal{D}\Phi\mathcal{D}\Psi e^{-S-\Lambda\delta(\int \Psi^\dagger \delta\Psi)}$$ is independent of Λ.

$$\overset{\Lambda\to\infty}{=} \sum_{(\Phi_0,\Psi_0)} e^{-S[\Phi_0,\Psi_0]} \frac{\det_{\Delta\Psi}(\delta^2(\Phi_0,\Psi_0))}{\det_{\Delta\Phi}(\delta^2(\Phi_0,\Psi_0))}$$

solutions of $|\delta\Psi|^2 = 0$

1-loop determinant of fluctuation around (Φ_0, Ψ_0)

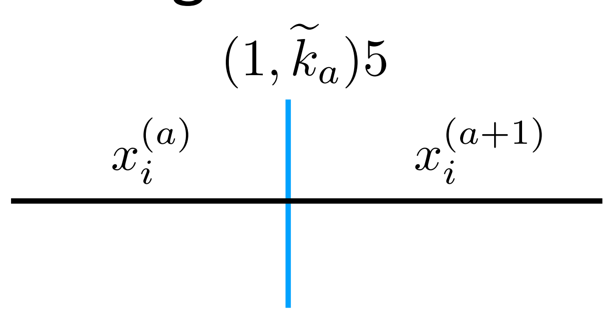
$(\Psi_0 = 0)$



$$Z = \frac{1}{(N!)^n} \int \frac{d^{Nn}x}{(2\pi)^{nN}} \prod_{a=1}^{n} e^{\frac{ik_a}{4\pi}(x_i^{(a)})^2} \frac{\prod_{i<j}(2\sinh\frac{x_i^{(a)}-x_j^{(a)}}{2})^2}{\prod_{i,j} 2\cosh\frac{x_i^{(a)}-x_j^{(a+1)}}{2}}$$

[Kapustin,Willett,Yaakov,'09]

building blocks in IIB brane setup:



$$e^{\frac{i\widetilde{k}_a(x^{(a)})^2}{4\pi}} \frac{\prod_{i<j} 2\sinh\frac{x_i^{(a)}-x_j^{(a)}}{2} \prod_{i<j} 2\sinh\frac{x_i^{(a+1)}-x_j^{(a+1)}}{2}}{\prod_{i,j} 2\cosh\frac{x_i^{(a)}-x_j^{(a+1)}}{2}} e^{-\frac{i\widetilde{k}_a(x^{(a+1)})^2}{4\pi}}$$

Fermi gas formalism

$$Z=\frac{1}{(N!)^n}\int\frac{d^{Nn}x}{(2\pi)^{nN}}\prod_{a=1}^{n}e^{\frac{i\tilde{k}_a}{4\pi}(x_i^{(a)})^2}\frac{\prod_{i<j}2\sinh\frac{x_i^{(a)}-x_j^{(a)}}{2}\prod_{i<j}2\sinh\frac{x_i^{(a+1)}-x_j^{(a+1)}}{2}}{\prod_{i,j}2\cosh\frac{x_i^{(a)}-x_j^{(a+1)}}{2}}e^{-\frac{i\tilde{k}_a}{4\pi}(x_i^{(a+1)})^2}$$

$$\det_{i,j}\frac{1}{2\cosh\frac{x_i^{(a)}-x_j^{(a+1)}}{2}}=\langle x_i^{(a)}|\frac{1}{2\cosh\frac{\widehat{p}}{2}}|x_j^{(a+1)}\rangle$$

Cauchy det. formula

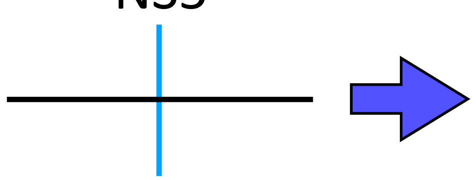
$$\frac{\prod_{i<j}(x_i-x_j)\prod_{i<j}(y_i-y_j)}{\prod_{i,j}(x_i+y_j)}=\det_{i,j}\frac{1}{x_i+y_j}$$

By using Cauchy-Binet formula $\int\frac{d^Nz}{N!}\det_{i,j}f_i(z_j)\det_{i,j}g_j(z_i)=\det_{i,j}\Big[\int dz f_i(z)g_j(z)\Big]$

$$Z(N)=\frac{1}{N!}\int\frac{d^Nx}{(2\pi)^N}\det_{i,j}\Big[\frac{1}{2\cosh\frac{\widehat{x}\text{ or }\widehat{p}}{2}}\frac{1}{2\cosh\frac{\widehat{x}\text{ or }\widehat{p}}{2}}\cdots\Big]$$

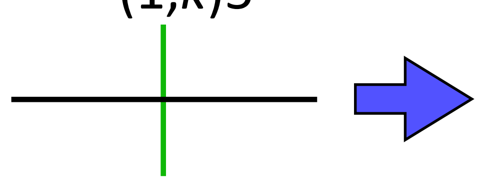
[Marino,Putrov,'11]

NS5



$$\frac{1}{2\cosh\frac{\widehat{p}}{2}}$$

(1,*k*)5



$$\frac{1}{2\cosh\frac{\widehat{x}}{2}}$$

$$[\widehat{x},\widehat{p}]=2\pi ik$$

All order 1/*N* perturbative expansion

$$Z = \frac{1}{N!}\int \frac{d^N x}{(2\pi)^N} \det_{i,j}\langle x_i|\widehat{\rho}|x_j\rangle$$

It is useful to define grand partition function *Ξ*(*κ*) and grand potential *J*(*μ*)

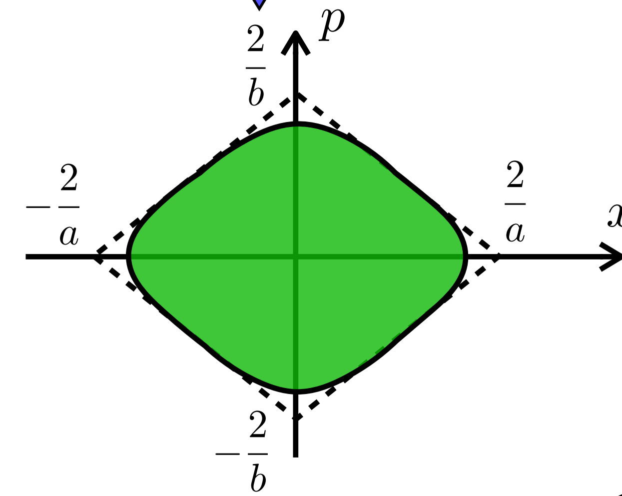
$$\Xi(\kappa) = \sum_{N=0}^{\infty} \kappa^N Z(N) = \mathrm{Det}(1+\kappa\widehat{\rho})$$

$$\Xi(\kappa) = e^{J(\mu)}, \quad \kappa = e^{\mu}$$

$$\frac{dJ}{d\mu} = \mathrm{Tr}\frac{1}{1+e^{-\mu}\widehat{\rho}^{-1}} \overset{\mu\gg 1}{\approx} \frac{1}{2\pi\hbar}\mathrm{vol}(\log\rho^{-1} < \mu)$$

$$\widehat{\rho} = \frac{1}{2\cosh\frac{\widehat{x}\text{ or }\widehat{p}}{2}}\cdots\frac{1}{2\cosh\frac{\widehat{x}\text{ or }\widehat{p}}{2}}$$

$$\log\rho^{-1} = a\log 2\cosh\frac{x}{2} + b\log 2\cosh\frac{p}{2}$$



$$J = \frac{C}{3}\mu^3 + B\mu + A + \mathcal{O}(e^{-\mu})$$

$$\left(C = \frac{2}{\pi^2 abk}\right)$$

Inverse trsf. $(\mu \sim \sqrt{N/C})$

$$Z(N) = \int \frac{d\mu}{2\pi i} e^{J-\mu N} = e^A C^{-\frac{1}{3}}\mathrm{Ai}[C^{-\frac{1}{3}}(N-B)](1+\mathcal{O}(e^{-\sqrt{N}}))$$

$-\log Z(N) \sim \frac{\pi\sqrt{2abk}}{3}N^{\frac{3}{2}} \sim \frac{R_{\mathrm{AdS}}^{11-2}}{abk}$: consistent with 11d SUGRA $\quad (R_{\mathrm{AdS}} \sim (abkN)^{\frac{1}{6}})$

1/*N* non-perturbative effects

1/*N* non-pert. = bulk closed M2-branes wrapped on 3-cycles in $\mathrm{AdS}_4 \times \underset{\sim\sim\sim}{(Y_7/\mathbb{Z}_k)}$

3-cycle with $S^1/\mathbb{Z}_k$ ➡ $e^{-\mathrm{vol(M2)}} = e^{-k^{-1}R_{\mathrm{AdS}}^3} \sim e^{-\sqrt{\frac{N}{k}}}$: worldsheet instanton

3-cycle without $S^1/\mathbb{Z}_k$ ➡ $e^{-\mathrm{vol(M2)}} = e^{-R_{\mathrm{AdS}}^3} \sim e^{-\sqrt{kN}}$: membrane instanton

$R_{\mathrm{AdS}} \sim (kN)^{\frac{1}{6}}$

In *J*(*μ*), MB: $e^{-\mu}$ $\longleftarrow$ $\hbar$-expansion

WS: $e^{-\frac{\mu}{k}}$ $\longleftarrow$ fitting of exact values of $Z_{k=1,2,\cdots}(N=1,2,\cdots)$

All order 1/*N* non-pert. effects by topological string

From the exact results it was found

$J_{\rm WS}(\mu)$ = unrefined topological string free energy

$$J_{\rm WS}(\mu) = \sum_{d=1}^{\infty}\sum_{g=0}^{\infty} N_g^d \sum_{n=1}^{\infty}\frac{1}{n}\left(2\sinh\frac{2\pi n}{k}\right)^{2g-2} e^{-\frac{4nd\mu}{k}}$$

$J_{\rm MB}(\mu)$ = Nekrasov-Shatashvili limit of refined topological string free energy

on the same target Calabi-Yau threefold determined by $\widehat{\rho}$

[Hatsuda,Marino,Moriyama,Okuyama,'13][Moriyama,TN,'14]

: topological string/spectral theory correspondence [Grassi,Hatsuda,Marino,'14]

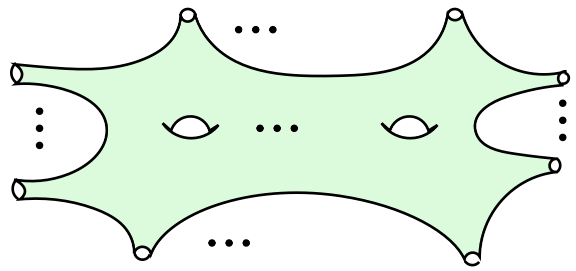
*Once we believe TS/ST correspondence, we obtain instantons to arbitrary order by calculating topological invariants ⇒ more tractable than "WKB + fitting"

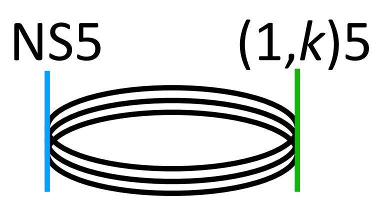
M2-brane = quantum curve

Fermi gas formalism: M2-brane $\longleftrightarrow$ quantization of curve $C(v,w)=const.$

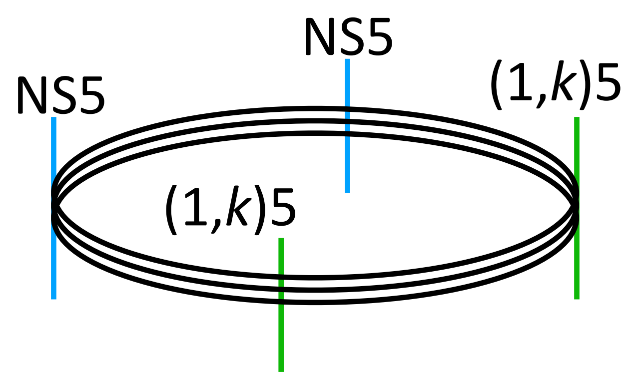
$$\sum_{N=0}^{\infty}\kappa^N Z(N)=\mathrm{Det}(1+\kappa C(e^{\widehat{x}},e^{\widehat{p}})^{-1})$$

$$C(v,w)=\sum_{m,n}c_{mn}v^m w^n$$

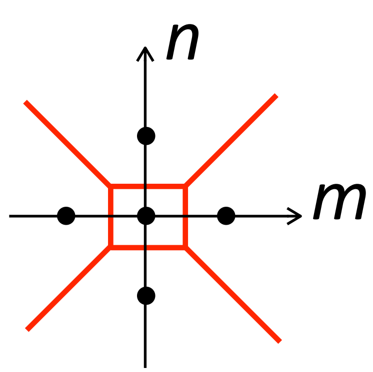




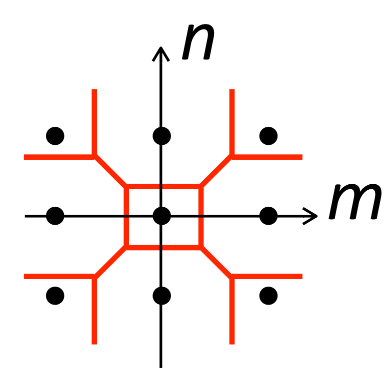
$$C(e^{\widehat{x}},e^{\widehat{p}})=2\cosh\frac{\widehat{x}}{2}2\cosh\frac{\widehat{p}}{2}$$



$$C(e^{\widehat{x}},e^{\widehat{p}})=\left(2\cosh\frac{\widehat{x}}{2}\right)^2\left(2\cosh\frac{\widehat{p}}{2}\right)^2$$



→ qPIII_3



→ qPVI

For genus 1, *C*(*v*,*w*) coincide with curves associated with qPX equations

*For general quiver CS, $\widehat{\rho}^{-1}$ is a higher genus curve (unless parameters are tuned)

Plan of talk

✓ 1. M2-branes and Fermi gas formalism

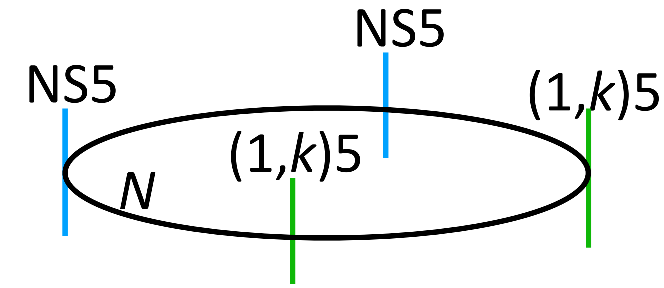
2. qPVI parameters from four node quiver

3. $\tau^{\mathrm{qPVI}} = \sum_{N=0}^{\infty} \kappa^N Z^{\text{four node}}(N)$

4. Summary

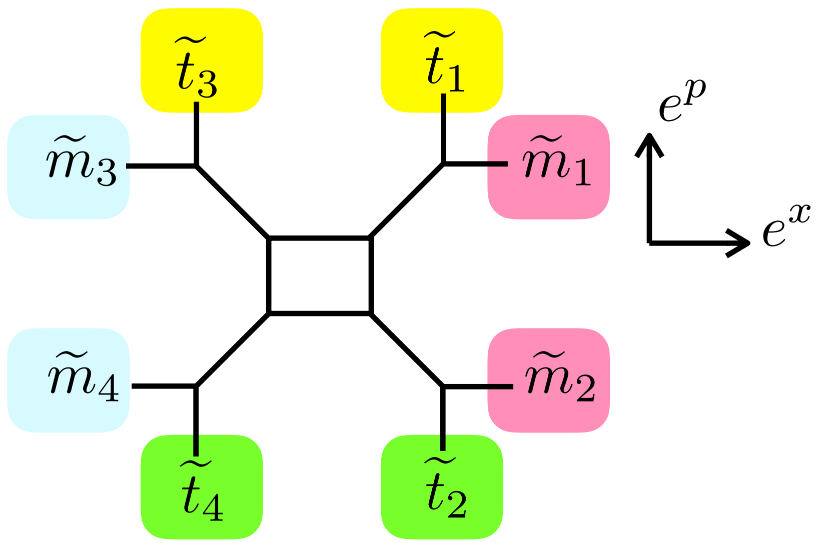
Parameters of quantum curve $C^{\rm VI}(e^{\widehat{x}}, e^{\widehat{p}})$

$$\tau^{\rm qPVI} \sim \sum_{N=0}^{\infty} \kappa^N Z(N)$$



To check qPVI equation, we have to turn on the moduli = generic coefficients c_{mn} of

$$\widehat{C}^{\rm VI} = \sum_{m=-1,0,1} \sum_{n=-1,0,1} c_{mn} e^{m\widehat{x}+n\widehat{p}}$$



moduli = asymptotic loci ➡ 8 – 2 – 1 = 5 parameters

$\widetilde{m}_i \to \alpha \widetilde{m}_i$

$\widetilde{t}_i \to \beta \widetilde{t}_i$

$\widetilde{m}_1 \widetilde{m}_2 \widetilde{t}_2 \widetilde{t}_4 = \widetilde{m}_3 \widetilde{m}_4 \widetilde{t}_1 \widetilde{t}_3$

$$2\cosh\frac{\widehat{x}}{2}\left(2\cosh\frac{\widehat{p}}{2}\right)^2 2\cosh\frac{\widehat{x}}{2} = e^{-\widehat{x}+\widehat{p}} + (e^{\pi i k} + e^{-\pi i k})e^{\widehat{p}} + e^{\widehat{x}+\widehat{p}} + 2e^{-\widehat{x}} + 4 + 2e^{\widehat{x}} + e^{-\widehat{x}-\widehat{p}} + (e^{\pi i k} + e^{-\pi i k})e^{-\widehat{p}} + e^{\widehat{x}-\widehat{p}}$$

➡ $\widetilde{m}_i = 1,\ \widetilde{t}_{1,2} = -e^{-\pi i k},\ \widetilde{t}_{3,4} = -e^{\pi i k}$

Natural coordinate of moduli space

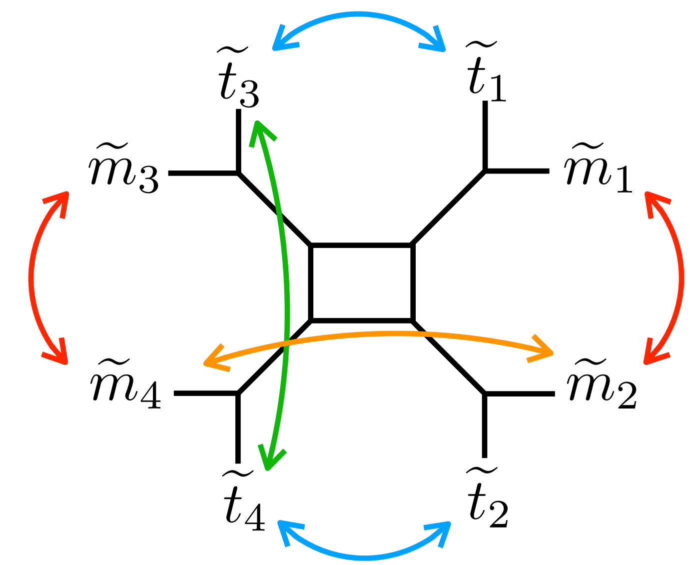
C^{VI} curve has $W(D_5)$ discrete symmetry exchanging asymptotic loci

generated by canonical trsf. $\widehat{x}, \widehat{p} \to \widehat{x}', \widehat{p}'$ ➡ preserves $\mathrm{Det}(1+\kappa(\widehat{C}^{\mathrm{VI}})^{-1})$

A natural coordinate $(t_1, t_2, t_3, t_4, t_5)$ is such that

$W(D_5)$ = reflections of two components + permutations

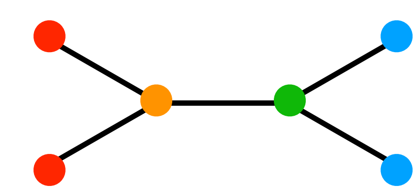
$$(t_1, t_2, t_3, t_4, t_5) = (M_0, M_1, M_3, Z_1, Z_2)$$



$$M_0 = \frac{1}{4\pi i} \log \frac{\widetilde{m}_1 \widetilde{m}_2}{\widetilde{m}_3 \widetilde{m}_4} \qquad Z_1 = \frac{1}{4\pi i} \log \frac{\widetilde{m}_2 \widetilde{m}_3}{\widetilde{m}_1 \widetilde{m}_4}$$

$$M_1 = \frac{1}{4\pi i} \log \frac{\widetilde{m}_2 \widetilde{m}_4}{\widetilde{m}_1 \widetilde{m}_3} \qquad Z_3 = \frac{1}{4\pi i} \log \frac{\widetilde{t}_1 \widetilde{t}_4}{\widetilde{t}_2 \widetilde{t}_3}$$

$$M_3 = \frac{1}{4\pi i} \log \frac{\widetilde{t}_3 \widetilde{t}_4}{\widetilde{t}_1 \widetilde{t}_2}$$

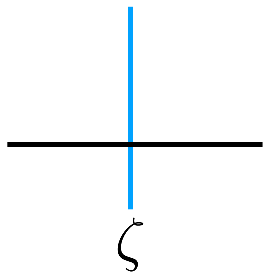


$$2\cosh\frac{\widehat{x}}{2}\left(2\cosh\frac{\widehat{p}}{2}\right)^2 2\cosh\frac{\widehat{x}}{2}$$ ➡ $$(M_0, M_1, M_3, Z_1, Z_3) = (0, 0, k, 0, 0)$$

How can we turn on general moduli parameters?

qPVI parameters from 3d

2 of 5 moduli can be turned on by FI parameters

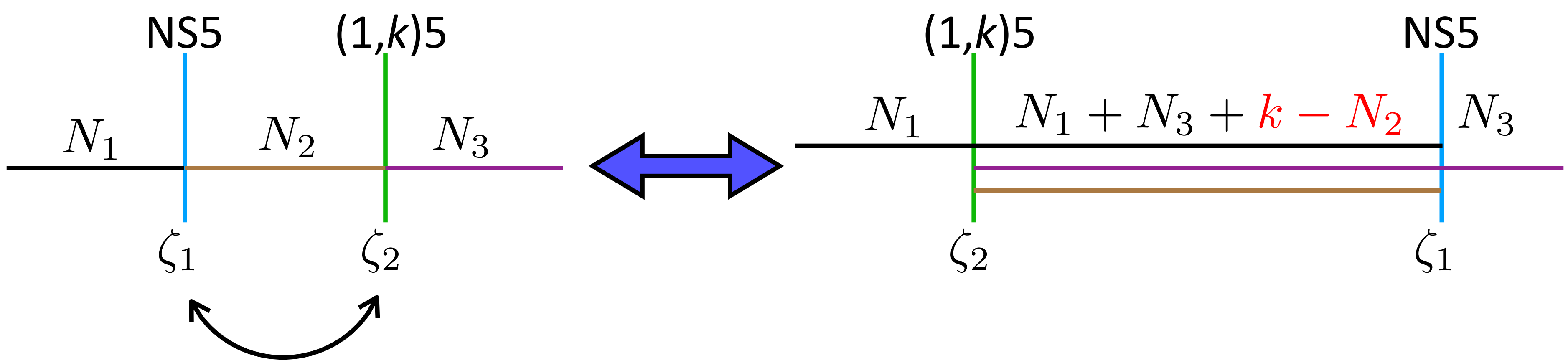


$$e^{-\frac{i\zeta\widehat{x}}{k}}\frac{1}{2\cosh\frac{\widehat{p}}{2}}e^{\frac{i\zeta\widehat{x}}{k}}$$

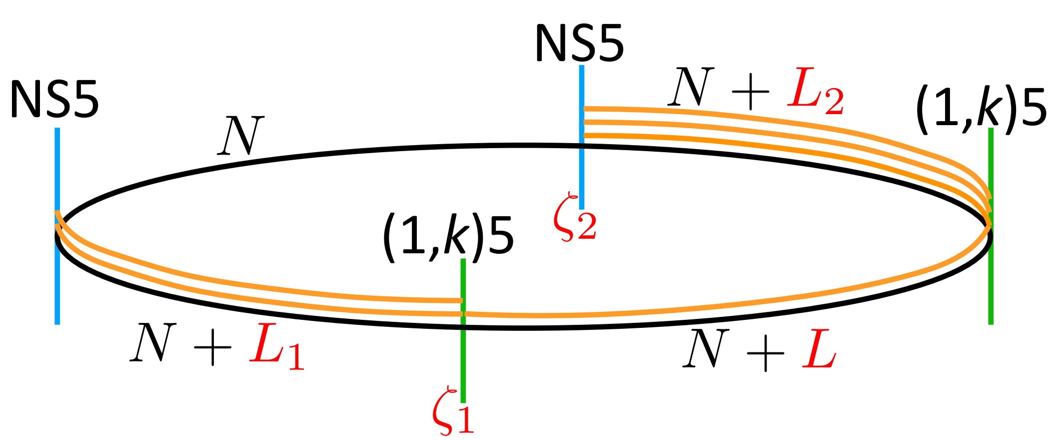
Other 3?

Hint: there is $W(D_5)$ discrete symmetry on C^{VI} curve

discrete symmetry in 3d = Hanany-Witten IR dualities [Hanany,Witten,'96]



5 moduli are realized by $(L_1, L_2, L, \zeta_1, \zeta_2)$



Parameter identification

Method 1 (possible only for $L = 0$)

generalize Fermi gas formalism to $(L_1, L_2, L) \neq (0,0,0)$

$$\frac{Z(N)}{Z(0)} = \frac{1}{N!}\int \frac{d^N x}{(2\pi)^N} \det_{i,j} \langle x_i | \widehat{\rho}(L_1, L_2, L, \zeta_1, \zeta_2) | x_j \rangle$$

and directly calculate $\widehat{\mathcal{O}} = \widehat{\rho}^{-1}$

Method 2 (hard)

calculate 1/*N* non-perturbative effects

→ match with topological string free energy (Gopakumar-Vafa formula)

Method 3 (quick)

interpolation from special points

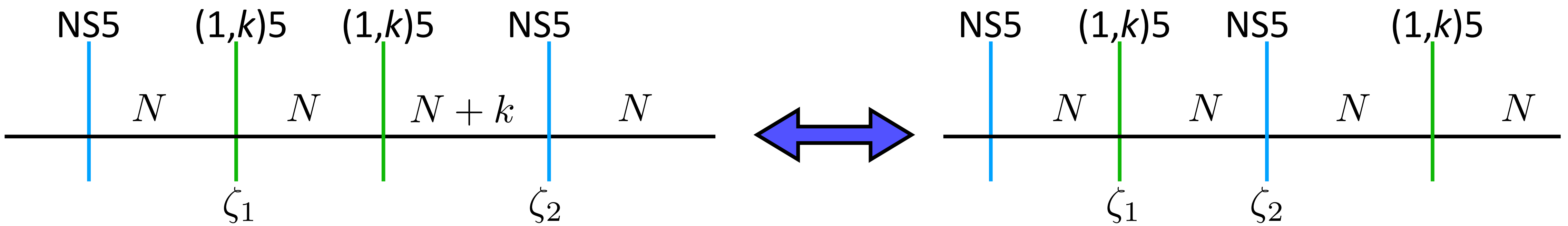
+ assumption: $t_{1,\cdots,5} = \text{1st order polnomial of } (L_1, L_2, L, \zeta_1, \zeta_2)$

*assumption was confirmed by Methods 1,2

Interpolation method

For special values of L_1, L_2, L , IIB brane setup can be transformed into another setup without rank deformations by Hanany-Witten move.

example:



$$\widehat{C}^{\mathrm{VI}} = \left(2\cosh\frac{\widehat{p}}{2}\right) e^{-i\zeta_2(\widehat{x}-\widehat{p})} \left(2\cosh\frac{\widehat{x}}{2}\right) e^{i\zeta_2(\widehat{x}-\widehat{p})} e^{-i\zeta_1(\widehat{x}-\widehat{p})} \left(2\cosh\frac{\widehat{p}}{2}\right) e^{i\zeta_1(\widehat{x}-\widehat{p})} \left(2\cosh\frac{\widehat{x}}{2}\right)$$

(L_1, L_2, L)	$(M_0, M_1, M_3, Z_1, Z_3)$
$(0,0,0)$	$(0,0,k,i\zeta_1,i\zeta_2)$
$(0,0,k)$	$(0,k,k,i\zeta_1,i\zeta_2)$
$(0,k,k)$	$\left(\frac{k}{2},\frac{k}{2},\frac{k}{2},i\zeta_1,i\zeta_2\right)$
$(k,0,0)$	$\left(-\frac{k}{2},-\frac{k}{2},\frac{k}{2},i\zeta_1,i\zeta_2\right)$

$$(M_0, M_1, M_3, Z_1, Z_3) = \left(-\frac{L_1}{2}+\frac{L_2}{2}, L-\frac{L_1}{2}-\frac{L_2}{2}, k-\frac{L_1}{2}-\frac{L_2}{2}, i\zeta_1, i\zeta_2\right)$$

[Bonelli,Globlek,Kubo,TN,Tanzini,'22]

Plan of talk

✓ 1. M2-branes and Fermi gas formalism

✓ 2. qPVI parameters from four node quiver

3. $\tau^{\mathrm{qPVI}} = \sum_{N=0}^{\infty} \kappa^N Z^{\text{four node}}(N)$

4. Summary

Structure of qPVI equations

Our conjecture:

$$\tau^{\mathrm{qPVI}}(M_0, M_1, M_3, Z_1, Z_3; \kappa) = \bigcirc \mathrm{Det}(1 + \kappa(\widehat{C}^{\mathrm{VI}})^{-1}) = \bigcirc \sum_{N=0}^{\infty} \kappa^N \frac{Z_k(N; L_1, L_2, L, \zeta_1, \zeta_2)}{Z_k(0; L_1, L_2, L, \zeta_1, \zeta_2)}$$

satisfies q-Painlevé VI bilinear equations

$$(\cdots)\prod_{\pm} \tau_{i_\pm j_\pm} + (\cdots)\prod_{\pm} \tau_{i_\pm j_\mp} + (\cdots)\prod_{\pm} \tau_{k_{\pm\sigma_1} l_{\pm\sigma_2} m_{\pm\sigma_3}} = 0 \qquad \tau_{i_\sigma j_{\sigma'} \cdots} = \tau\Big|_{t_i + \frac{\sigma}{2}, t_j + \frac{\sigma'}{2}, \cdots}$$

with $\vec{t} = (M_0, M_1, M_3, Z_1, Z_3) = \left(-\frac{L_1}{2} + \frac{L_2}{2}, L - \frac{L_1}{2} - \frac{L_2}{2}, k - \frac{L_1}{2} - \frac{L_2}{2}, i\zeta_1, i\zeta_2\right)$

$\bigcirc$: some $(L_1, L_2, L, \zeta_1, \zeta_2)$-dependent factor

overall factor of τ fcn. can be absorbed to the coefficients of bilinear eq

➡ we have to fix either $\bigcirc$ or coefs of bilinear eqs (...) by hand.

Strategy: choose $\bigcirc$ by hand → fix coefs of bilinear eqs → check higher order in *κ*

Candidates of overall factor

Choose $\bigcirc = 1$?

➡ could not find simple coefficients of bililnear eqs. for $\kappa \neq 0$

Another natural choice: $\bigcirc = Z(0; L_1, L_2, L, \zeta_1, \zeta_2)$?

cf. $U(N)_k \times U(N+M)_{-k}$

➡ $Z(N=0) = Z_k^{\mathrm{CS}}(M) = \frac{1}{k^{\frac{M}{2}}} \prod_{i>j} 2\sin\frac{\pi(i-j)}{k}$

$$\boxed{Z_k^{\mathrm{CS}}(M+1) Z_k^{\mathrm{CS}}(M-1) = 2\sin\frac{\pi M}{k} Z_k^{\mathrm{CS}}(M)^2}$$

: essentially qPIII$_3$ equation!

[Bonelli,Grassi,Tanzini,'17]

It works!

$U(L_1)_{k,\zeta_1}$ — $U(L)_{0,-\zeta_1}$ — $U(L_2)_{-k,\zeta_2}$

$$Z_k(0;L_1,L_2,L,\zeta_1,\zeta_2) = Z_k^{\rm CS}(L_1)Z_k^{\rm CS}(L_2)(e^{-\pi i\zeta_1} - (-1)^{L_1+L_2}e^{\pi\zeta_1})^{-L}$$
$$\times \det_{i,j}^{L}\left[\sum_{r'=1}^{L_1}\frac{e^{\frac{2\pi i}{k}(L+1-r-s+i\zeta_1)(\frac{L_1+1}{2}-r')}}{\prod_{r''(\neq r')}^{L_1}2\sin\frac{\pi(r'-r'')}{k}\prod_{r''}^{L_2}2\sin\frac{\pi(r'-r''+i\zeta_2)}{k}} + \sum_{r'=1}^{L_2}\frac{e^{\frac{2\pi i}{k}(L+1-r-s+i\zeta_1)(\frac{L_2+1}{2}-r'+i\zeta_2)}}{\prod_{r''}^{L_1}2\sin\frac{\pi(r'-r''-i\zeta_2)}{k}\prod_{r''(\neq r')}^{L_2}2\sin\frac{\pi(r'-r'')}{k}}\right]$$

(similar calculation as [TN,Yokoyama,'17])

➡ satisfies 40 bilinear eqs with simple coefficients

$$e^{-\frac{\pi i}{2k}(\sigma_k v_k+\sigma_l v_l+\sigma_m v_m)}S^{(1)}\prod_{\pm}\tau_{i_\pm j_\pm} + e^{\frac{\pi i}{2k}(\sigma_k v_k+\sigma_l v_l+\sigma_m v_m)}S^{(2)}\prod_{\pm}\tau_{i_\pm j_\mp} + S^{(3)}\prod_{\pm}\tau_{k_{\pm\sigma_1}l_{\pm\sigma_2}m_{\pm\sigma_3}} = 0$$

$$i,j = M_1,Z_1 \to S^{(1)},S^{(2)} = 2\sin\frac{\pi(M_1\pm Z_1)}{k},\quad S^{(3)} = 2\sin\frac{\pi(M_3+\sigma_{M_0}Z_3)}{k}$$
$$i,j = M_3,Z_3 \to S^{(1)},S^{(2)} = 2\sin\frac{\pi(M_3\pm Z_3)}{k},\quad S^{(3)} = 2\sin\frac{\pi(M_1+\sigma_{M_0}Z_1)}{k}$$
$$i = M_0, j = M_1 \text{ or } Z_1 \to S^{(3)} = 2\sin\frac{\pi(M_3+\sigma_{(Z_1\text{ or }M_1)}Z_3)}{k}$$
$$i = M_0, j = M_3 \text{ or } Z_3 \to S^{(3)} = 2\sin\frac{\pi(M_1+\sigma_{(Z_3\text{ or }M_3)}Z_1)}{k}$$

[Moriyama,TN, in preparation]

Bilinear equations are also satisfied at higher order in κ with

$$(\cdots)\prod_{\pm}\tau_{\vec{v}\pm\frac{1}{2}(\delta_\alpha^i+\delta_\alpha^j)}(\kappa) + (\cdots)\prod_{\pm}\tau_{\vec{v}\pm\frac{1}{2}(\delta_\alpha^i-\delta_\alpha^j)}(-\kappa) + (\cdots)\prod_{\pm}\tau_{\vec{v}\pm\frac{1}{2}(\sigma_k\delta_\alpha^k+\sigma_l\delta_\alpha^l+\sigma_m\delta_\alpha^m)}(\mp i\kappa) = 0$$

Quartic relations of *Z*(*N*=0)

Viewed as the equations for $\mathrm{Det}(1+\kappa(\widehat{C}^{\mathrm{VI}})^{-1})$, 40 bilinear equations

$$f_1^{(ij;\sigma_1\sigma_2\sigma_3)}\prod_{\pm}\mathrm{Det}(1+\kappa(\widehat{C}^{\mathrm{VI}}_{i_\pm,j_\pm})^{-1})+f_2^{(ij;\sigma_1\sigma_2\sigma_3)}\prod_{\pm}\mathrm{Det}(1-\kappa(\widehat{C}^{\mathrm{VI}}_{i_\pm,j_\mp})^{-1})+f_3^{(ij;\sigma_1\sigma_2\sigma_3)}\prod_{\pm}\mathrm{Det}(1\mp\kappa(\widehat{C}^{\mathrm{VI}}_{k_{\pm\sigma_1},l_{\pm\sigma_2},m_{\pm\sigma_3}})^{-1})=0$$

can be generated from single $(ij;\sigma_1\sigma_2\sigma_3)$ by Weyl symmetry.

example: $(12;+++)$ evaluated at (M_0,M_3,M_1,Z_1,Z_3)

$$f_1^{(12;+++)}(M_0,M_3,M_1,Z_1,Z_3)\prod_{\pm}\mathrm{Det}(1+\kappa(\widehat{C}^{\mathrm{VI}}(M_0\pm\frac{1}{2},M_3\pm\frac{1}{2},M_1,Z_1,Z_3)^{-1})$$
$$+f_2^{(12;+++)}(M_0,M_3,M_1,Z_1,Z_3)\prod_{\pm}\mathrm{Det}(1-\kappa(\widehat{C}^{\mathrm{VI}}(M_0\pm\frac{1}{2},M_3\mp\frac{1}{2},M_1,Z_1,Z_3)^{-1})$$
$$+f_3^{(12;+++)}(M_0,M_3,M_1,Z_1,Z_3)\prod_{\pm}\mathrm{Det}(1\mp i\kappa(\widehat{C}^{\mathrm{VI}}(M_0,M_3,M_1\pm\frac{1}{2},Z_1\pm\frac{1}{2},Z_3\pm\frac{1}{2})^{-1})=0$$

$= \mathrm{Det}(1+\kappa(\widehat{C}^{\mathrm{VI}}(M_0\pm\frac{1}{2},M_1,M_3\pm\frac{1}{2},Z_1,Z_3))^{-1})$

$= \mathrm{Det}(1+\kappa(\widehat{C}^{\mathrm{VI}}(M_0\pm\frac{1}{2},M_1,M_3\mp\frac{1}{2},Z_1,Z_3))^{-1})$

$= \mathrm{Det}(1\mp i\kappa(\widehat{C}^{\mathrm{VI}}(M_0,M_1\pm\frac{1}{2},M_3,Z_1\pm\frac{1}{2},Z_3\pm\frac{1}{2}))^{-1})$

➡ same as (13;+++)

In order this to be consistent with original (13;+++) at all order in *κ*, it follows

$$\frac{f_1^{(12;+++)}(M_0,M_3,M_1,Z_1,Z_3)}{f_1^{(13;+++)}(M_0,M_1,M_3,Z_1,Z_3)}=\frac{f_2^{(12;+++)}(M_0,M_3,M_1,Z_1,Z_3)}{f_2^{(13;+++)}(M_0,M_1,M_3,Z_1,Z_3)}=\frac{f_3^{(12;+++)}(M_0,M_3,M_1,Z_1,Z_3)}{f_3^{(13;+++)}(M_0,M_1,M_3,Z_1,Z_3)}$$

➡ quartic identities for $Z(0;L_1,L_2,L,\zeta_1,\zeta_2)$ (stronger than qPVI eqs!)

Plan of talk

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✓ 3. $\tau^{\text{qPVI}} = \sum_{N=0}^{\infty} \kappa^N Z^{\text{four node}}(N)$

4. Summary

Summary

M2-branes ⟷ q-discrete Painlevé systems

$$\tau^{\mathrm{qPIII}_3}(t)=\sum_{N=0}^{\infty}\kappa^N Z^{U(N)_k\times U(N+M)_{-k}}\left(=Z^{U(M)_k}\mathrm{Det}(1+\kappa(\widehat{C}^{\mathrm{III}_3})^{-1})\right)$$

$$\tau^{\mathrm{qPVI}}(t_1,\cdots,t_5)=\sum_{N=0}^{\infty}\kappa^N Z^{U(N+L_1)_{k,\zeta_1}\times U(N+L)_{0,-\zeta_1}\times U(N+L_2)_{-k,\zeta_2}\times U(N)_{0,-\zeta_2}}\left(=Z^{U(L_1)_{k,\zeta_1}\times U(L)_{0,-\zeta_1}\times U(L_2)_{-k,\zeta_2}}\mathrm{Det}(1+\kappa(\widehat{C}^{\mathrm{VI}})^{-1})\right)$$ ← new

- Fredholm determinants give new expansion of q-Painlevé τ fcns.

(cf. $\tau=Z^{\mathrm{5d}}_{\mathrm{Nek}}$ → early time expansion)

- M2-theories are efficient (essential) ways to define $\mathrm{Det}(1+\kappa C^{-1})$ with all parameters.

- Painlevé bilinear equations are also satisfied by $Z(N=0)$, which is out of the original reasoning for $\tau\sim\mathrm{Det}(1+\kappa C^{-1})$ through 5d/spectral theory correspondence.

➡ new explanation of q-Painlevé eq from 3d ?

cf. matrix model and integrable hierarchy

Future works

Implication to other theories related to (q-)Painlevé?

→ (2d (q-)Liouville/Toda CFT, 4d/5d Yang-Mills, ...)

Application to physics problems governed by non-linear ordinary differential equations?

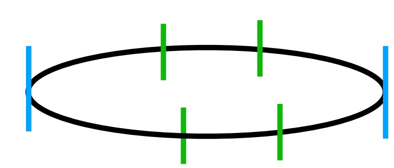
Proof (beyond $q \to 1$ limit of aPIII$_3$) [Tracy,Widom,'95]

Further generalizations:

- higher q-Painlevé
- higher rank (PIII$_3$ → $\widehat{A}_{n\geq 2}$ Toda) ⟷ ?
- BCDEFG Toda

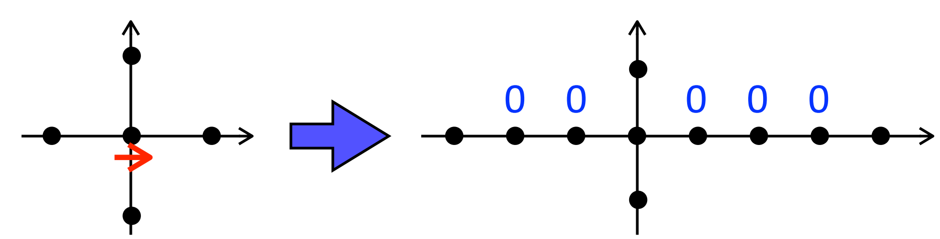
⋮

- longer circular quivers



→ $\mathrm{qP}(E_7)$

- mass deformations [Bonelli,Grassi,Tanzini,'17][TN,'20]



- D/E-type quivers