# Half-wormholes and Ensemble Averages

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# Outline

- **Half-wormholes and Ensemble averages**

1. Introduction & Motivation

   i. SYK model

   ii. Factorization problem

2. Recent progress

   i. Half-wormhole

3. Our work

   i. Generalized half-wormhole

   ii. Half-wormhole from decomposition

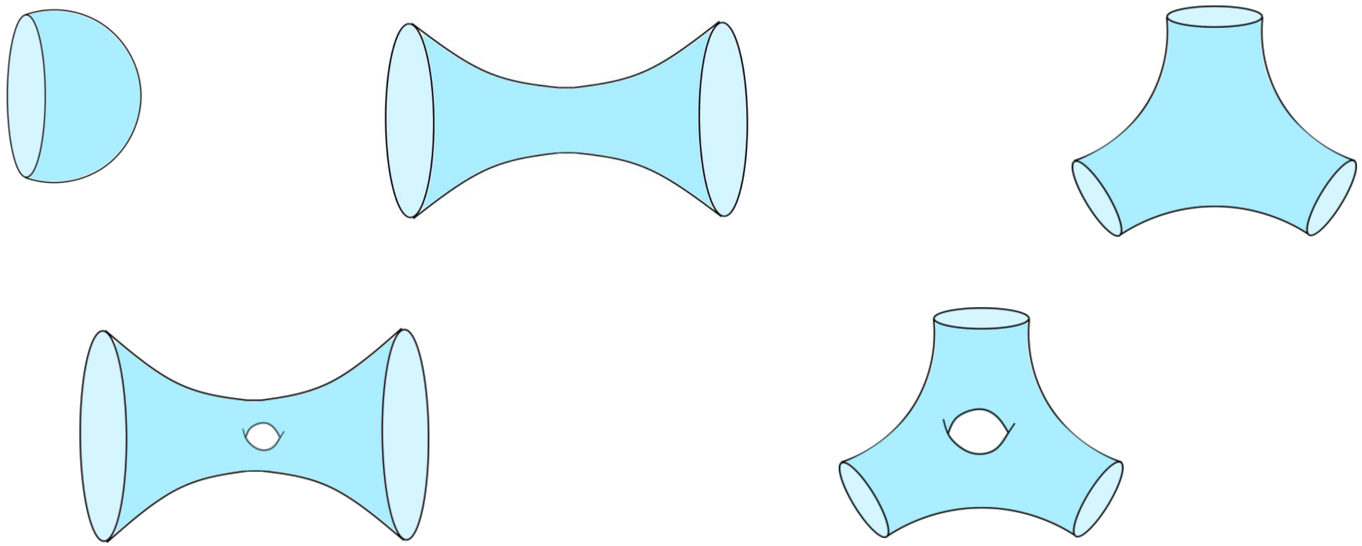
# Introduction

- Consider the integral

$$\int [dX\, dg]\exp(-S)$$

$g_{\mu\nu}$: the metric

- There're many kinds of geometries, such as black holes, wormholes.



# Introduction

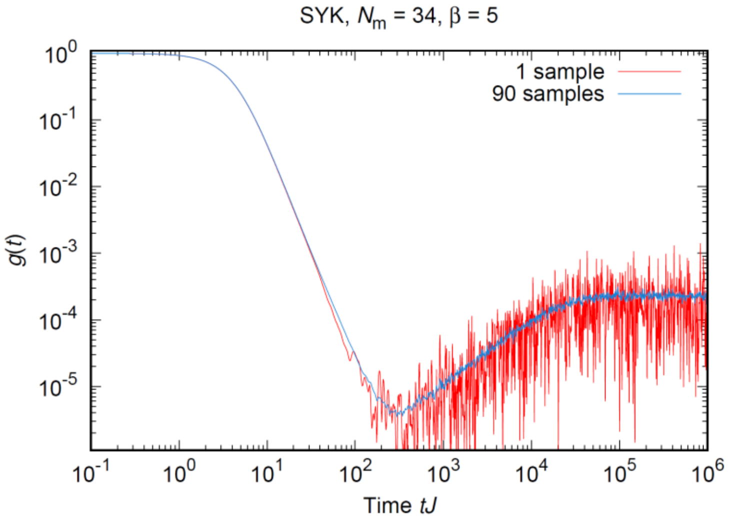
- Wormhole is useful in: spectral form factor [cotler,etc,1611.04650][Saad,Shenker,Stanford,2018], Page Curve [Penington,Shenker,Stanford,Yang,2019], Correlation function [Saad,2019]

- Spectral form factor

$$Z(\beta,t) \equiv Tr\left(e^{-\beta H - iHt}\right), g(t;\beta) \equiv \frac{\langle Z(\beta,t)Z^{\star}(\beta,t)\rangle_J}{\langle Z(\beta)\rangle_J^2}$$

slope ↔ disconnected ↔ black hole

ramp ↔ connected ↔ wormhole

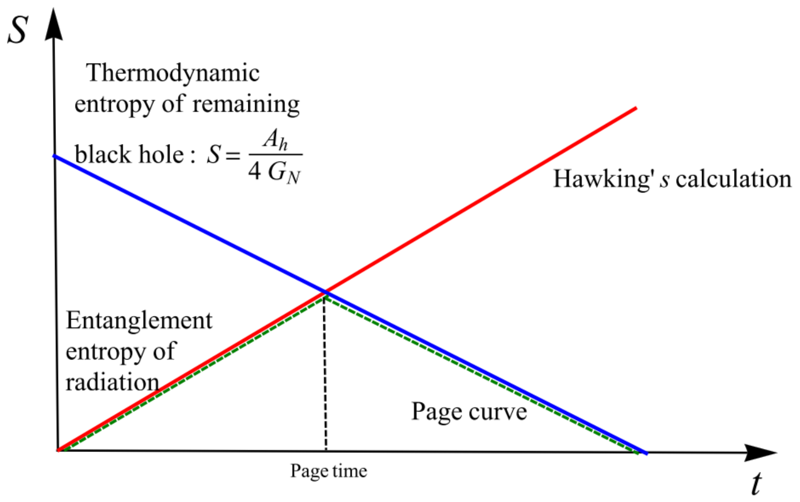


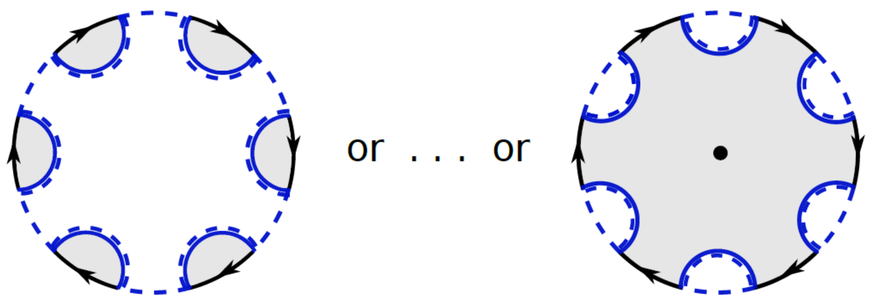

[cotler,etc,1611.04650]

# Introduction

- Page Curve

$$S_R = -Tr(\rho_R \log \rho_R) = -\lim_{n\to 1}\frac{1}{n-1}\log Tr(\rho_R^n)$$




[Penington,Shenker,Stanford,Yang,2019]

# Introduction

- Correlation function [Saad,2019]

late time behavior of the boundary two point functions in JT gravity coupled to matter

$$G_{2,\beta}^{L,R}(T) = \int dE\, dE' \rho(E,E') e^{-\frac{\beta}{2}(E+E')} e^{-iT(E-E')} \left|O_{E,E'}\right|^2$$

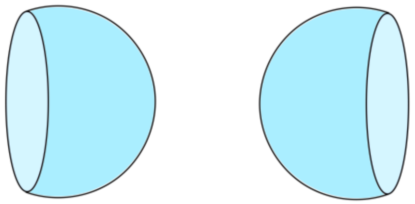
ramp and plateau

- Summary

wormhole ↔ connected contribution

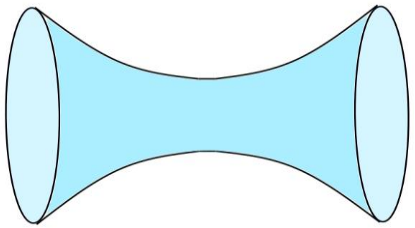
# Factorization problem

- Problem:  wormhole can lead to factorization puzzle



In AdS/CFT,  two decoupled field theories L,R

boundary view: $Z_{LR} = Z_L Z_R$



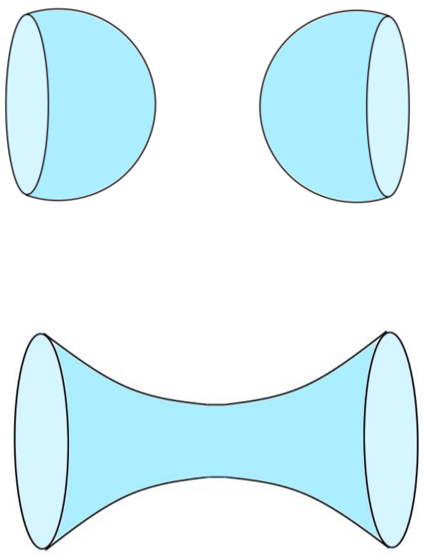
bulk view: $Z_{LR} \neq Z_L Z_R$   due to the wormhole contribution

# Factorization problem

- A plausible solution: ensemble average

  bulk is dual to an ensemble in boundary

  it's natural $\langle Z_L Z_R \rangle \neq \langle Z_L \rangle \langle Z_R \rangle$



- Question: Bulk/Ensemble is different to AdS/CFT

  What happens when we consider a single element in the ensemble

# SYK

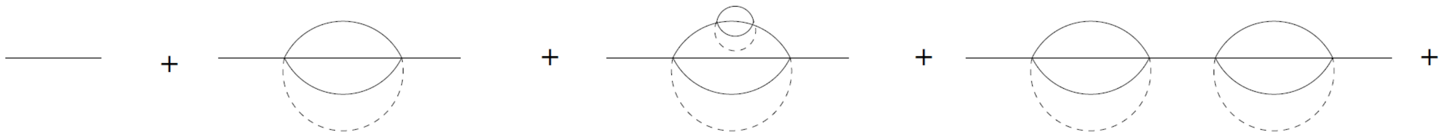
- Example: SYK [Maldacena,Stanford,2016]

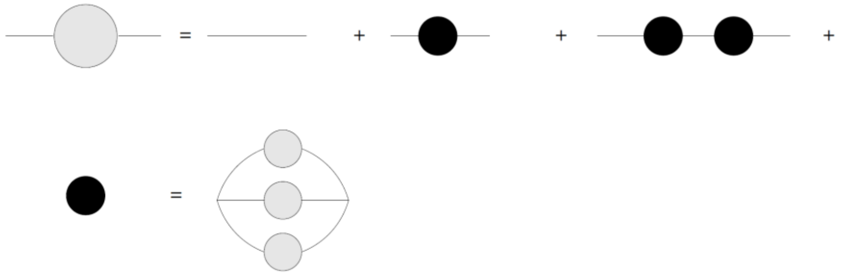
$H = i^{q/2}\sum_{i_1 \ldots i_q} J_{i_1\cdots i_q}\psi_{i_1}\cdots\psi_{i_q}$

$\psi$: N Majorana fermions in (0+1)d

$J_{i_1\cdots i_q}$: Gaussian random all to all coupling with zero mean

it's solvable in large N limit





$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \Sigma(\tau) = J^2[G(\tau)]^{q-1}$$

[Maldacena,Stanford,2016]

# SYK

- Path integral in SYK

$$Z=\int D\psi\,\exp\left(\int_0^\beta d\tau\left(-\frac{1}{2}\sum_j\psi_j\partial_\tau\psi_j+\mathrm{i}^{q/2}\sum_{i_1\ldots i_q}J_{i_1\cdots i_q}\psi_{i_1}\cdots\psi_{i_q}\right)\right)$$

Ensemble average

$$\left\langle J_{a_1\cdots a_q}J_{b_1\cdots b_q}\right\rangle=\delta_{a_1b_1}\cdots\delta_{a_qb_q}\frac{J^2(q-1)!}{N^{q-1}}$$

$$\langle Z\rangle=\int D\psi\,\exp\left(-\frac{1}{2}\int d\tau\sum_j\psi_j\partial_\tau\psi_j+\frac{NJ^2}{2q}\iint d\tau d\tau'\left(\frac{1}{N}\sum_j\psi_j(\tau)\psi_j(\tau')\right)^q\right)$$

Insert an identity

$$1=\int DG\,\delta\left(G(\tau,\tau')-\frac{1}{N}\sum_j\psi_j(\tau)\psi_j(\tau')\right)\exp\left(\frac{NJ^2}{2q}\left(G^q-\left(\frac{1}{N}\psi_i^L\psi_i^R\right)^q\right)\right)$$

$$=\iint DGD\Sigma\,\exp\left(\frac{1}{2}\Sigma(\tau,\tau')\left(G(\tau,\tau')-\frac{1}{N}\sum_j\psi_j(\tau)\psi_j(\tau')\right)\right)\exp\left(\frac{NJ^2}{2q}\left(G^q-\left(\frac{1}{N}\psi_i^L\psi_i^R\right)^q\right)\right)$$

# SYK

Integrate over $\psi$, we get an effective action

$$\frac{S}{N} = -\frac{1}{2}\log\det(\partial_\tau - \Sigma) + \frac{1}{2}\int d\tau d\tau' \left[\Sigma(\tau,\tau')G(\tau,\tau') - \frac{J^2}{q}G(\tau,\tau')^q\right]$$

Variation over $G, \Sigma$ gives the same equations from Feynman diagrams

$$\partial_\tau G(\tau) - \Sigma(\tau) * G(\tau) = \delta(\tau), \qquad \Sigma(\tau) = J^2[G(\tau)]^{q-1}$$

$$\downarrow$$

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \qquad \Sigma(\tau) = J^2[G(\tau)]^{q-1}$$

We'll use path integral method and saddle point analysis.

# Wormhole in SYK

- Consider two copies of SYK

$$Z_L Z_R = \int \mathrm{D}\psi^a \, e^{\int_0^\beta d\tau\left(-\frac{1}{2}\psi_i^a\partial_\tau\psi_i^a + \mathrm{i}^{\frac{q}{2}} J_{i_1\cdots i_q}\left(\psi_{i_1}^L\cdots\psi_{i_q}^L + \psi_{i_1}^R\cdots\psi_{i_q}^R\right)\right)}, a = L, R$$

$$\langle Z_L Z_R\rangle = \int \mathrm{D}G\mathrm{D}\Sigma \, e^{-N\left(-\log \mathrm{Pf}\,(\delta_{ab}\partial_\tau - \Sigma_{ab}) + \frac{1}{2}\int_0^\beta \mathrm{d}\tau_1 \mathrm{d}\tau_2\left(\Sigma_{ab}G_{ab} - \frac{J^2}{q}G_{ab}^q\right)\right)}, \quad a, b = L, R$$

Generally $\langle Z_L Z_R\rangle \neq \langle Z_L\rangle\langle Z_R\rangle$, but if $G_{LR} \equiv 0, \langle Z_L Z_R\rangle = \langle Z_L\rangle\langle Z_R\rangle$

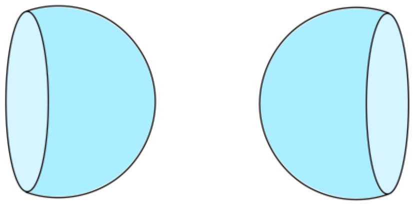
$G_{LR} \neq 0 \leftrightarrow$ wormhole

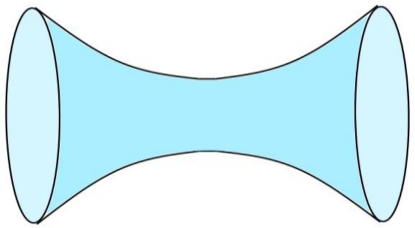
# Factorization problem

- A plausible solution:  ensemble average

  bulk is dual to an ensemble in boundary

  it’s natural   $\langle Z_L Z_R \rangle \neq \langle Z_L \rangle \langle Z_R \rangle$





- Question:  Bulk/Ensemble is different to AdS/CFT

  What happens when we consider a single element in the  ensemble

# Half-wormhole

- Progress: Half-wormhole [Saad,Shenker,Stanford,Yao,2021]

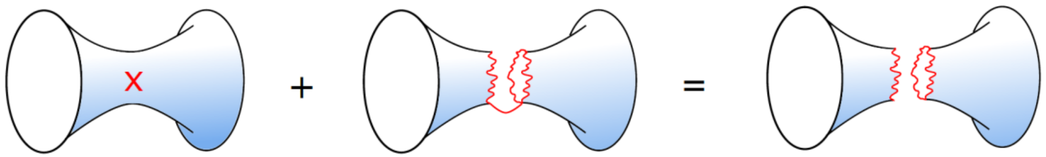
wormhole exists

factorization is restored with new saddles called half-wormholes

$z_L z_R \approx$ wormhole saddle + half-wormhole saddle

wormhole is self-averaged, half-wormhole is not

$\langle z_L z_R \rangle \approx$ wormhole saddle




[Saad,Shenker,Stanford,Yao,2021]

# 0d SYK

- To introduce the idea, we consider a simpler model

- (0+0)-dimensional SYK  or  SYK at one time point

now $\psi_i$ is a Grassmann number,    partition function is a Grassmann integral

$$z = \int \mathrm{d}^N\psi\, e^{\mathrm{i}^{q/2} J_{i_1\cdots i_q}\psi_{i_1}\cdots\psi_{i_q}} = \mathrm{PF}(J_A)$$

hyperpfaffian in [Mukhametzhanov,2021]

$$= \sum_{A_1<\cdots<A_p} \mathrm{sgn}(A) J_{A_1}\cdots J_{A_p}, \quad p = \frac{N}{q}, \quad A = \{i_1, \cdots, i_p\}$$

$$\langle z\rangle_J = 0, \qquad \langle z_L z_R\rangle_J = \int \mathrm{d}G\, \frac{\mathrm{d}\Sigma}{2\pi\mathrm{i}/N}\, e^{N\left(\log\Sigma - \Sigma G + \frac{1}{q}G^q\right)} = \frac{N!(N/q)^{N/q}}{N^N (N/q)!}$$

# Wormhole in 0d SYK

- Wormhole saddles

action $$S = \log \Sigma - \Sigma G + \frac{1}{q} G^q$$

redefine $$\Sigma = i e^{-i\phi}\sigma, \quad G = e^{i\phi} g$$

saddle point equations $$\frac{1}{\sigma} - ig = 0, \quad -i\sigma - g^{q-1} = 0 \qquad \Longrightarrow \qquad g^q = -1$$

there are q solutions, including one-loop correction the saddles contribute

$$\frac{1}{\sqrt{q}} e^{-\left(1-\frac{1}{q}\right)N} e^{2\pi i m N/q}, \quad m = 0, \dots, q-1$$

from Lefschetz thimble analysis [Witten, 1001.2933] all the saddles contribute to integral

$$\langle z_L z_R \rangle = \sqrt{q} e^{-\left(1-\frac{1}{q}\right)N}$$

it reproduces the result of the integration in large N

# Half-wormhole in 0d SYK

- the half-wormhole saddle [Saad,Shenker,Stanford,Yao,2021]

consider the unaveraged integral to find the half-wormhole saddle

$$z_L z_R = \int \mathrm{d}^{2N}\psi \exp\left(\mathrm{i}^{\frac{q}{2}} J_{i_1\cdots i_q}\left(\psi^L_{i_1\cdots i_q} + \psi^R_{i_1\cdots i_q}\right)\right)\int_R \mathrm{d}G\,\delta\left(G - \frac{1}{N}\psi_i^L\psi_i^R\right)\exp\left(\frac{N}{q}\left(G^q - \left(\frac{1}{N}\psi_i^L\psi_i^R\right)^q\right)\right)$$

$$= \int_R \mathrm{d}\sigma\,\Psi(\sigma)\Phi(\sigma)$$

$$\Psi(\sigma) = \int_R \frac{\mathrm{d}g}{2\pi/N}\exp\left(N\left(-\mathrm{i}\sigma g - \frac{1}{q}g^q\right)\right)$$

$$\Phi(\sigma) = \int \mathrm{d}^{2N}\psi \exp\left(\mathrm{i}e^{-\frac{\mathrm{i}\pi}{q}}\sigma\psi_i^L\psi_i^R + \mathrm{i}^{q/2}J_{i_1\cdots i_q}\left(\psi^L_{i_1\cdots i_q} + \psi^R_{i_1\cdots i_q}\right) - \frac{N}{q}\left(\frac{1}{N}\psi_i^L\psi_i^R\right)^q\right)$$

# Half-wormhole in 0d SYK

the coupling $J_{i_1 \cdots i_q}$ only in $\Phi(\sigma)$,

$$\langle \Phi(\sigma) \rangle = \left( i e^{-i\pi/q} \sigma \right)^N$$

$$\left\langle \Phi(\sigma)^2 \right\rangle = \left( -e^{-\frac{2i\pi}{q}} \right)^N \sum \frac{N!}{N^{2q(n_2+n_3)}} \left( \frac{N}{q} \right)^{2(n_2+n_3)} \frac{\sigma^{2qn_1} (qn_2)! (qn_3)!}{(qn_1)! (n_2!)^2 (n_3!)^2}$$

then compare $\left\langle \Phi(\sigma)^2 \right\rangle$ and $\langle \Phi(\sigma) \rangle^2$

the self-averaged region of $\Phi(\sigma)$ includes wormholes, the other is dominated by half-wormholes

$z^2 \approx (\text{wormhole saddles with } |\sigma| = 1) + (\text{half} - \text{wormhole saddle at } \sigma = 0)$

or $\quad z^2 \approx \left\langle z^2 \right\rangle + \Phi(0)$ [Saad,Shenker,Stanford,Yao,2021]

# Half-wormhole in 0d SYK

- Saddle point analysis [Peng,Tian,Yang,2022]

$$\langle \Phi(\sigma)^2 \rangle = \int \frac{d^4\sigma_{AB} d^4 g_{AB}}{(2\pi/N)^4} e^{N\left(\log\left(-e^{-2i\pi/q}\left(\sigma^2+\sigma_{14}\sigma_{23}-\sigma_{13}\sigma_{24}\right)\right)-i\sigma_{AB}g_{AB}-\frac{1}{q}g_{AB}^q\right)}$$

We can have saddle point and Lefschetz analysis with different $\sigma$

The dominant saddles are self-averaged when $\sigma \neq 0$

When $\sigma = 0$ the dominant saddles (half-wormhole) are not self-averaged

- From it we find half-wormholes always exist, but for most of time it's subleading

# Generalized half-wormhole

- previous cases: Gaussian with zero mean in 0d SYK

  our cases: other distributions in more models [Peng,Tian,Yang,2022]

- First a simple case: Gaussian with non-zero mean in 0d SYK

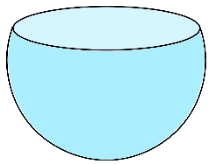
$$\langle J_A \rangle = J_A^0 = u, \ \langle J_A^2 \rangle - \langle J_A \rangle^2 = t^2$$

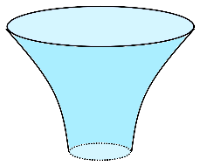
$$\langle z \rangle = \mathrm{PF}\left(J_A^0\right), \qquad \langle z^2 \rangle = \int \mathrm{d}^{2N}\psi \exp\left(\mathrm{i}^q t^2 \psi_A^L \psi_A^R + \mathrm{i}^{q/2} J_A^0 \left(\psi_A^L + \psi_A^R\right)\right)$$

$$= \sum_{A,B} \mathrm{sgn}(A)\mathrm{sgn}(B)\left(J_{A_1}^0 J_{B_1}^0 + \delta_{A_1 B_1} t^2\right) \cdots \left(J_{A_p}^0 J_{B_p}^0 + \delta_{A_p B_p} t^2\right)$$

- Half-wormholes in $z$

  following the previous way (may not be right with non-zero $u$)





$$z \approx \langle z \rangle + \Theta_1$$

$$\Theta_1 = \Phi(0) = \mathrm{PF}\left(J_A - J_A^0\right) = \sum_A \mathrm{sgn}(A)\left(J_{A_1} - J_{A_1}^0\right) \cdots \left(J_{A_p} - J_{A_p}^0\right)$$

# Generalized half-wormhole

- But the approximation should be corrected

$$z = \mathrm{PF}(J_A) = \mathrm{PF}(u + J_A - J_A^0)$$
$$= \sum_A \mathrm{sgn}(A)\left(u + J_{A_1} - J_{A_1}^0\right)\cdots\left(u + J_{A_p} - J_{A_p}^0\right) = \sum_{n=0}^{p} \Theta^{(n)}$$
$$\Theta^{(p-1)} = \sum_A \mathrm{sgn}(A)\left(J_{A_1} - J_{A_1}^0\right)\cdots J_{A_i}^0 \cdots\left(J_{A_p} - J_{A_p}^0\right), \quad \Theta^{(0)} = \langle z\rangle, \quad \Theta^{(p)} = \Theta_1$$

from $\langle z^2\rangle$ we can find which $n$'s are dominant, it depends on $u, t$

if $n = 0, k$ dominate $\quad z \approx \langle z\rangle + \Theta^{(k)}$

- Half-wormholes in $z^2$

Similarly we have

$$z^2 = \int \mathrm{d}\sigma_w \mathrm{d}\sigma_L \mathrm{d}\sigma_R \Psi(\sigma_w, \sigma_L, \sigma_R)\widehat{\Lambda}(\sigma_w, \sigma_L, \sigma_R)$$
$$= \sum_n \Lambda^{(n)}, \quad \Lambda^{(0)} = \langle z^2\rangle, \quad \Lambda^{(p)} = \Lambda_{u=0}$$

if $n = 0, k$ dominate $\quad z^2 \approx \langle z^2\rangle + \Lambda^{(k)}$

# Generalized half-wormhole

- Brownian SYK [Saad,Shenker,Stanford,2018]

The only difference to SYK is

$$\left\langle J_{a_1\cdots a_q}(t_a) J_{b_1\cdots b_q}(t_b)\right\rangle = \delta_{a_1 b_1}\cdots\delta_{a_q b_q}\delta(t_a - t_b)\frac{J(q-1)!}{N^{q-1}}$$

which leads to a local effective action

we consider the time evolution operator

$$U(T) = \mathrm{T}\, e^{-\mathrm{i}\int_0^T \mathrm{d}t H(t)}$$

in $G,\Sigma$ formulation

$$\langle |z|^2\rangle \equiv \langle |\mathrm{Tr}U(T)|^2\rangle = \int \frac{\mathrm{D}G_{LR}\mathrm{D}\Sigma_{LR}}{2\pi\mathrm{i}/(TN)}\, e^{N\left(\log\left(2\cos\frac{T\Sigma_{LR}}{4}\right)-\frac{JT}{q2^q}+\mathrm{i}^q\frac{JT}{q}G_{LR}^q-\frac{T}{2}\Sigma_{LR}G_{LR}\right)}$$

# Generalized half-wormhole

- $|z|^2$ at fixed coupling in Brownian SYK

  similar to

$$|\mathrm{Tr}U(T)|^2=\int \mathrm{D}\psi_a^L\,\mathrm{D}\psi_a^R\; e^{\mathrm{i}\int_0^T \mathrm{d}t\left(\frac{\mathrm{i}}{2}\psi_a^j\partial_t\psi_a^j-J_{a_1\cdots a_q}(t)\left(\mathrm{i}^{q/2}\psi_{a_1\cdots a_q}^L-(-\mathrm{i})^{q/2}\psi_{a_1\cdots a_q}^R\right)\right)}$$

we introduce

$$1=\int_R \mathrm{D}\,G_{LR}\;\int_{\mathrm{i}R}\frac{\mathrm{D}\Sigma_{LR}}{4\mathrm{i}\pi/N}\;e^{-\int \mathrm{d}t\mathrm{d}t'\frac{\Sigma_{LR}}{2}\left(NG_{LR}(t,t')-\psi_a^L(t)\psi_a^R(t')\right)}e^{f_{LR}(NG_{LR})-f_{LR}(\psi_a^L\psi_a^R)}$$

then we have

$$|\mathrm{Tr}U(T)|^2=\int_R\frac{\mathrm{D}\Sigma_{LR}}{2\pi/N}\Phi(\Sigma_{LR})\Psi(\Sigma_{LR})$$

$$\Psi(\Sigma_{LR})=\int_R \mathrm{D}G_{LR}\,e^{-\int \mathrm{d}t\mathrm{d}t' N\frac{\Sigma_{LR}(t,t')}{2}G_{LR}(t,t')}e^{f_{LR}(NG_{LR})},\qquad f_{LR}(NG_{LR})=\frac{2J}{N^{q-1}q}(-1)^{\frac{q}{2}}(NG_{LR})^q$$

$$\Phi(\Sigma_{LR})=\int \mathrm{D}\psi_a^L\mathrm{D}\psi_a^R e^{\frac{1}{2}\int \mathrm{d}t\mathrm{d}t'\Sigma_{LR}(t,t')\,\psi_a^L(t)\psi_a^R(t')-f_{LR}(\psi_a^L\psi_a^R)}$$
$$\times\, e^{\mathrm{i}\int_0^T \mathrm{d}t\left(\frac{\mathrm{i}}{2}\psi_a^j\partial_t\psi_a^j-J_{a_1\cdots a_q}(t)\left(\mathrm{i}^{q/2}\psi_{a_1\cdots a_q}^L-(-\mathrm{i})^{q/2}\psi_{a_1\cdots a_q}^R\right)\right)}$$

# Generalized half-wormhole

- $|z|^2$ at fixed coupling in Brownian SYK

we compare $\langle \Phi(\sigma)^2 \rangle$ and $\langle \Phi(\sigma) \rangle^2$ to find the self-averaged region and the wormhole and half-wormhole exist

our conclusion is

$|z|^2 \approx \langle |z|^2 \rangle + \Phi(0)$

the approximation is good in large $T$ limit

we also have the saddle point analysis

# Decomposition

- We also propose a new way to detect half-wormholes [Tian,Yang,2022]

$$z = \sum_{n=0}^{p} \Theta^{(n)}$$

which $n$'s are dominant, it depends on $u, t$

but it may not only contain two of them

- we decompose the observable into a sum of many sectors

$$Z^n = \sum_i c_i \theta^{(i)}$$

and find some of the sectors dominant, related to wormholes and half-wormholes

# Decomposition

- In Gaussian ensemble

$$Z=\theta^{(1)}+\langle Z\rangle,\qquad Z^2=\theta^{(2)}+2\langle Z\rangle\theta^{(1)}+\langle Z^2\rangle,\qquad \dots$$

$$\langle\theta^{(i)}\theta^{(j)}\rangle_Z=i!\,t^{2i}\delta_{ij},\qquad \langle Z^nZ^n\rangle_Z=\sum_i c_i^2 i!\,t^{2i}$$

it is similar to the Sturm-Liouville theory

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=-\lambda\omega(x)y$$

$$\int y_n(x)y_m(x)\omega(x)dx=\delta_{mn}$$

$$\omega(x)\leftrightarrow \mathrm{f(Z)},\qquad y(x)\leftrightarrow\theta^{(i)}$$

but only simple in Gaussian case

# Decomposition

in 0d SYK model

$$z = \int d^N\psi e^{i^{q/2}\sum_A J_A\psi_A} = \int d^N\psi e^{i^{q/2}\sum_A u\psi_A}\, e^{i^{q/2}\sum_A \theta_A^{(1)}\psi_A}$$

$$z \equiv \sum_k \Theta_k = \int d^N\psi \sum_{k=0} \frac{\left(i^{q/2}\sum_A \theta_A^{(1)}\psi_A\right)^k}{k!} e^{i^{q/2}u\sum_A \psi_A}$$

by matching the power of $\theta_i^{(1)}$, we can find out $\Theta_k$

$$\Theta_k = \int d^N\psi \frac{\left(i^{q/2}\sum_A \theta_A^{(1)}\psi_A\right)^k}{k!} e^{i^{q/2}u\sum_A \psi_A}$$

$$\Theta_k = u^{p-k}\sum'_{I_1<\cdots<I_{p-k}} Pf\left(\theta_A^{(1)(I_1,\ldots,I_{p-k})}\right)$$

# Decomposition

- By computing $\langle z^2 \rangle$ we can find the dominant $\Theta_i$

$$\langle z^2 \rangle = \sum_i z_2^{(i)} = \sum_i \langle \Theta_i \Theta_i \rangle$$

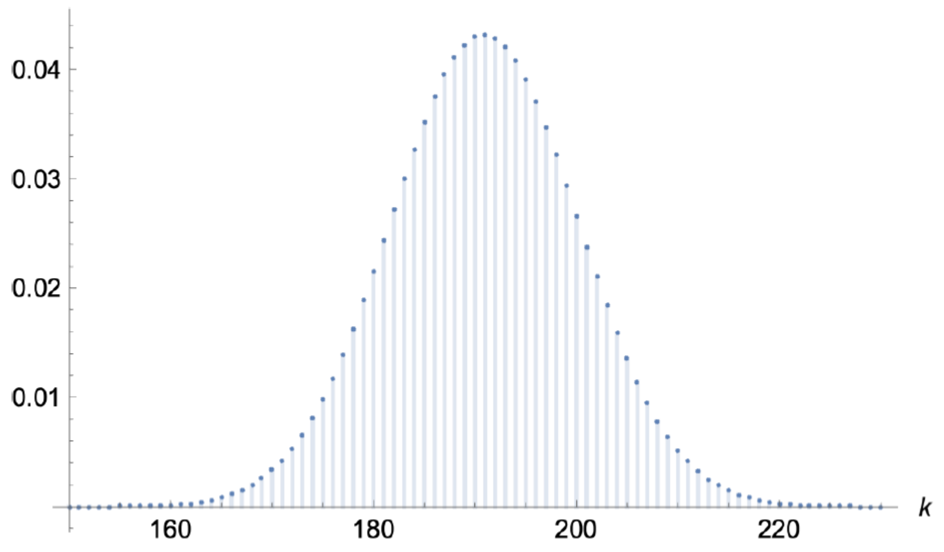
- An example with $q = 2$

the figure shows the ratio $\mathrm{s_k} = z_2^{(k)} / z_2^{(0)}$

y axis labels $\frac{s_k}{\sum_{i=0}^{p} s_i}$

$N = 1000, q = 2$

the peak implies a half-wormhole saddle



# Decomposition

- Beyond Gaussian ensemble

$$\langle Z\rangle=\kappa_1,\langle Z^2\rangle-\langle Z\rangle^2=\kappa_2,\langle Z^3\rangle-3\langle Z\rangle\langle Z^2\rangle+2\langle Z\rangle^2=\kappa_3$$

we can still have the decomposition, but

$$\langle\theta^{(i)}\theta^{(j)}\rangle\neq\delta_{ij}$$

$$M_3\equiv\langle\theta^{(i)}\theta^{(j)}\rangle=\begin{pmatrix}\kappa_2 & \kappa_3 & 0\\ \kappa_3 & 2\kappa_2^2 & 6\kappa_2\kappa_3\\ 0 & 6\kappa_2\kappa_3 & 6\kappa_2^3+9\kappa_3^2\end{pmatrix}$$

- We can have similar computation in 0d SYK model and so on.

# Thank you !