Polytopes and Scattering Amplitudes

scalar field Amplitudes from Positive geometries

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Based on earlier work with Aneesh, Pinaki, Mrunmay, Renjan and Sujoy

Recent work with Mrunmay Jagadale

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Motivation

- Scattering Amplitudes are the fundamental objects in Quantum field theory.
- But a complete understanding of Scattering amplitude escapes us.
- In perturbation theory, one definition of Scattering Amplitude is via Feynman diagrams. (i.e. a $n \rightarrow m$ particle scattering can be computed via summing over all the Feynman diagrams which contribute upto the desired order.)
- However Feynman diagrams are known to obscure the simplicity and in many cases "symmetries" of scattering Amplitudes.
- The most well known examples of this deception is when we compute Maximal Helicity violating Amplitudes in pure Yang-Mills in four dimensions.
- The answer is a compact rational expression in terms of spinor-helicity variables for arbitrary number of gluons. But from diagrammatics, we would need 50,000 odd diagrams even to compute amplitude for 9 gluon scattering. So clearly something is amiss.

Motivation contd.

- Leaving aside the question of efficiency, how much do the Feynman techniques tell us about the precise definition of an S-matrix?
- To answer this question, we need to look at what are the **core** postulates which any definition of S-matrix has to satisfy.

The Postulates from 60s : Analytic S-matrix program

Core postulates

- Unitarity
- Poincare invariance
- Crossing (amplitude in one channel can be obtained by analytic continuation of amplitude in another channel)
- **Maximal Analyticity**

The postulates from 60s

Key postulate : Maximal analyticity

- In essence : In the full complex hyper-plane spanned by Mandelstam invariants, the only singularities of an S-matrix are those which are consistent with unitarity and causality (locality).

- But the singularities of a *perturbative* S-matrix are hard to classify even in perturbation theory

Analytic structure of the S-matrix

- Consider a 1-loop scattering Amplitude $\mathcal{M}_4(s,t)$ in ϕ^4 theory. Where are all the singularities in the (complex) s plane at fixed t ?
- From Unitariy considerations, we know : some poles between $s = 0$ and $s = 4m^2$ and branch cut starting at $s = 4m^2$. But this is not all !
- In the complex s plane, there is already a singularity at $s = 0$ which is not "physical" (known as Pseudo threshold) $\mathcal{M}_4 \sim \sqrt{\frac{s-4m^2}{s}}$.
- Hence the full structure of singularities even in perturbation theory of scalar field amplitudes is complicated. ("Study of complex singularities in scattering amplitudes is complex"-Itzykson, Zuber)
- Thus we do not even have a satisfactory answer to the following question : Does the Feynman diagram representation provide us with a perturbative definition of the S-matrix.

S matrix and it's representations

- Feynman diagrams are a representation for perturbative scattering amplitudes, but perhaps not always the most suitable representation (yang-Mills amplitude)
- They obscure the simplicity of many amplitudes because they do not respect the symmetry inherent in the scattering amplitudes (Yangian invariance for MHV amplitudes, Color kinematics duality in large class of theories with color)
- They rely on auxiliary structures like manifest locality of fields and gauge degrees of freedom which often complicate the computations rather then simplify them. (Yang-Mills, Gravity, QCD, SYM, ...)
- Deeper structures relating one theory to another is completely obscured. (Double copy relations between gauge theories and gravity)
- The representation does not lead us to a structural definition of a perturbative scattering Amplitude.

So finally

- We do not know what a "perturbative S-matrix" structurally is, but we have many representations.
- In AdS, a convinient representation is via CFT correlation functions.
- Properties manifest in one representation become corollaries in other representation.
- **Q : Is there a framework of scattering amplitudes in which all these disparate revelations can be unified?**
- And if there is, what are the fundamental postulate on which it is based.

- Positive Geometries/Amplituhedron program : **A new perspective on amplitudes which encompasses different representations and radically changes our understanding of the S-matrix**

Main ideas : Tree-level amplitude

- We start with a very bare-bone structure. For example, Space of Mandelstam invariants $\mathcal{K}_n$ and want to build a theory of S-matrix from scratch.
- In a remarkable paper, Arkani-Hamed, Bai, He and Yun showed that for a massless colored ϕ^3 theory in any space-time dimension, there are convex polytopes (known as Associahedron) in $\mathcal{K}_n^{\geq 0}$ whose boundaries correspond to all singularities of the tree-level scattering amplitude.
- Associated to each of these polytopes is a *unique* differential form, and this form *is* the (tree-level) scattering Amplitude.

Key ideas

- Associahedron has many fascinating properties. One of them is that if it is k dimensional, then it has boundaries of all co-dimensions. Such objects are now called Positive geometries.
- Perhaps the most relevant property from scattering amplitude perspective is : Any boundary of associahedron is a product of lower dimensional associahedra and nothing else.
- **example** : One dimensional associahedron is a line with two vertices. Two dimensional associahedron is a Pentagon. So a three dimensional associahedron is either a pentagon or a square but can not be any other polygon!
- Unitarity and Locality are consequences of the above properties.

Our main results in this talk

- We show that for all ϕ^p interactions, a natural perspective is provided by considering a class of polytopes known as Accordiohedra.
- These polytopes are extremely simple to construct : via dissections of Polygons. (Pilaud, Maneville, Padrol, Palu,Plamondon and many others)

Disclaimer

- We will work with color ordered (planar) Amplitudes. So we implicitly assume that all our scalars have color degrees of freedom in the adjoint (or bi-adjoint) representation. Although we will never display the color indices explicitly.
- This is because these ideas are perhaps best developed for planar Amplitudes. (**Here planar does not mean large N !**)
- Although we will focus on tree-level scattering amplitudes, the polytope perspective is now well understood even for 1-loop integrands. (For ϕ^3 theory, there are upcoming results by Arkani-Hamed et al which extend these ideas to all loops.)

Kinematic space

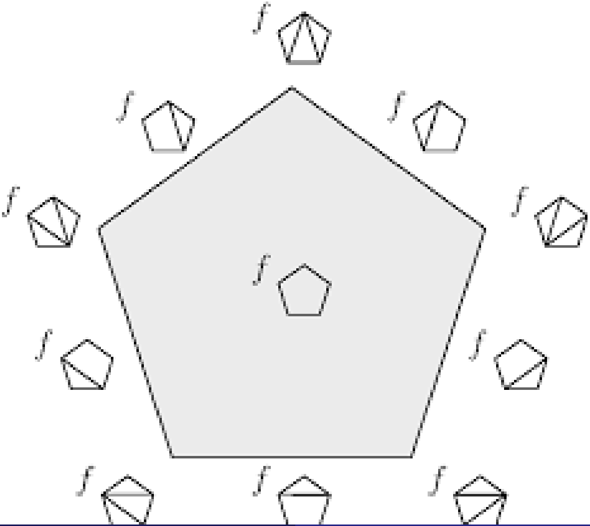
- We start with the Kinematic space, spanned by Mandelstam invariants for planar tree-level Amplitudes. defined by *planar* variables $X_{ij},\ i < j$.

$$X_{ij} = (p_i + \ldots + p_n + \ldots + p_{j-1})^2$$

- Consider Polygon formed by n vertices where the vertices are labelled in clock-wise direction.
- Consider any set of X_{ij}s which dissect the polygon into cells like triangles, quadrilaterals or any other n-gon.
- **The combinatorics of these dissections can be used to contstruct (abstract) polytopes (higher dimensional generalisation of a polygon). We give two examples.**
- triangulation of a pentagon ($n = 5$ case) and quadrangulation of an octagon ($n = 8$) case

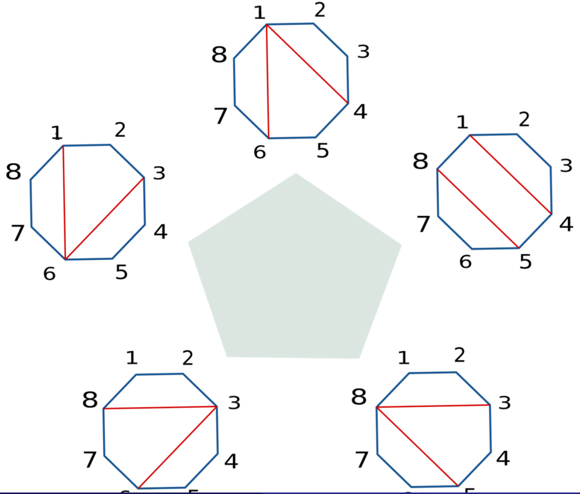
An example of Associahedron with triangulation

- If the original dissection of the polygon is into triangles, then the resulting combinatorial polytope is called Associahedron.



An example of Accordiohedra with quadrangulation

- If the dissection is into quadrilaterals, then the combinatorial structure is called Accordiohedron $\mathcal{AC}_Q$: Let $Q = \{(14, 16)$.



Accordiohedra Polytopes

Some Properties

- For **any dissection** Q, we get one (combinatorial) polytope.
- If Q is a triangulation, the corresponding polytope is unique and is independent of the choice of Q. (That is, you can start with any triangulation to build the polytope.) This polytope is the Associahedron.
- If Q is anything else then a triangulation, then the polytope retains the memory of Q. That is, for different quadrangulation that we start with, we get distinct polytopes.
- A happy fact : dimension of the polytope built out of a p-gulation is same as number of propogators in an n particle amplitude with ϕ^p interactions. Associahedron : $n-3$

From Combinatorial Polytope to Scattering amplitudes

- Assign a sign to each vertex of the polytope $\mathcal{AC}_Q$, starting with any given vertex such that two adjacent vertices have opposite signs.
- Define a $\dim(\mathcal{AC}_Q)$ form on the planar kinematic space as,

$$\Omega_Q = \sum_{I=1}^{|\text{vertices}|} (-1)^v \wedge d \ln X_{ij} \tag{1}$$

- **A new postulate for S-matrix?** : For any Q, the scattering form is **projective** : Invariant under $X_{ij} \to f(\{X\})\, X_{ij}$. (This is a simple analog of dual conformal invariance in $\mathcal{N} = \triangle$ SYM amplitude.)
- If Q is a triangulation, there is a unique (projective) scattering form, as associahedron does not retain memory of Q. (Arkani-Hamed et al)

From Accordiahedra to amplitudes : An example

- Consider quadrangulation of a hexagon (6 pt. Scattering with ϕ^4 interactions.)
- Choose a quadrangulation Q out of (14), 25 or 36

$$\begin{aligned} \Omega_{Q=(14)} &= d\ln X_{14} - d\ln X_{36} \\ \Omega_{Q=(25)} &= d\ln X_{25} - d\ln X_{14} \\ \Omega_{Q=(36)} &= d\ln X_{36} - d\ln X_{25} \end{aligned} \tag{2}$$

- So we have three distinct projective forms , one for each Q
- But this form seems to have nothing to do with amplitude. There are relative minus signs between various channels which are absent in scalar field amplitudes.
- Related question : $\mathcal{AC}_Q$ is an abstract combinatorial polytope. What does it have to do with space of Mandelstam invariants?

From Combinatorial to convex Polytope

- These combinatorial polytopes are in fact sitting in the positive region of Kinematic space $\mathcal{K}_n^+$
- And all we have to do is locate them (ABHY, Padrol, Palu, Pilaud, Plamondon)
- That is, we need equations to locate them in the kinematic space.
- **remarkable algebraic structures known as Cluster algebra/gentle algebra precisely locate these polytopes in kinematic space such that the boundaries of the polytope coincide with singularity locations of the amplitude.**
- To write the equations, consider standard Mandelstam invariants s_{ij}

$$s_{ij} = (p_i + p_j)^2 = X_{ij} + X_{i+1,j+1} - X_{ij+1} - X_{i+1j} \quad (3)$$

- If T is a triangulation and if T^c is a triangulation obtained by rotating T counter-clockwise, then the constraints are simply

$$s_{ij} = -c_{ij} \, \forall \, (ij) \notin T^c, \; |i-j| \geq 2 \tag{4}$$

- Intuitively, these constraints freeze all the Mandelstam variables which will never occur as poles in a ϕ^3 amplitude. (ABHY)
- If Q is a quandrangulation (or any other dissection), choose any T that contains Q. $\mathcal{AC}_Q$ is realised in the (positive) region of Kinematic space simply by adjoining to 4, a following set of constraints

$$X_{ij} = d_{ij} \, \forall \, (ij) \, \mathrm{in} \, T/Q. \tag{5}$$

- Intuitively, the extra constraints freeze all the propagators in the cubic channels which will not occur ϕ^4 channels.

Some nice properties

- If Q is a quadrangulation, then the convex realisation is a polytope which (modulo some technical conditions) can always be embedded inside an associahdron $\mathcal{A}_T$ as long as $Q \subset T$. (Aneesh et al)
- **The Last property shows why Associahedra are the "master polytopes" in kinematic space.**

- We now have projective forms in $\mathcal{K}_n$ and convex polytopes in $\mathcal{K}_n^+$. This is all we need to construct amplitudes.

From Scattering forms to Scattering Amplitudes

- Consider a specific example of 6 point scattering with ϕ^4 interactions.
- There are three possible quadrangulations of a hexagon labelled by $Q = \{(14)\,(25)\,(36)\}$.
- For each choice of Q, we have a one dimensional polytope (a line with two end points) sitting inside the Kinematic space.

$$\begin{aligned} C(14) &: X_{14} + X_{36} = c_1 \\ C(25) &: X_{25} + X_{14} = c_2 \\ C(36) &: X_{36} + X_{25} = c_3 \end{aligned}$$

(All c_is are positive)

From Scattering forms to Amplitudes

- Recall the projective scattering form

$$\Omega^6_{(14)} \;=\; d\ln X_{14} - d\ln X_{36}$$

$$\Omega_{(14)}|_{\mathcal{AC}_{14}} \;=\; \left(\frac{1}{X_{14}} + \frac{1}{X_{36}}\right)dX_{14} \;=:\; m_{(14)}\,dX_{14} \tag{6}$$

- We then immediately see that a weighted sum over. three forms produce the complete tree-level amplitude which has no non-planar channels.

$$\sum_Q \alpha_Q\, m_Q \;=\; \mathcal{M}_6$$

where $\alpha_Q \;=\; \frac{1}{2}\,\forall\, Q$.

General Result for n point Amplitude

- So we showed that canonical form on $\mathcal{AC}_Q$ determine tree-level amplitude of ϕ^4 theory if Q is quadrangulation of an n-gon.
- In fact, more is true. And this is a striking result in many ways.
- Take the form Ω_Q^n and project it onto any Associahedron $\mathcal{A}_T$ such that $Q \subset T$.

$$\Omega_Q|_{\mathcal{A}_T} = m_Q^n \wedge_{(ij)\in Q} dX_{ij}$$

- We thus see that a weighted sum over all such lower forms on Associahedra produce the complete Amplitude.

$$\mathcal{M}_n = \sum_Q \alpha_Q \, m_Q^n$$

n-point Amplitudes

- We thus see that the complete Amplitude is not determined by a single Polytope but by summing over all the polytopes of a given dimensions.
- But the sum depends on pesky weight factors.
- The Weights are determined from the combinatorics of the Polytope and use a key property of all such Accordiohedras, namely Facet of any Accordiohedra is a product of lower dimensional Accordiohedra of the same type. (Prashanth Raman, Ryota Kojima)

Main results

- For any ϕ^p interaction, there is a set of projective planar scattering forms in the Kinematic space.
- for $p = 3$, this form is unique but for all $p > 3$ there is a whole set of them parametrized by dissections Q.
- These forms when restricted to the Associahedra produce lower forms which have singularities associated to poles in the Feynman diagrams with ϕ^p vertices.
- **Locality and Unitarity emerge due to the remarkable property of all Accordiohedra that all their faces belong to the same family of polytopes.**
- **The entire picture holds up for 1-loop integrands of scattering amplitudes.**

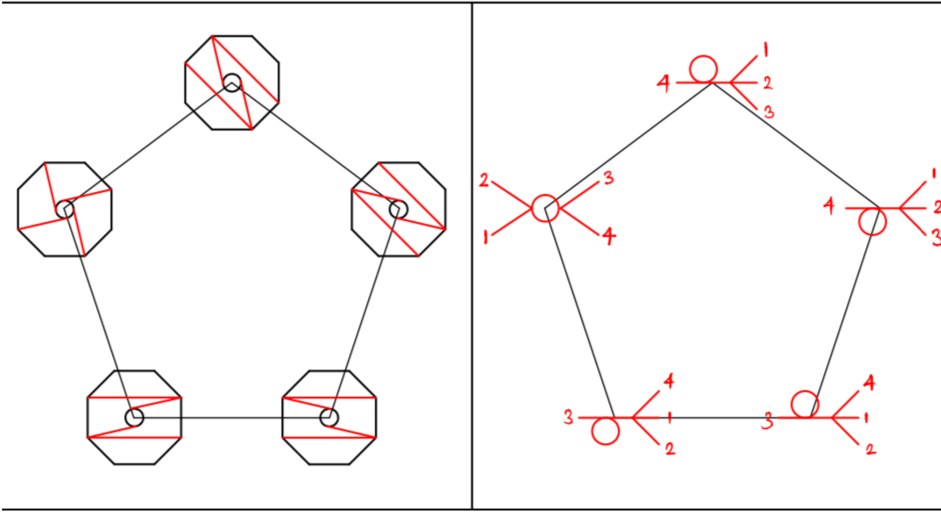
Towards 1-loop integrand for quartic interactions

Builds on the work of Arkani-Hamed, He, Salvatori and Thomas for cubic interactions.

Pseudo-accordiohedra and projective forms

- Dissections of an n-gon $\rightarrow$ Dissections of an n-gon with a hole at the center.
- All such dissections can be used to form combinatorial polytopes that we call (pseudo)-accordiohedron. $\mathcal{AC}_Q \rightarrow \mathcal{PAC}_Q$.
- Kinematic space in 1-loop case is, $\mathcal{K}_n \rightarrow \mathcal{K}_n \cup \text{loop-variables}$.
- Fundamental principle : Projectivity of the form, fixes the form uniquely.
- There is a canonical realisation of $\mathcal{PAC}_Q$ in the kinematic space such that the boundaries of the polytope coincide with singularities of the integrand.
- Top forms on the (realised) polytopes= integrand for 1-loop amplitude.

An example



A new class of polytopes?

- For "triangulations" of a holed n-gon, the polytope is well studied in mathematics and amplitudes literature and is known as a cluster polytope. (Arkani-Hamed, He, Thomas and Salvatori)
- For generic dissections, $\mathcal{PAC}_Q$ has not been analysed in tge literature and may lead us to interesting new algebraic structures.

Polytopes and Cluster Algebras

Associahedra : **type-A Cluster Algebraof triangulation Quiver**
Accordiohedra : **Gentle algebras of dissection quivers**

Type-D Cluster Polytope : **type-D Cluster algebra**

Polytopes associated to Pseudo-quadrangulations.
Are they realisations of any Algebra?

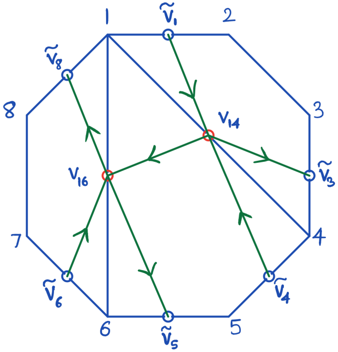
Our proposal : Colored Gentle algebras

Conclusions

- Although we focused on quartic interactions at 1-loop, we believe that everything should generalise to other monomial (and perhaps even Polynomial) interactions.
- Hence in a nut-shell, the principle of projectivity (dual conformal invariance) and existence of positive geometries in momentum space lead us to a novel restructuring of S-matrix program, **where unitarity and locality are consequences and not fundamental postulates**!
- It will be interesting to understand the algebraic origins of the new polytopes we have found in the context of 1-loop quartic amplitudes.
- The tree level lower forms on Associahedra have interesting connection with lower forms on CHY Moduli space. (Can this connection be refined using Binary geometries for Accordiohedra? He, Li, Raman, and Zhang.)

Thank you all for listening !

Combinatorial polytopes



- Consider any path along the quiver that (1) begins and ends on the edges of the polygon, (2) that does not cross adjacent edges in any cell and (3) that does not begin and end on adjacent edges of the polygon. ($\tilde{v}_1\tilde{v}_6$, $\tilde{v}_3\tilde{v}_8$ are allowed paths. $\tilde{v}_6\tilde{v}_8$ is not an allowed path.)

Polytopes from Dissection Quiver

- For each allowed path $v_{\tilde{i}} v_{\tilde{j}}$ draw a linear chord $\mathcal{D}_{i_0 j_0}$.
- These chords dissect a polygon made from hollow vertices.
- So we now forget the original polygon and only focus on the "hollow polygon" and these chords which dissect it.
- **We can use the collection of all such chords to form a polytope.**

- Each vertex of the polytope is labelled by the maximal set of chords that do not intersect except in end points.
- Two vertices in this poytope are adjacent if they differ in one and only one chord. We call such an operation a flip.
- The edge connecting such vertices is labelled by all the chords common to both of them.

Cluster Polytopes for ϕ^3 integrand.

- In a recent work, Arkani-Hamed, He, Salvatori and Thomas extended the "Associahedron program" to include 1-loop integrands for cubic interactions.
- The striking aspect of their work is not only a discovery of polytopes in a Kinematic space whose canonical form determines ϕ^3 integrand, but the fact that these polytopes are geometric realisations of type-D cluster algebras just as ABHY Associahedra are geomtric realisations of type-A Cluster algebras.

Questions

- Is there a generalisation of Accordiohedra which determine ϕ^4 1-loop integrands?
- Can one also understand these integrands as lower forms on type-D Cluster polytopes?

Dissection Quivers for type-D Polytopes

- Recall : Combinatorially, Associahedron is determined by triangulation of a polygon.
- It turns out that One “dissection model" that determines the (combinatorial) type-D polytope is “. centrally symmetric Pseudo-triangulations" of a polygon with Annulus. (Ceballos and Pilaud)
- A Pseudo-triangle is any (curved) polygon which has precisely three convex coerners

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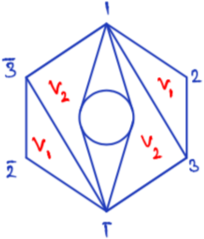


Kinematic space for 1-loop integrands

- All the dissections of a centrally symmetric Pseudo-triangulation form the kinematic space for 1-loop amplitude.
- For $n = 3$, Kinematic Space $= \{X_{13}, X_{2\bar{1}}, X_{3\bar{2}}, Y_1, \tilde{Y}_1, \dots\}$.
- Y_1 is an arc which emerges from 1 and touches the Annulus on the left and $\tilde{Y}_1$ is the one that touches on the right.

From X_{ij} to s_{ij}

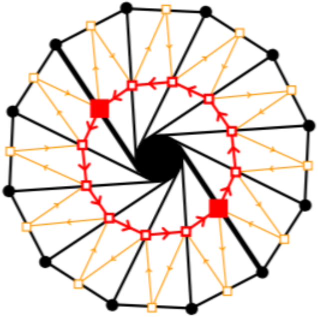
- For any two vertices i, j on the (doubled) polygon, we can define s_{ij} in terms of X_{ij} as in the case of Planar Kinematics space.

$$s_{i\overline{i+1}} = X_{i\overline{i+1}} + X_{i+1\overline{i+2}} - X_{i\overline{i+2}} - Y_i - \tilde{Y}_{i+1} \tag{7}$$

From Pseudo-triangulation to Convex Polytope

Various Realisations of the type-D polytope

- Consider a reference Pseudo triangulation and the corresponding dissection Quiver drawn below.



Convex realisations of type-D polytope from the Quiver

- We can again consider an algebra generated by the paths of the quiver which begin and end on an external edge.
- Precisely mirroring the Associahedron scenario, this algebra generates certain linear constraints.

$$s_{ij} = -c_{ij} \; \forall \; (ij) \; \notin \; PT^c$$

- **One caveat** : A single quiver leads to too many constraints ! But there is a canonical split into a optimum set of constraints (namely, $n^2 - n$ in number).
- Solving these constraints produce *various* realisations of the type-D polytope one of which is the realisation obtained by Arkani-Hamed, He, Salvatori and Thomas

A Polytope for 1-loop ϕ^4 Amplitude

- We can now consider a centrally symmetric Pseudo-Quandrangulations.
- Proceeding exactly as in the case of Accordiohedra, we append to the constraints that produce convex realisation of type-D polytope,

$$X_{ij} = d_{ij} \, \forall \, (ij) \, \in \, \mathrm{a \; PT}$$

- These constraints can be consistently solved and we obtain a convex polytope of dimension $\frac{n}{2}$.
- Vertices of this convex polytope are in 1-1 correspondence with a set of Feynman diagrams of ϕ^4 amplitude at 1-loop level.
- The polytope retains memory of reference Pseudo-Quadrangulation.