

# Fluid-gravity correspondence: a new cut-off surface approach

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# PLAN OF THE TALK

- Introduction and earlier approaches.
- Further motivations.
- New approach: through parallel transport.
- Possible aspects of present approach.
- Final remarks.

# Introduction and earlier approaches

Two of the most famous non-linear PDEs in nature.

- The Navier-Stokes (NS) equation:

$$\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] + \vec{\nabla} P - \eta \nabla^2 \vec{v} - \left( \zeta + \frac{1}{3} \eta \right) \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) = 0.$$

- Einstein's field equation:

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab}$$

Issue of finding globally regular solutions remains an open challenge.

IS THERE ANY DUALITY OR CORRESPONDENCE BETWEEN  
THEM?

(ANY CORRESPONDENCE OR DUALITY BETWEEN THEM IS  
BOUND TO BE INTERESTING AND NON TRIVIAL.)

THE FLUID-GRAVITY CORRESPONDENCE

# Hints of connection

- Thesis of Damour (1979) - Suggesting a relation between horizon dynamics and fluid dynamics (DNS equation).

(NULL SURFACE)

- Membrane paradigm approach by Price and Thorne (1986).

(STRETCHED HORIZON)

- AdS/CFT.

Bhattacharyya, Hubeny, Minwalla, Rangamani: JHEP  
02 (2008) 045

Emergent paradigm of gravity. – Paddy et. al.

## TIMELIKE CUT-OFF SURFACE APPROACH

Bredberg et al, JHEP 1207 (2012) 146

Notations:  $a, b, c \dots$  – Spacetime indices.

$\mu, \nu, \alpha \dots$  – hypersurface indices.

$A, B, C, \dots$  – Transverse indices.

# Cut-off surface approach

- **Scale invariance of NS:**

Scaling of the fluid parameters

$$v_A^{(\epsilon)} = \epsilon v_A(\epsilon^2 \tau, \epsilon x_A); \quad P^{(\epsilon)} = \epsilon^2 P(\epsilon^2 \tau, \epsilon x_A);$$

The incompressible NS equation ( $\partial_A v^A = 0$ )

$$\partial_\tau v^A + v^C \partial_C v^A - \partial^A P - \eta \partial^2 v^A = 0$$

is invariant.

(De, Majhi: JHEP 1901 (2019) 044)

THE INCOMPRESSIBLE NS EQUATION GOVERN THE  
HYDRODYNAMICS OF FLUID WITH A CLASS OF SOLUTIONS  
( $v_A^{(\epsilon)}(\tau, x_A), P^{(\epsilon)}(\tau, x_A)$ ) PARAMETERIZED BY  $\epsilon$ .

- **ADS/CFT approach: the backbone.**

Thermal state  $\leftrightarrow$  AdS spacetime

Relativistic hydrodynamics + corrections  $\leftrightarrow$  Relativistic gradient expansion bulk solution

- **Underlying idea:**

Using the hydrodynamic scaling, find a  $(p + 2)$ -dim. bulk solution of vacuum Einstein's equations corresponding to thermal fluid living in  $(p + 1)$ -dim.

with the following information:

Thermal fluid state  $\leftrightarrow$  Rindler spacetime

Incompressible NS with corrections  $\leftrightarrow$  Non-relativistic hydrodynamic expansion of bulk solution

# The solution

- Hydrodynamic scaling as order of parameters:  $v_A \sim \mathcal{O}(\epsilon)$ ;  $P \sim \mathcal{O}(\epsilon^2)$ ;  $\partial_A \sim \mathcal{O}(\epsilon)$ ;  $\partial_\tau \sim \mathcal{O}(\epsilon^2)$ .

- $$ds^2 = -r d\tau^2 + 2d\tau dr + dx_A dx^A$$
$$- 2(1 - r/r_c) v_A dx^A d\tau - \frac{2v_A}{r_c} dx^A dr$$
$$+ (1 - r/r_c) \left[ (v^2 + 2P) d\tau^2 + \frac{v_A v_B}{r_c} dx^A dx^B \right] + \left( \frac{v^2 + 2P}{r_c} \right) d\tau dr$$
$$+ \mathcal{O}(\epsilon^3).$$

$R_{ab} \sim \mathcal{O}(\epsilon^3)$  provided  $\partial_A v^A \sim \mathcal{O}(\epsilon^3)$ : Solution to the vacuum Einstein's equation to order  $\mathcal{O}(\epsilon^2)$ .

Cut-off surface:  $r = r_c$  (constant) hypersurface.

Metric is curved.

- Constraint Einstein's equations of motion in ADM decomposition on the cut-off surface

Momentum constraint:  $D_\mu T_{(BY)}^{\mu\nu} = \gamma^{\nu b} N^a T_{ab} = 0$ .

Hamiltonian constraint:

$$^{(p+1)}R + K^2 - K_{ab}K^{ab} = T_{ab}N^aN^b = 0$$

where  $T_{\mu\nu}^{(BY)} = 2(\gamma_{\mu\nu}K - K_{\mu\nu})$  is Brown-York stress-tensor and extrinsic curvature  $K_{ab} = (1/2)\mathcal{L}_N\gamma_{ab}$ .

- Momentum constraint equation yields:

$$\partial_A v^A = 0 \text{ at } \mathcal{O}(\epsilon^2);$$

$$\partial_\tau v^A + v^C \partial_C v^A - \partial^A P - \eta \partial^2 v^A = 0 \text{ at } \mathcal{O}(\epsilon^3).$$

with  $\eta = r_c$  (Location of cut-off surface).

The metric can be constructed by using constant scaling and Lorentz transformations on the Rindler metric and then upgrading the transformation parameters as spacetime dependent variables.

(Compere, McFadden, Skenderis, Taylor: JHEP 1107 (2011) 050))

- Einstein's equation is interpreted as NS equation  $\Rightarrow$  On-shell approach.
- Dynamics of the fluid is represented by gravitational constraint equation.

IS IT ALWAYS NECESSARY?

OR

SOMETHING DIFFERENT CAN BE PRESCRIBED TO THIS  
DUALITY!!

# Further motivation

Old approach to gravitational theory (gravity as a scalar field):

- Action for interacting particle with a scalar field, representing the gravitational potential

$$\mathcal{A} = -m \int d\tau (1 + \Phi(x)),$$

which is consistent with [equivalence principle](#).

$$\text{EOM: } \frac{du^a}{d\tau} = -\frac{\partial^a \Phi}{1+\Phi} - \frac{1}{1+\Phi} u^a u^b \partial_b \Phi.$$

- Equivalently, one can introduce free particle moving on a spacetime, represented by the metric

$$g_{ab} = \eta_{ab}(1 + \Phi(x))^2.$$

At the weak field limit, this is identical to that of Rindler frame.

Lesson: The interaction can be encoded in non-trivial spacetime and a free particle moving on this represents the same interaction picture.

Can we use the similar idea?

A free fluid is moving on the cut-off hypersurface of a properly chosen bulk metric can represent the NS fluid equation!!

Non-rotating frame: De, Dey, Majhi: PRD99, 124024  
(2019)

Rotating frame: Dey, De, Majhi: arXiv:2002.06801

# Metric construction

- Idea is similar to earlier one.

For thermal fluid we keep our background as Rindler one.

Using hydrodynamic scaling construct the metric order by order such a way that the acceleration of the particle on the chosen hypersurface is zero;

i.e.  $\gamma_{bc} v^a \nabla_a v^b|_{r_c} = 0,$

with  $v^a$  is suitably chosen as  $v^a = (1, 0, v^A).$

- Proposed metric:

$$ds^2 = -rd\tau^2 + 2drd\tau + dx_A dx^A \\ - \left( \frac{2a_1}{r_c} \partial_A P + 2a_2 \partial^2 v_A \right) dx^A dr \\ + \mathcal{O}(\epsilon^4).$$

Induced metric on timelike slice  $r = r_c$  is flat.

$v_A(\tau, x^A)$  and  $P(\tau, x^A)$  are independent of  $r$ .

- $\gamma_{bc} v^a \nabla_a v^b|_{r_c} = 0$  is trivially satisfied upto order  $\epsilon^3$  for free index  $c = \tau$  and  $c = r$ .
- For  $c = A$ , this is valid till  $\mathcal{O}(\epsilon^2)$ . But at  $\mathcal{O}(\epsilon^3)$ , LHS yields
 
$$\partial_\tau v_A + v^C \partial_C V_A + \frac{a_1}{2} \partial_A P + \frac{a_2}{2} r_c \partial^2 v_A.$$

Therefore satisfaction of our condition gives the incompressible NS equation

$$\partial_\tau v_A + v^C \partial_C V_A + \frac{a_1}{2} \partial_A P + \frac{a_2}{2} r_c \partial^2 v_A = 0.$$

This with the choice  $a_1 = 2 = -a_2$  leads to NS equation.

- The incompressibility condition is achieved by demanding that  $\Theta = \gamma^{ab} \nabla_a v_b = 0$  upto  $\mathcal{O}(\epsilon^3)$ .

It is checked that all orders till  $\epsilon^3$  are trivially satisfied, expect  $\mathcal{O}(\epsilon^2)$ .

Demanding this to be satisfied in this order as well one finds that  $\partial_A v^A = 0$ .

# More about the metric

- The bulk metric can be derived by a coordinate transformation over the base Rindler metric:

$$\tilde{x}^A = x^A + 2r\partial_r\Gamma_{\tau\tau}^{A(3)}$$

$$\tilde{\tau} = \tau$$

$$\tilde{r} = r;$$

$$\text{with } \Gamma_{\tau\tau}^{A(3)} = \frac{r}{2}\delta^{AB}\left(\frac{a_1}{r_c}\partial_B P + a_2\partial^2 v_B\right).$$

$$\text{Then } ds^2 = -\tilde{r}d\tilde{\tau}^2 + 2d\tilde{r}d\tilde{\tau} + d\tilde{x}_A d\tilde{x}^A$$

$$= -rd\tau^2 + 2drd\tau + dx_A dx^A$$

$$- \left(\frac{2a_1}{r_c}\partial_A P + 2a_2\partial^2 v_A\right) dx^A dr + \mathcal{O}(\epsilon^4).$$

The metric is flat.

- Trivially solves Einstein's equation of motion as  $G_{ab} \sim \mathcal{O}(\epsilon^4)$ .
- Suggestive analogy: Viscous, incompressible fluid residing in flat space is essentially equivalent to a free fluid residing in a hypersurface of flat spacetime but with a different coordinate system defined by a unique choice of the metric.
- **Equivalence Principle !!**
- Fluid property is interpreted as manifold property – geometrical description.
- The approach has been extended to fluid in a rotating frame. Contrary to this, the base metric is intrinsically curved and the intrinsic rotation parameter of spacetime has been identified as the rotation of the frame. [Ref. arXiv: 2002.06801]
- This is an off-shell approach.

# Possible prospects of the approach

- Possibility of constructing an action for the fluid system.
- Off-shell approach  $\Rightarrow$  Quantum many particle theory in fluid side  $\leftrightarrow$  Quantum theory of free fluid on the hypersurface.

No need to quantize gravity; a semiclassical analysis.

- The knowledge of microscopic level of fluid may shed light in the same for gravity theory.

Since null surface has thermodynamic structure (Ref. PRD 87 (2013) 124011), an off-shell analysis is very much required.

- Higher order corrections to NS equation in this metric formalism.

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*Thank You*