IR physics from the holographic RG flow

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Based on

J. H. Lee and CP, arXiv:2406.17221

Motivation

Can we understand nonperturbative IR physics from the fundamental theory point of view?

- To understand IR (macroscopic) physics by the fundamental (microscopic) QFT, we need to figure out a non-perturbative RG flow, involving all quantum effects.

non-perturbative RG flow
(with full quantum corrections)

Motivation

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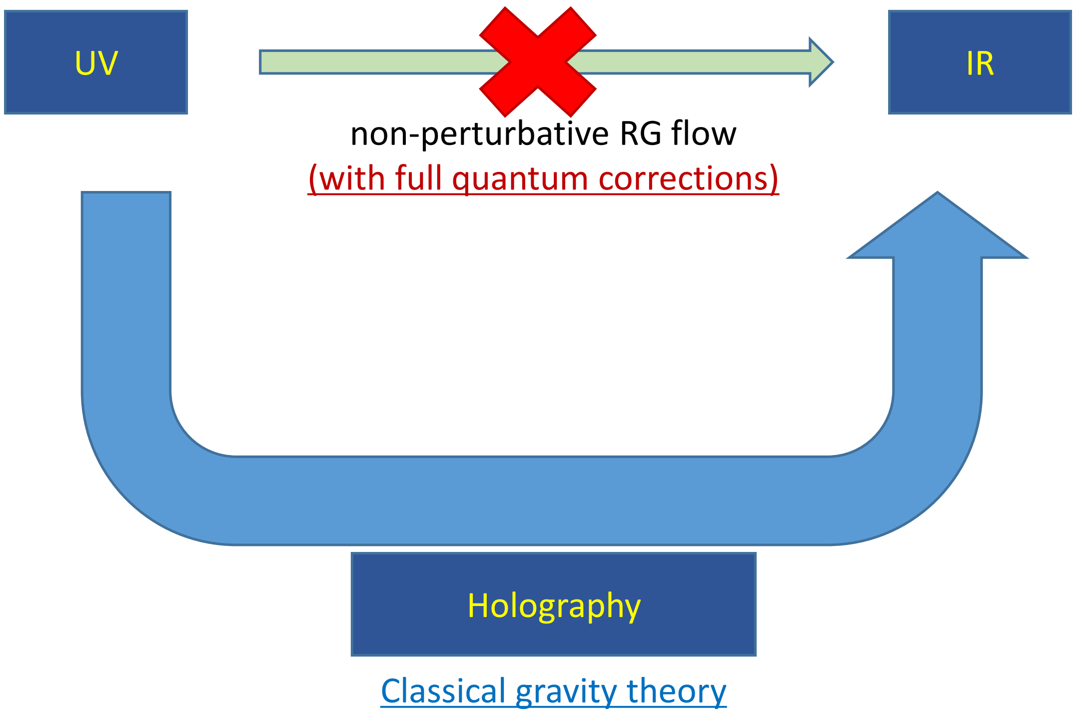


AdS/CFT correspondence (Holography)

Classical SUGRA on AdS space-time ⟷ (1 to 1) **Super-CFT** at the AdS boundary (**in a strong coupling regime**)

Due to the conformal symmetry, the RG flow is trivial.

How about a non-conformal and non-supersymmetric QFT like nuclear and condensed matter theories?

IR ⟵ non-trivial RG flow ⟵ UV

- EFT in IR (CMT)

- not UV complete (the connection to the fundamental theory is not clear)

Holography

- IR physics (?)

- AdS space (UV complete)

Due to restoration of a conformal symmetry at critical (or fixed) points, some physical quantities like entanglement entropy and correlation functions are constrained even at an IR region. This leads to a universal feature of IR physics like critical exponents. Here, we investigate such universality in the IR fixed points by applying the holographic methods.

Transverse field Ising model (EFT in CMt)

Hamiltonian

$$H_I = -\sum_{n=1}^{L-1} \sigma_n^x - g \sum_{n=1}^{L-1} \sigma_n^z \sigma_{n+1}^z,$$

1) for $g = 0$, paramagnet with $\langle \sigma_n^z \rangle = 0$
2) for $g = \infty$ ferromagnet with $\langle \sigma_n^z \rangle = \pm 1$
3) at $g = 1$ 2nd-order phase transition occurs (conformal at the critical point)

Entanglement entropy

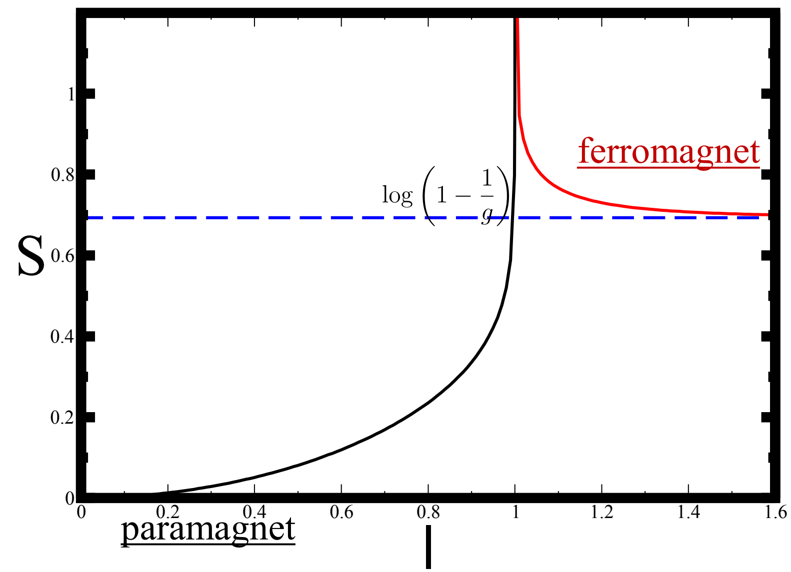
$$S_E = \epsilon \sum_{j=0}^{\infty} \frac{2j+1}{1+e^{(2j+1)\epsilon}} + \sum_{j=0}^{\infty} \log\left(1+e^{-(2j+1)\epsilon}\right) \quad \text{for } g < 1,$$

$$= \epsilon \sum_{j=0}^{\infty} \frac{2j}{1+e^{2j\epsilon}} + \sum_{j=0}^{\infty} \log\left(1+e^{-2j\epsilon}\right) \quad \text{for } g > 1,$$

with energy gap between energy levels

$$\epsilon = \pi \frac{K(\sqrt{1-1/g^2})}{K(1/g)},$$

where K means a elliptic integral of the first kind.



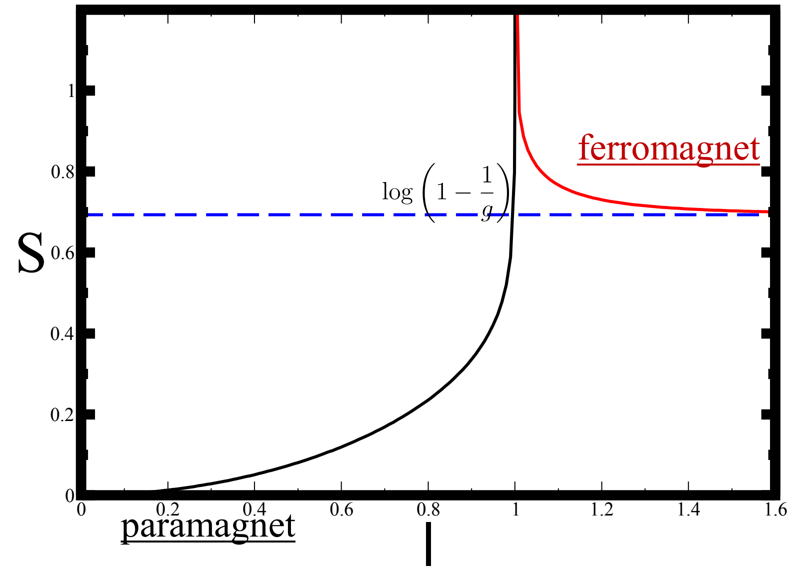
Entanglement entropy for massive field theory is given by

$$S_E = -\frac{c}{6}\log\frac{a}{\xi}$$

For the transverse field Ising model, the entanglement entropy near $g=1$ reduces to

$$S_E \approx -\frac{1}{12}\log\left(1-\frac{1}{g}\right)$$

This corresponds to $c=\frac{1}{2}$ with a long-range correlation which is a typical feature of fermionic conformal field theory.



- <u>logarithmic divergence</u> due to the conformal symmetry
- This is a typical feature of a two-dimensional CFT
- <u>A similar feature can also appear at IR fixed points because of restoration of a conformal symmetry.</u>

Holographic IR physics

3-dimensional gravity theory representing the RG flow from a UV to IR fixed point

$$S = \frac{1}{2\kappa^2}\int d^3X\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}\partial_M\phi\partial^M\phi - \frac{V(\phi)}{R_{UV}^2}\right)$$

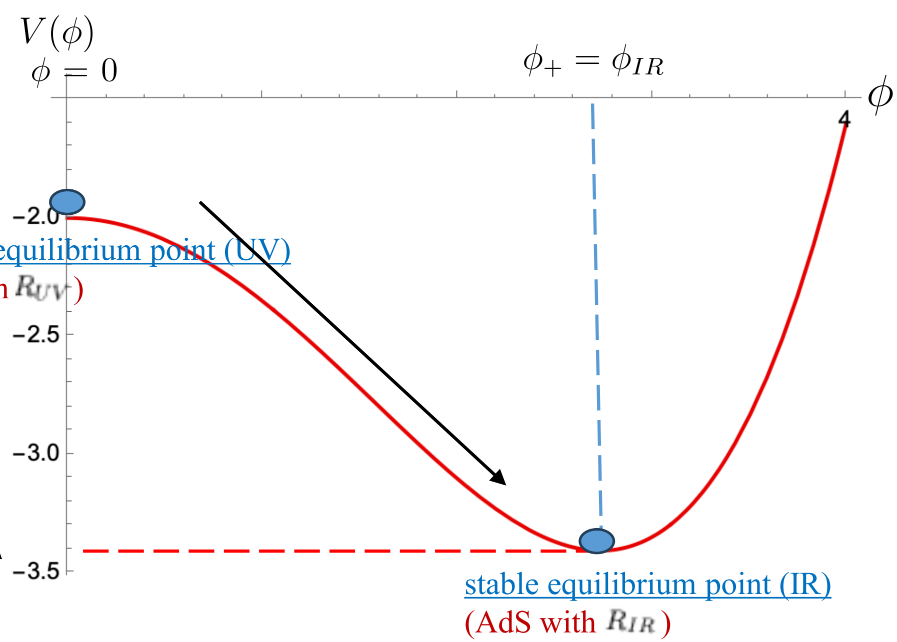
with the following scalar potential

$$V(\phi) = 2R_{UV}^2\Lambda_{UV} + \frac{M_\phi^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 = 2R_{UV}^2\Lambda_{UV} + \frac{\lambda}{4}\phi^2\left(\phi^2 - 2\frac{m_\phi^2}{\lambda}\right)$$

where $M_\phi^2 = -m_\phi^2 < 0$ and $\lambda > 0$.

This gravity theory allows one local maximum at $\phi = 0$ and two degenerated local minima at $\phi_\pm = \pm m_\phi/\sqrt{\lambda}$.

$V(\phi) = 2R_{UV}^2\Lambda_{UV}$



$V = 2R_{UV}^2\Lambda_{UV} - m_\phi^4/4\lambda$

$$R_{IR} = \sqrt{\frac{8\lambda}{8\lambda + m_\phi^4}}\,R_{UV}$$

1) Rolling of the scalar field from a unstable to stable equilibrium point describes the RG flow from a UV to IR fixed point on the dual QFT side.

2) Since $R_{IR} < R_{UV}$

$c_{UV} > c_{IR}$ (c-theorem)

where

$$c_{UV} = \frac{3R_{UV}}{2G} \quad \text{and} \quad c_{IR} = \frac{3R_{IR}}{2G}$$

Intermediate energy scale

The geometric solution interpolating a local maximum and a local minimum describes the RG flow of a dual QFT.

To see this, we consider the following metric ansatz in the normal coordinate

$$ds^2 = dy^2 + e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu$$

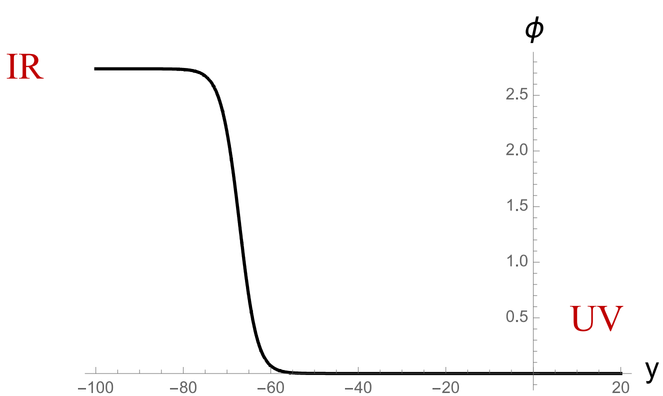
where a scale transformation of the dual QFT is represented as a translation in y-coordinate

Equations of motion

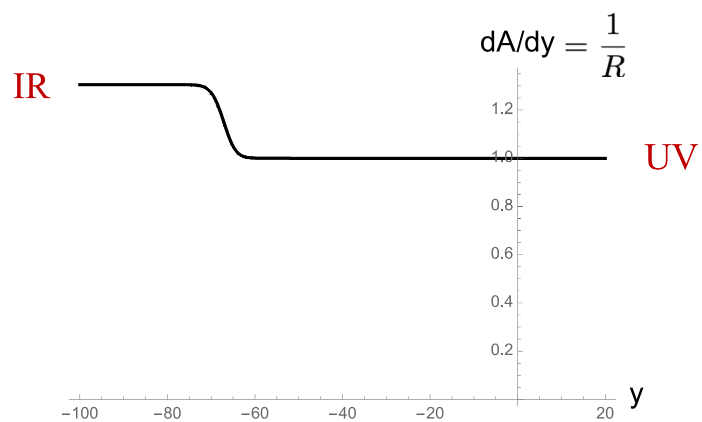
$$\begin{aligned}
0 &= A'^2 - \frac{1}{4}\phi'^2 - \frac{m_\phi^2}{4R_{UV}^2}\phi^2 + \frac{\lambda}{8R_{UV}}\phi^4 - \frac{1}{R_{UV}^2},\\
0 &= A'' + A'^2 + \frac{1}{4}\phi'^2 - \frac{m_\phi^2}{4R_{UV}^2}\phi^2 + \frac{\lambda}{8R_{UV}}\phi^4 - \frac{1}{R_{UV}^2},\\
0 &= \phi'' + 2A'\phi' + \frac{m_\phi^2}{R_{UV}^2}\phi^2 - \frac{\lambda}{R_{UV}}\phi^4.
\end{aligned}$$

In this case, the UV and IR fixed points appear at $y = \infty$ and $-\infty$, respectively.

Numerical solution



(a) Scalar field $\phi(y)$



(b) Metric factor $A(y)$

Entanglement entropy

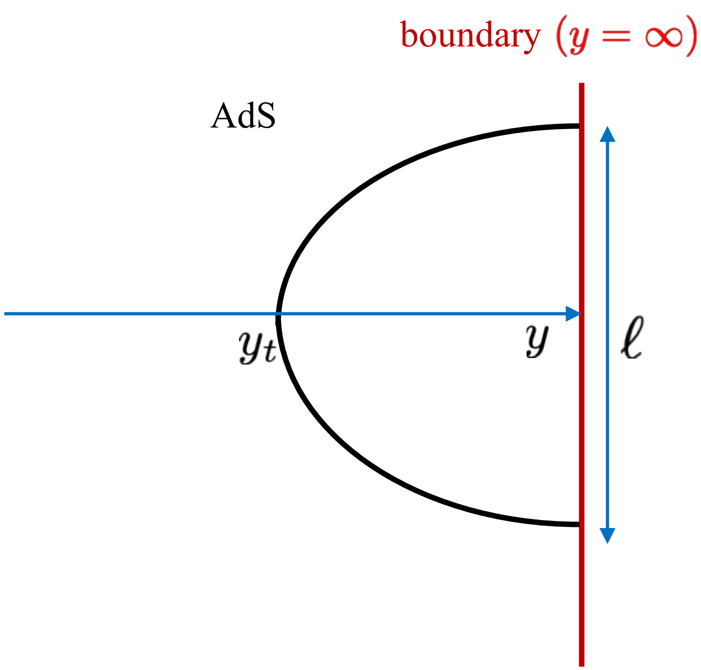
Now, we investigate the entanglement entropy in the previous interpolation geometry.

According to the Ryu-Takayanagi proposal,

the entanglement entropy of the boundary theory can be described by the area of a minimal surface extending to the dual geometry

$$S_E = \frac{1}{4G}\int_{-\ell/2}^{\ell/2} dx \sqrt{y'^2 + e^{2A}}$$

$$\ell = \int_{y_t}^{\Lambda_y} dy \, \frac{2}{e^{2A}\sqrt{e^{-2A_t} - e^{-2A}}},$$
$$S_E = \frac{1}{2G}\int_{y_t}^{\Lambda_y} dy \, \frac{e^{-A_t}}{e^{2A}\sqrt{e^{-2A_t} - e^{-2A}}},$$



(1) Introducing a turning point y_t

the entanglement entropy and subsystem size are given by functions of the turning point

(2) Removing the turning point, the entanglement entropy is rewritten as a function of the subsystem size

$$S_E = S_E(\ell)$$

In this case, the subsystem size is reinterpreted as the inverse of the RG scale, which describes the real-space RG flow.

(3) What is a relation between the RG scale and a coupling constant?

The interpolation function can be understood as follows:

Gravity		**Dual QFT**
AdS	$\Longleftrightarrow$	CFT
+		+
scalar field fluctuation		deformation by a scalar operator $\int d^2x\,\phi\,\mathcal{O}$

In addition, the radial coordinate is reinterpreted as the RG scale of the dual QFT (momentum-space RG flow).

- Recalling that the subsystem size is uniquely fixed by the turning point $\ell = \ell(y_t)$,

the radial position of a turning point is also associated with the RG scale.

- In the normal coordinate system

$$ds^2 = dy^2 + e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu$$

the RG scale of the dual QFT is given by

$$\mu = e^{A(y_t)}$$

In the UV and IR limits,

$$ds^2 = dy^2 + e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu$$

At the UV fixed point $(y \to \infty)$ at $z \to 0$

$$z = R_{UV}\, e^{-y/R_{UV}}$$

$$ds^2_{UV} = \frac{R^2_{UV}}{z^2}\left(-dt^2 + dx^2 + dz^2\right)$$

$$\boxed{\phi(z) = c_1 z^{2-\Delta_{UV}} + c_2 z^{\Delta_{UV}} \quad \text{with} \quad \Delta_{UV} = 1 + \sqrt{1 - m_\phi^2}}$$

$$\boxed{\beta_\phi = \mu\frac{d\phi}{\delta\mu} = -(2-\Delta_{UV})\phi + \cdots\,,}$$

in the IR region $(y \to -\infty)$ for $\bar{z} \to \infty$

$$\bar{z} = R_{IR}\, e^{-y/R_{IR}}$$

$$ds^2_{IR} = \frac{R^2_{IR}}{\bar{z}^2}\left(-d\bar{t}^2 + d\bar{x}^2 + d\bar{z}^2\right) \quad \text{with} \quad R_{IR} = \sqrt{\frac{8\lambda}{8\lambda + m_\phi^4}}\, R_{UV}$$

$$\phi = \phi_{IR} + \delta\phi \quad \text{with} \quad \phi_{IR} = m_\phi/\sqrt{\lambda}$$

$$\delta\phi = d_1\, \bar{z}^{2+\delta_{IR}} - d_2\, \bar{z}^{-\delta_{IR}}$$

$$\delta_{IR} = \sqrt{1 + 2m_\phi^2\frac{R^2_{IR}}{R^2_{UV}}} - 1 = \frac{\sqrt{8\lambda + m_\phi^4 + 16 m_\phi^2\lambda}}{\sqrt{8\lambda + m_\phi^4}} - 1 > 0 \longrightarrow d_1 = 0$$

Finally, $\boxed{\phi(\bar{z}) = \phi_{IR} - d_2\, \bar{z}^{-\delta_{IR}} + \cdots\,, \text{with} \quad \Delta_{IR} = 2}$

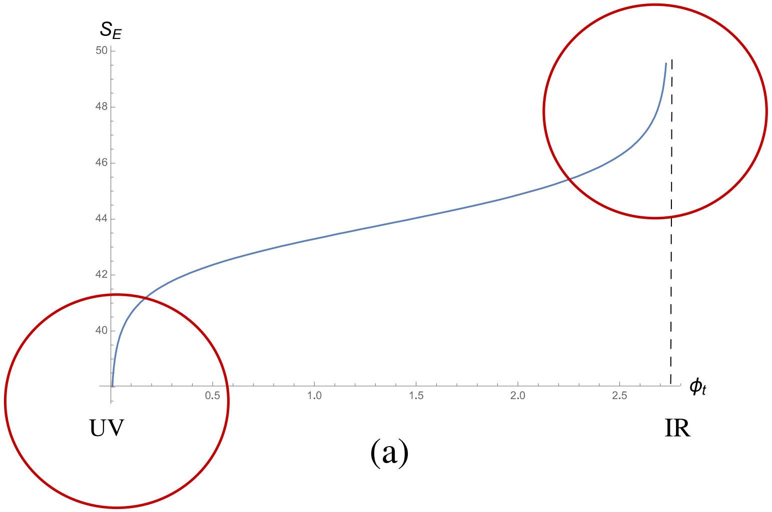
$$\boxed{\beta_\phi = \mu\frac{d\phi}{d\mu} = -\delta_{IR}\Big(\phi_{IR} - \phi(\bar{z})\Big) + \cdots\,.}$$

$\beta_\phi = \mu\dfrac{\partial\phi}{\partial\mu} = -(2-\Delta_{UV})\phi(\mu) + \cdots$ in the UV region (relevant for $\Delta_{UV} < 2$)

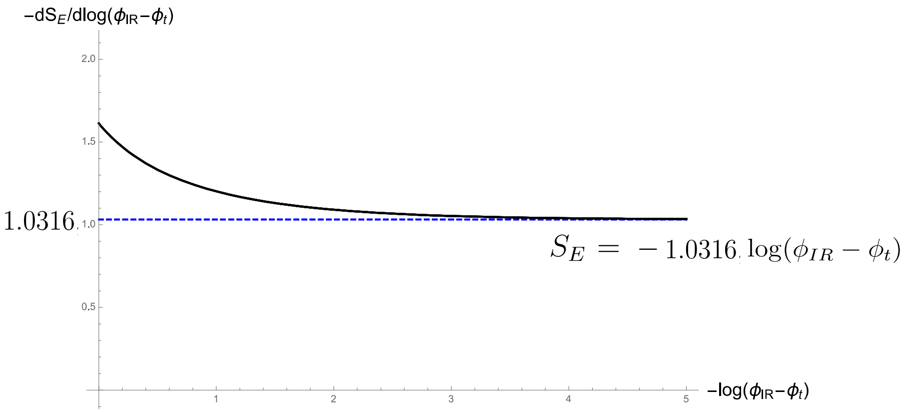
$\beta_\phi = \mu\dfrac{\partial\phi}{\partial\mu} = 0 - \delta_{IR}(\phi_{IR} - \phi(\mu)) + \cdots$ in the IR region (marginal with $\Delta_{IR} = 2$) $\longrightarrow$ conformal

Now, we evaluate the entanglement entropy numerically for $R = G = 1$, $m_\phi = \sqrt{3}/2$ and $\lambda = 0.1$

Numerical entanglement entropy



(a)



(b)

As expected, the entanglement entropy shows a logarithmic divergence at UV and IR fixed points.

Analytic estimation of the leading entanglement entropy at fixed points

In the UV region,

the entanglement entropy is governed by

$$S_E \approx \frac{R_{UV}}{4G} \int_{-\ell/2}^{\ell/2} dx \, \frac{\sqrt{z'^2 + 1}}{z},$$

which gives rise to

$$\ell(z_t) = \int_0^{z_t} dz \frac{2z}{\sqrt{z_t^2 - z^2}} = 2z_t + \cdots ,$$

$$S_E(z_t) = \frac{R_{UV}}{2G} \int_{2\epsilon}^{z_t} dz \frac{z_t}{z\sqrt{z_t^2 - z^2}} = \frac{R_{UV}}{2G} \log \frac{z_t}{\epsilon'} + \cdots$$

Since the subsystem size is determined as a function of the turning point, we can reinterpret the turning point as the energy scale of the dual QFT. Then, the turning point (or energy scale) is reexpressed in terms of the coupling constant

$$z_t \approx \left(\frac{\phi_t}{c_1} \right)^{1/(2-\Delta_{UV})}$$

Then, the entanglement entropy in terms of the coupling constant in the UV region is given by

$$S_E = \frac{c_{UV}}{3\,(2 - \Delta_{UV})} \log \frac{\phi_t}{\phi_{\epsilon'}} + \mathcal{O}\left(\phi_t^{1/(2-\Delta_{UV})} \right)$$

This shows a logarithmic behavior near the UV fixed point due to the conformal symmetry.

In the IR region,

the main contribution to the entanglement entropy comes from the geodesic extending the IR geometry

$$S_E \approx \lim_{\ell \to \infty} \frac{R_{IR}}{4G} \int_{-\ell/2}^{\ell/2} dx \, \frac{\sqrt{\bar{z}'^2 + 1}}{\bar{z}},$$

Then, the IR entanglement entropy becomes

$$\begin{aligned} \ell(\bar{z}_t) &\approx \lim_{z_t \to \infty} 2\bar{z}_t, \\ S_E(\bar{z}_t) &\approx \lim_{z_t \to \infty} \frac{R_{IR}}{2G} \log\left(\bar{z}_t\right) = \frac{c_{IR}}{3} \log\left(\bar{z}_t\right). \end{aligned}$$

Using the IR profile of the scalar field

$$\phi(\bar{z}) = \phi_{IR} - d_2 \, \bar{z}^{-\delta_{IR}}$$

The IR entanglement entropy is in terms of the coupling constant

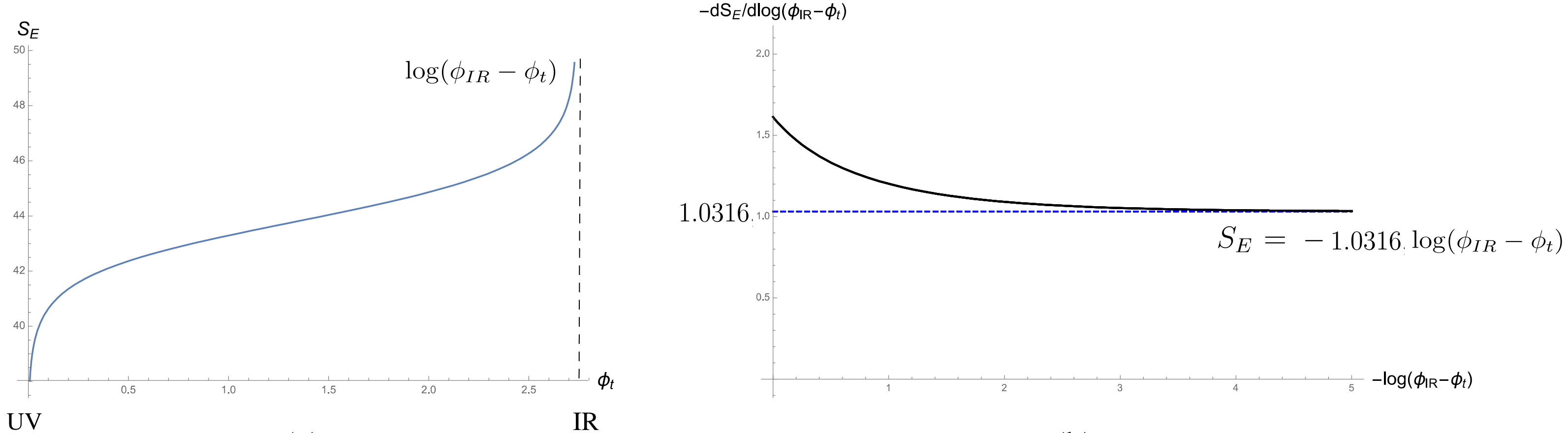
$$\begin{aligned} S_E(\bar{z}_t) &\approx -\frac{R_{IR}}{2G\delta_{IR}} \log\left(\phi_{IR} - \bar{\phi}_t\right) \\ &= -\frac{c_{UV}\sqrt{8\,\lambda}}{3\left(\sqrt{8\lambda + m_\phi^4 + 16 m_\phi^2 \lambda} - \sqrt{8\lambda + m_\phi^4}\right)} \log\left(\phi_{IR} - \bar{\phi}_t\right) \end{aligned}$$

For $R = G = 1$, $m_\phi = \sqrt{3}/2$ and $\lambda = 0.1$

$$-\frac{dS_E}{d\log(\phi_{IR}-\phi_t)} = \frac{R\sqrt{8}\,\lambda}{2G\left(\sqrt{8\lambda + m_\phi^4 + 16 m_\phi^2\lambda} - \sqrt{8\lambda + m_\phi^4}\right)} = 1.0316,$$

which is perfectly matched with the previous numerical result

Numerical entanglement entropy



(a) (b)

Anomalous dimension of a local operator

The RG scale dependent two-point function of a local operator

Two-point function at UV and IR fixed points

$$\langle O_\chi(t_1,x_1)\, O_\chi(t_2,x_2)\rangle = \frac{1}{(-|t_1-t_2|^2+|x_1-x_2|^2)^{\Delta^\chi_{UV}}} \qquad \text{at a UV fixed point}$$

$$\langle O_\chi(t_1,x_1)\, O_\chi(t_2,x_2)\rangle = \frac{1}{(-|t_1-t_2|^2+|x_1-x_2|^2)^{\Delta^\chi_{IR}}} \qquad \text{at an IR fixed point}$$

The conformal dimensions at UV and IR fixed points are generally different due to the RG flow.

In the holographic setup in the probe limit, we consider a bulk field χ as the dual of O_χ

This operator is affected by the change of the ground state. We study how a two-point function of a local operator is modified under a nontrivial RG flow of the ground state

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-G}}\partial_M\left(\sqrt{-G}G^{MN}\partial_M\chi\right) + \frac{m_\chi^2}{R_{UV}^2}\chi \\ &= \frac{z^2}{R_{UV}^2}\left(\partial_z^2\chi - \frac{1}{z}\,\partial_z\chi + \frac{m_\chi^2}{z^2}\chi\right). \end{aligned}$$

The metric contains information about the RG flow of the ground state.

According to the AdS/CFT correspondence, the conformal dimension at the UV fixed point is given by

$$\chi = c_1 \, z^{2-\Delta_{UV}^{\chi}} + c_2 \, z^{\Delta_{UV}^{\chi}}$$

with

$$\Delta_{UV}^{\chi} = 1 + \sqrt{1 - m_{\chi}^2}.$$

Therefore, the UV two-point function becomes

$$\langle O_{\chi}(t_1, x_1) \, O_{\chi}(t_2, x_2) \rangle = \frac{1}{(-|t_1 - t_2|^2 + |x_1 - x_2|^2)^{\Delta_{UV}^{\chi}}}$$

In the IR region where the background metric is modified, the equation in terms of the IR coordinate is

$$0 = \partial_{\bar{z}}^2 \chi + \left(1 - \frac{2R_{IR}}{R_{UV}}\right) \frac{1}{\bar{z}} \, \partial_{\bar{z}} \chi + \frac{R_{IR}^2}{R_{UV}^2} \frac{m_{\chi}^2}{\bar{z}^2} \chi$$

which allows the following solution

$$\chi = \bar{c}_1 \, \bar{z}^{2R_{IR}/R_{UV} - \Delta_{IR}^{\chi}} + \bar{c}_2 \, \bar{z}^{\Delta_{IR}^{\chi}}$$

with

$$\Delta_{IR}^{\chi} = \frac{R_{IR}}{R_{UV}} \, \Delta_{UV}^{\chi} = \frac{\sqrt{8\lambda}}{\sqrt{8\lambda + m_{\phi}^4}} \, \left(1 + \sqrt{1 - m_{\chi}^2}\right)$$

As a result, the IR two-point function becomes

$$\langle O_{\chi}(t_1, x_1) \, O_{\chi}(t_2, x_2) \rangle = \frac{1}{(-|t_1 - t_2|^2 + |x_1 - x_2|^2)^{\Delta_{IR}^{\chi}}}$$

Therefore, the anomalous dimension is

$$\gamma_\chi \equiv \Delta^\chi_{IR} - \Delta^\chi_{UV} = -\left(1 - \sqrt{\frac{8\lambda}{8\lambda + m_\phi^4}}\right) \Delta^\chi_{UV} < 0.$$

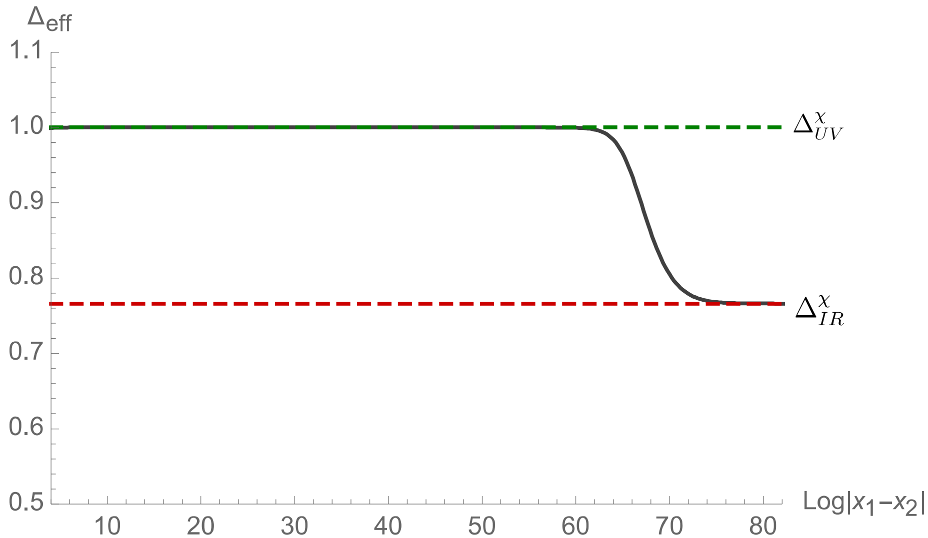
It is also possible to calculate the conformal dimension relying on the RG scale by

$$\langle O_\chi(t, x_1)\, O_\chi(t, x_2)\rangle \sim e^{-\Delta^\chi_{UV}\, L(t,x_1;t,x_2)/R_{UV}}$$

with the geodesic length

$$L(t, x_1; t, x_2) = \int_{y_t}^{\Lambda_y} dy\, \frac{2e^{-A_t}}{e^{2A(y)}\sqrt{e^{-2A_t} - e^{-2A(y)}}}$$

We perform this integral numerically and find the conformal dimension depending on the RG scale



$$\frac{\Delta^\chi_{IR}}{\Delta^\chi_{UV}} = \frac{\sqrt{8\lambda}}{\sqrt{8\lambda + m_\phi^4}} = 0.7663$$

The numerical result is consistent with the previous theoretical expectation.

Discussion

- Using the holographic method, we describe the RG flow of the dual QFT correctly.

- At fixed points of 2-dimensional QFTs, the entanglement entropy generally shows a logarithmic behavior due to the restoration of a conformal symmetry.

- Using the geodesic description, we also studied the change of the conformal dimension in the probe limit.

- These holographic descriptions well describe the expected RG flow of QFTs.

- It would be interesting to investigate the RG flow of correlation functions beyond the probe limit.

Thank you!