Position-Dependent Mass Quantum systems and ADM formalism

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International Conference on Holography, String Theory and Discrete Approaches in Hanoi
6 August 2020

Contents

Introduction

- Classical mechanics is Galilean invariant, i.e, time parameter t and position coordinate q(t) are explicitly functions of each other.
- Non existence of a locally Lorentz invariant quantum theory with single particle interpretation.
- In GR, we have difficulty to interpret time as we did in classical mechanics.
- GR is locally Lorentz invariant (time t is just a coordinate and is no longer considered as a parameter).
- In the language of the variational calculus, the analogue to coordinate $q(t)$ is the Riemannian metric $g_{ab}(x^c),, a, b = 0...3$ is a function of the all coordinates $x^c = (t, x^A), A = 1,..3$.

Introduction

- That makes quantum gravity difficult to build. Basically we have several QGs,
- Semiclassical quantum gravity (quantum fields on classical backgrounds)
- Loop Quantum gravity
- String Theory
- Non commutativity as a way to address quantum effects in gravity (singularities,..)
- Lorentz symmetry breaking (Lifshitz theories(stochastically renormalizable),..)
- It seems we have the problem of “disappearance of time”(Timeless quantum gravity)

Introduction

we are still looking for a fully covariant canonical quantum theory for gravity!

- My observations initiated when I studied GR as a classical gauge theory [M. Carmeli, "Claasical fields: General Relativity and Gauge Theory" (1982)]
- During studying non standard classical dynamical systems I found a class of Lagrangian models with second order time derivative of the position $L(q, \dot{q}, \ddot{q})$.
- A wide class of such models reduce to the position dependence mass (PDM) models[1507.05217,1605.06829, 1710.02135].
- It is obviously interesting to show that whether GR reduces to such models!.
- The classical Einstein-Hilbert (EH) Lagrangian reduces to the position-dependent -mass (PDM) model up to a boundary term.

Introduction
Analogy between classical mechanics and GR

- GR espects gauge transformations (any type of arbitrary change in the coordinates, from one frame to the other $x^a \rightarrow \tilde{x}^a$)
- We can rewrite EH Lagrangian in terms of the metric, first and second derivatives of it. It looks like a classical system in the form of $L(q, \dot{q}, \ddot{q})$.
- In this formal analogy, the classical acceleration term $\ddot{q}$ in the classical models under study is now replaced with the second derivative for metric i.e,$\partial_e \Gamma_{dab}$.
- we will need to define a super mass tensor as a function of the metric instead of the common variable mass function $m(q(t))$ in classical mechanic.

Introduction
Analogy between classical mechanics and GR

Theory	Position	Velocity	Acceleration	Mass
Classical Mechanics with PDM	$q(t)$	$\dot{q}(t)$	$\ddot{q}(t)$	Scalar mass $m(q(t))$
GR	g_{ab}	$\partial_d g_{ab}$	$\partial_e \partial_d g_{ab}$	M^{abldeh}

Introduction

- The notation for Einstein-Hilbert action is:$S_{EH} = \int d^4x \mathscr{L}_{GR}$
- Here $\mathscr{L}_{GR} = \mathscr{L}_{GR}(|g|, \partial |g|, g, \overline{g}, \partial g, \partial \overline{g})$
- Notations:$g \equiv g_{ab}$,
  $\overline{g} \equiv g^{ab}, |g| \equiv \det(g_{ab}) = \dfrac{1}{\det(\overline{g})}, \dfrac{1}{2}\overline{g} \, . \, \partial g \equiv \Gamma^a_{bc}, \partial g = \Gamma_{bla} \equiv g_{bl,a} + g_{la,b} - g_{ba,l} \, .$
- We adopt metricity condition $\nabla_a g^{bc} = |g|^{-1/2} \partial_a (|g|^{1/2} g^{bc}) \equiv |g|^{-1/2} \partial (|g|^{1/2} \overline{g}) = 0$ along $\partial \, . \, \overline{g} = - \, \overline{g} \, . \, \partial g \, . \, \overline{g}$.
- I showed that GR Lagrangian reduces to a PDM fully classical system with a super mass tensor of rank six.
- I built a consistent super phase space as well as a set of Poisson brackets.
- I show that gravitational field equations reduced top a set of first order Hamilton's equations.

Super Mass Tensor for GR as PDM classical system

Equivalent form for Lagrangian of the GR

- One can eliminate the second derivative term $\partial_{de}g_{ab}$ simply by integrating by part and using the metricity condition $\nabla_a g^{bc} = 0$, by taking into the account all the above requirements a possible: $\mathscr{L}_{GR} = \frac{1}{2}\sqrt{|g|}\left(g^{al}g^{be}\Gamma_{bla}\partial^h g_{eh} + g^{be}g^{dh}\Gamma_{bld}\partial^l g_{eh} + \frac{1}{2}g^{al}g^{bd}g^{te}\Gamma_{tld}\Gamma_{bea} - \frac{1}{2}g^{al}g^{bd}g^{te}\Gamma_{tla}\Gamma_{bed}\right)$ and we have $S_{EH} = \int d^4x \mathscr{L}_{GR} + B\,.\,T$
- By B.T we mean boundary term defined as: $B\,.\,T = \int_{\partial\mathscr{M}} \sqrt{|h_{AB}|}\,h^{BD}h^{AL}\Gamma_{BLA}|_{x^D=constant} + \int_{\partial\mathscr{M}} \sqrt{|h_{AB}|}\,h^{BD}h^{AL}\Gamma_{BLD}|_{x^A=constant}$
- We can re express the above GR Lagrangian in our convenient notations as: $\mathscr{L}_{GR} = \frac{1}{2}\sqrt{|g|}\left(\overline{g}\,.\,\partial g\,.\,\overline{g}\,.\,\overline{\partial}g + \overline{g}\,.\,\overline{g}\,.\,\partial g\,.\,\overline{\partial}g + \frac{1}{2}\overline{g}\,.\,\overline{g}\,.\,\overline{g}\,.\,\partial g\,.\,\partial g - \frac{1}{2}\overline{g}\,.\,\partial g\,.\,\overline{g}\,.\,\overline{g}\,.\,\partial g\right)$
- Note that by dot” we mean tensor product(we adopt Einstein summation rule).

Super Mass Tensor for GR as PDM classical system

- From the above representation we can realize $\{g, \overline{g}\}$ as two fields , in analogy to the Dirac Lagrangian where the fermionic pairs $\psi, \bar{\psi}$ appeared .

- The difference here is due to the fact that the pair of objects $\{g, \overline{g}\}$ depend on each other as we know $g\,.\,\overline{g} = \delta$, the Kronecker delta, however in the Dirac Lagrangian the norm $\overline{\psi}\dot{\psi} \neq I$.

- In our program we won't use this duality and we will focus on the coordinates representation of the GR Lagrangian.

- If we substitute the definition of Gamma terms and combine the theory, we obtain the final form for the Lagrangian as a PDM system for coordinate g_{ab}:

$$\mathscr{L}_{GR} = \frac{1}{2}\sqrt{|g|}\,M^{abldeh}\partial_a g_{bl}\partial_d g_{eh}$$

Super Mass Tensor for GR as PDM classical system

Definition of super mass tensor:

$$
\begin{aligned}
&|g|^{1/2}\overline{M} = |g|^{1/2} M^{a_1 b_1 l_1 d_1 e_1 h_1} = \frac{1}{4} g^{al}\, g^{bd}\ g^{te}\ \times \\
&\left(\delta^{a_1}_a\,\delta^{b_1}_b\,\delta^{l_1}_e + \delta^{b_1}_a\,\delta^{a_1}_b\,\delta^{l_1}_e - \delta^{l_1}_a\,\delta^{b_1}_b\,\delta^{a_1}_e\right)\left(\delta^{d_1}_d\,\delta^{h_1}_l\,\delta^{e_1}_t - \delta^{h_1}_d\,\delta^{d_1}_l\,\delta^{e_1}_t + \delta^{e_1}_d\,\delta^{h_1}_l\,\delta^{d_1}_t\right) \\
&-\frac{1}{4} g^{al}\, g^{bd}\ g^{te}\ \left(\delta^{a_1}_b\,\delta^{b_1}_d\,\delta^{l_1}_e + \delta^{b_1}_b\,\delta^{a_1}_d\,\delta^{l_1}_e - \delta^{b_1}_b\,\delta^{l_1}_d\,\delta^{a_1}_e\right)\left(\delta^{d_1}_a\,\delta^{h_1}_l\,\delta^{e_1}_t - \delta^{h_1}_a\,\delta^{d_1}_l\,\delta^{e_1}_t + \delta^{e_1}_a\,\delta^{h_1}_l\,\delta^{d_1}_t\right) \\
&+\frac{1}{4} g^{al}\, g^{bd}\, g^{te}\ \left(\delta^{d_1}_a\,\delta^{e_1}_b\,\delta^{h_1}_e + \delta^{e_1}_a\,\delta^{d_1}_b\,\delta^{h_1}_e - \delta^{h_1}_a\,\delta^{e_1}_b\,\delta^{d_1}_e\right)\left(\delta^{a_1}_d\,\delta^{l_1}_l\,\delta^{b_1}_t - \delta^{l_1}_d\,\delta^{a_1}_l\,\delta^{b_1}_t + \delta^{b_1}_d\,\delta^{l_1}_l\,\delta^{a_1}_t\right)) \\
&-\frac{1}{4} g^{al}\, g^{bd}\, g^{te}\ \left(\delta^{d_1}_b\,\delta^{e_1}_d\,\delta^{h_1}_e + \delta^{e_1}_b\,\delta^{d_1}_d\,\delta^{h_1}_e - \delta^{e_1}_b\,\delta^{h_1}_d\,\delta^{d_1}_e\right)\left(\delta^{a_1}_a\,\delta^{l_1}_l\,\delta^{b_1}_t - \delta^{l_1}_a\,\delta^{a_1}_l\,\delta^{b_1}_t + \delta^{b_1}_a\,\delta^{l_1}_l\,\delta^{a_1}_t\right)) \\
&+g^{al}\,\delta^{d_1}_a\, g^{be}\, g^{dh}\,\delta^{e_1}_e\,\delta^{h_1}_h\ \left(\delta^{b_1}_b\,\delta^{a_1}_d\,\delta^{l_1}_l + \delta^{a_1}_b\,\delta^{l_1}_d\,\delta^{b_1}_l - \delta^{b_1}_b\,\delta^{l_1}_d\,\delta^{a_1}_l\right) \\
&+g^{al}\, g^{be}\, g^{dh}\,\delta^{d_1}_d\,\delta^{e_1}_e\,\delta^{h_1}_h\ \left(\delta^{a_1}_a\,\delta^{b_1}_b\,\delta^{l_1}_l + \delta^{l_1}_a\,\delta^{a_1}_b\,\delta^{b_1}_l - \delta^{l_1}_a\,\delta^{b_1}_b\,\delta^{a_1}_l\right) \\
&+g^{al}\,\delta^{a_1}_a\, g^{be}\, g^{dh}\,\delta^{b_1}_e\,\delta^{l_1}_h\ \left(\delta^{e_1}_b\,\delta^{d_1}_d\,\delta^{h_1}_l + \delta^{d_1}_b\,\delta^{h_1}_d\,\delta^{e_1}_l - \delta^{e_1}_b\,\delta^{h_1}_d\,\delta^{d_1}_l\right) \\
&+g^{al}\, g^{be}\, g^{dh}\,\delta^{a_1}_d\,\delta^{b_1}_e\,\delta^{l_1}_h\ \left(\delta^{d_1}_a\,\delta^{e_1}_b\,\delta^{h_1}_l + \delta^{h_1}_a\,\delta^{d_1}_b\,\delta^{e_1}_l - \delta^{h_1}_a\,\delta^{e_1}_b\,\delta^{d_1}_l\right).
\end{aligned}
$$

Super Mass Tensor for GR as PDM classical system

Having the Lagrangian of GR, one can define a canonical pair of position conjugate momentum $(g, \overline{p})$ and construct a phase space.

Super phase space

- Defining the super conjugate momentum tensor :
$$p^{rst} = \frac{\partial \mathscr{L}_{GR}}{\partial(\partial_r g_{st})} = \frac{\sqrt{g}}{2}\left(M^{rstdeh}\partial_d g_{eh} + M^{ablrst}\partial_a g_{bl}\right)$$
- Note that for the mass tensor $M^{rstdeh}\partial_d g_{eh} = M^{ablrst}\partial_a g_{bl}$.
- A possible classical Hamiltonian: $\mathscr{H}_{GR} = \dfrac{1}{2\sqrt{|g|}} M^{abldeh} M_{rstabl} p^{rst} M_{uvwdeh} p^{uvw}$
- A possible Poisson's bracket $\{F, G\}_{P.B}$adopted to this system is:
$$\{F(g_{mn}, p^{stu}), G(g_{mn}, p^{stu})\}_{P.B} = \sum \left(\frac{\partial F}{\partial g_{ab}}\frac{\partial G}{\partial p^{rst}} - \frac{\partial F}{\partial p^{rst}}\frac{\partial G}{\partial g_{ab}}\right)$$
- In our notation it simplifies to the following expression
$$\{F(g, \overline{p}), G(g, \overline{p})\}_{P.B} = \sum \left(\frac{\partial F}{\partial g}\frac{\partial G}{\partial \overline{p}} - \frac{\partial F}{\partial \overline{p}}\frac{\partial G}{\partial g}\right)$$
- For our super phase coordinates (g_{ab}, p^{rst}), I postulate that$\{g_{ab}, p^{rst}\}_{P.B} = c^r \delta^{rs}_{ab}$

Super phase space

- Here δ^{rst}_{ab} is the generalized Kronecker defined as $\delta^{rst}_{ab} = 2!\delta^{s}_{[a}\delta^{t}_{b]}$.

- In the above Poisson's bracket, with structure constants c^r provide a classical minimal volume for super phase space (zero for Poisson's bracket same objects).

- We have now full algebraic structures in the super phase space and canonical Hamiltonian.

- We can write down Hamilton's equations as first order reductions of the Euler-Lagrange equations derived from the Lagrangian .

- I will show that how Einstein equation reduces to a type of covariant Hamilton equations.

Reduction of the Einstein Field Equations (EFE) to Hamilton's equation via covariant Hamiltonian

- The set of Hamilton's equations derived from the Hamiltonian $\partial_a g_{bl} = \{g_{bl}, \mathscr{H}_{GR}\}_{P.B} = \dfrac{\partial \mathscr{H}_{GR}}{\partial p_{abl}}$, $\partial_a p^{abl} = \{p^{abl}, \mathscr{H}_{GR}\}_{P.B} = -\dfrac{\partial \mathscr{H}_{GR}}{\partial g_{bl}}$

- We explicitly can write this pair of Hamilton's equation given as follows:

$$\frac{2}{\sqrt{g}}\partial_a g_{bl} = M^{a'b'l'deh} M_{rsta'b'l'} M_{uvwdeh}\left(\delta_a^u \delta_b^v \delta_l^w p^{rst} + \delta_a^r \delta_b^s \delta_l^t p^{uvw}\right)$$

$$-\frac{2}{\sqrt{g}}\partial_a p^{abl} = M^{a'b'l'deh} p^{rst} M_{rsta'b'l'} p^{uvw} \frac{\partial M_{uvwdeh}}{\partial g_{bl}} + M^{a'b'l'deh} p^{rst} M_{uvwdeh} \frac{\partial M_{rsta'b'l'}}{\partial g_{bl}}$$
$$+\frac{\partial M_{a'b'l'deh}}{\partial g_{bl}} p^{rst} M_{rsta'b'l'} p^{uvw} M_{uvwdeh} + M^{a'b'l'deh} \frac{\partial p^{rst}}{\partial g_{bl}} M_{rsta'b'l'} p^{uvw} M_{uvwdeh}$$
$$+M^{a'b'l'deh} p^{rst} M_{rsta'b'l'} \frac{\partial p^{uvw}}{\partial g_{bl}} M_{uvwdeh}.$$

- I believe that one can integrate this system as a general non autonomous dynamical system .

Quantization of GR

- I'm going to define appropriate forms for Dirac brackets:
$\hat{\pi}^{rst} \equiv -i\hbar^{r}\frac{\partial}{\partial \hat{g}_{st}}, \left[\hat{g}_{ab}, \hat{\pi}^{rst}\right] = i\hbar^{r}\delta^{st}_{ab}$

- The classical phase space spanned by the $\left(g_{ab}, p^{rst}\right)$one has more degrees of freedom (dof), basically is $10^5(=10\times 10^4)$ dimensional for a Riemannian manifold.

- The Dirac constant $\hbar$ is proportional to the minimum volume of the phase space V_0 defined as $\omega_0 = \int D(g_{ab}, p^{rst})$,$D(g_{ab}, p^{rst})$ is a covariant volume element.

- The super mass tensor $\overline{M} = M^{abldeh}$is a homogeneous (order 6) of the metric tensor

Quantization of GR

- Using the formalism of quantization for PDM systems the canonical quantized Hamiltonian for GR is: $\hat{\mathscr{H}}_{GR}(\hat{g}_{ab}, \frac{\partial}{\partial \hat{g}_{st}}) = -\frac{1}{2} f^{1/2}_{rstuvw} \hbar^r \frac{\partial}{\partial \hat{g}_{st}} \left[f^{1/2}_{rstuvw} \hbar^u \frac{\partial}{\partial \hat{g}_{vw}} \right]$
  here the auxiliary, scaled super mass tensor f_{rstuvw} is
  $f_{rstuvw} \equiv |g|^{-1/4} M^{abldeh} M_{rstabl} M_{uvwdeh}$.

- It is adequate to write the quantum Hamiltonian in the following closed form:
  $\hat{\mathscr{H}}_{GR}(\hat{g}_{ab}, \hat{\pi}^{mnp}) = \frac{1}{2} \left[|g|^{-1/2} \overline{M} M M \right]^{1/2} \bar{\hat{\pi}} \left[|g|^{-1/2} \overline{M} M M \right]^{1/2} \bar{\hat{\pi}}$
  where $\bar{\pi}$ is contravariant component of the super momentum π , etc.

- The above quantization of Hamiltonian is covariant since we didn't specify time t from the other spatial coordinates x^A.

Quantization of GR

- The model is considered as a timeless model, i.e, there is no first order time derivative in the final wave equation like $\dfrac{\partial}{\partial t}$.

- The associated functional second order wave equation which is fully locally Lorentz invariant as well as general covariant:

$$-\frac{1}{2} f^{1/2}_{rstuvw} \hbar^{r} \frac{\partial}{\partial \hat{g}_{st}} \left[f^{1/2}_{rstuvw} \hbar^{u} \frac{\partial}{\partial \hat{g}_{vw}} \Psi(\hat{g}_{ab}) \right] = E \Psi(\hat{g}_{ab})$$

- In our suggested functional wave equation for $\Psi(\hat{g}_{ab})$, we end up by the covariant (no first order derivative) of the functional Hilbert space.

- This model is a subclass of the timeless models of QG

Quantum cosmology

- We consider flat, Friedmann-Lemaître-Robertson-Walker (FLRW) model with Lorentzian metric $g_{ab} = diag(1, -a(t)\Sigma_3)$ where Σ_3 is the unit metric tensor for flat space, in coordinates $x^a = (t, x, y, z)$.

- The non vanishing elements of the super mass tensor: $M^{abldeh} = -12a^{-2}\delta^{a0}\delta^{d0}\delta^{BL}\delta^{EH}, B, L, E, H = 1,2,3$

- The auxiliary scaled super mass tensor $f_{rstuvw} = -\dfrac{3a^{1/2}}{4}\delta_{u0}\delta_{r0}\delta_{VW}\delta_{ST}$

- The functional wave equation reduces to the hypersurfaces Σ_3 coordinates $X^A = (x, y, z)$:

$$\frac{3\hbar_0^2 a^{1/2}}{8}\frac{\partial^2\Psi(\hat{g}_{AB})}{\partial\hat{g}_{SS}\partial g_{VV}} = E\Psi(\hat{g}_{AB})$$

Quantum cosmology

- it reduces simply to the following ordinary differential equation
  $$a\Psi''(a) - \Psi'(a) - \frac{32Ea^{5/2}}{3h_0^2}\Psi(a) = 0$$
- It can be reduce to a standard second order differential equation for wave function
  $$\Psi(a) = \sqrt{a}\phi(a), \phi''(a) - \left(\frac{32Ea^{1/2}}{3h_0^2} + \frac{3}{4a^2}\right)\phi(a) = 0.$$
- There are exact solutions for asymptotic regimes:
  $a^{3/2}$if $a \to 0, \exp[\frac{16\sqrt{2E}}{5\sqrt{3}h_0}a^{5/4}]$if $a \to \infty$.
- By suggesting $\phi(a) = \zeta(a)a^{3/2}\exp[\frac{16\sqrt{2E}}{5\sqrt{3}h_0}a^{5/4}]$ and $\zeta(a)$ will come as a transcendental (hypergeometric) function.

Quantum cosmology

Complete wave function:

- $$\Psi(a) = a^{5/2}\exp[\frac{16\sqrt{2E}}{5\sqrt{3}\hbar}a^{5/4}]\frac{3^{2/5}\hbar^{4/5}e^{-\frac{2}{15}\left(\frac{8\sqrt{6e}a^{5/4}}{\hbar}+3\right)}\left(5c_1\Gamma\left(\frac{1}{5}\right)I_{-\frac{4}{5}}\left(\frac{16\sqrt{\frac{2e}{3}}a^{5/4}}{5\hbar}\right)+32\sqrt[5]{2}c_2\Gamma\left(\frac{4}{5}\right)I_{\frac{4}{5}}\left(\frac{16\sqrt{\frac{2e}{3}}a^{5/4}}{5\hbar}\right)\right)}{64\sqrt[5]{5}a}$$

  The eigenvalue E (positive, negative or zero) can be discrete as well as continuous (bound states for $E < 0$).

- Remarkable is for vanishing energy state, $E = 0$
  $\Psi(a) = N_0 + Na^2$.

Note about ADM decomposition formalism and reduced phase space

- In our formalism If one adopt the ADM decomposition of the metric g_{ab} as follows
$$ds^2 = g_{ab}dx^a dx^b = h_{AB}dx^A dx^B + 2N_A dx^A dx^0 + (-N^2 + h^{AB}N_A N_B)(dx^0)^2$$
here x^0 is time, $A, B = 1,2,3$ refer to the spatial coordinates and h_{AB} is spatial metric, we recall the super conjugate momentum
$$\pi^{rs} = \frac{\partial \mathscr{L}_{GR}}{\partial \dot{g}_{st}} = \frac{\sqrt{g}}{2}\left(M^{0stdeh}\partial_d g_{eh} + M^{abl0st}\partial_a g_{bl}\right)$$
Building the Hamiltonian in a standard format as:
$$\mathscr{H}^{ADM}_{GR} = \frac{1}{2\sqrt{|g|}}M^{abldeh}M_{0stabl}\pi^{st}M_{0vwdeh}\pi^{vw}$$

- Again we can recover ADM Hamiltonian!

Final remarks

- The canonical covariant qunatization which I proposed here is a consistent theory.
- The GR action in a suitable form the Lagrangian reduced to a purely kinetic theory with position dependence mass term.
- With such a simple quadratic Lagrangian, I defined a conjugate momentum corresponding to the metric tensor.
- I developed a classical Hamiltonian using the metric and its conjugate momentum.
- One can write classical Hamilton's equations for metric and momentum are analogous to the second order nonlinear Einstein field equations.
- We replaced Poisson's brackets with Moyal(Dirac) and we defined a quantum Hamiltoninan for GR.

Final remarks

- I notice here that even if we didn't remove second derivative terms using integration part by part, it was possible to define a second conjugate momentum: $r^{abcd} = \dfrac{\partial \mathscr{L}_{GR}}{\partial(\partial_a \partial_b g_{cd})}$.
- If we impose a Bianchi identity between $\left(g_{ab}, p^{rst}, r^{abcd}\right)$, it is possible to fix this new momentum in terms of the other one and the metric.
- with the Bianchi identity, $\{g_{ab}, \{p^{cde}, r^{fghi}\}_{P.B}\}_{P.B} + \{p^{cde}, \{r^{fghi}, g_{ab}\}_{P.B}\}_{P.B} + \{r^{fghi}, \{g_{ab}, p^{cde}\}_{P.B}\}_{P.B} = 0$
- A suitable Legendre transformation from the GR Lagrangian $\mathscr{H}_{GR} = p^{rst}\partial_r g_{st} + r^{abcd}(\partial_a \partial_b g_{cd}) - \mathscr{L}_{GR}$
- One obtains a standard Hamiltoninan without this new higher order momentum.

THANK YOU !

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