Entanglement Harvesting in Accelerated Systems: Field Temperature and Boundary Effects

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Plan of the talk

1. Background
2. What is entanglement harvesting or leakage?
3. Entanglement Harvesting in Accelerated Systems
4. Role of thermal fields
5. Role of Reflecting boundaries
6. Entanglement Leakage
7. Conclusion

Background

Background

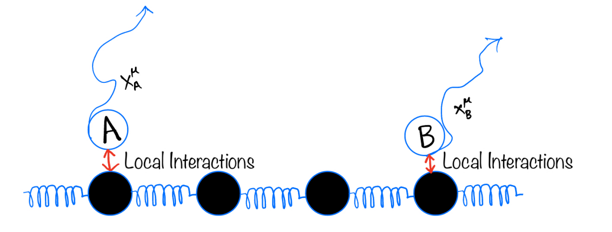
- *Hawking radiation:* black holes can radiate at the quantum level.
  (S.W. Hawking, Nature, 1974)
- *Equivalence principle:* accelerated frames can mimic gravity locally.
- Unruh effect: an accelerated observer will see particles in the Minkowski vacuum. (W.G. Unruh, PRD, 1976)
  – Analogous to Hawking effect.
- *QM:* information can not be created or destroyed (unitarity).
- *Black hole information paradox:* the thermal nature of Hawking radiation is a mixed state. (S.W. Hawking, CMP, 1975)
- *Present Situation:* quantum entanglement (information theory) may be important to explore the quantum nature of gravity and so the information paradox.
  (S. Bose et al, PRL, 2017; C. Marletto and V. Vedral, PRL, 2017)

Exploration of quantum entanglement in accelerated and gravitational systems is very important.

What is entanglement harvesting or leakage?

What is "Entanglement Harvesting (or Leakage)"?

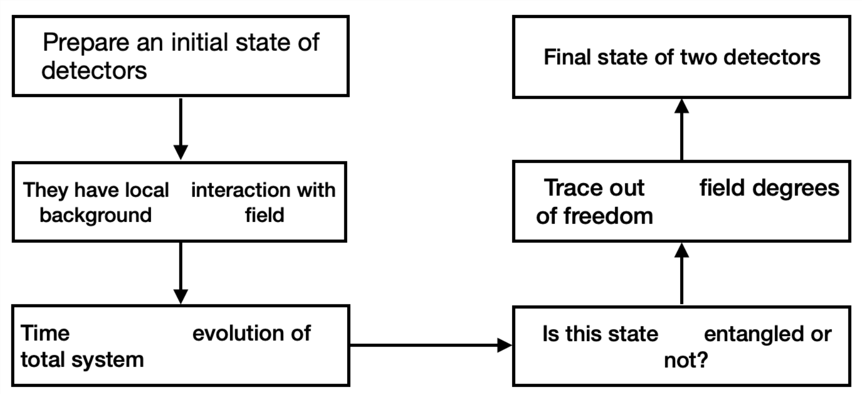
- Two two-level detectors (Unruh De-Witt detectors) and allow them to locally interact with a free quantum field.
- *Entanglement Harvesting (or leakage):* After some time, these two detectors may become more (or less) entangled, even if they are remain spacelike separated.
- Nature of entanglement is depends on the background quantum fields and motion of the detectors.



[1]

[1] **Credit: L. Henderson**.

Methodology for our study



System of two detectors

- Let us consider two two-level detectors: A and B.
  They interacts with the background fields with

$$S_{int} = \int d\tau_j \sum_{j=A,B} \lambda_j \chi_j(\tau_j) m_j(\tau_j) \phi(x_j) \tag{1}$$

  where, $m_j(0) = |g_j\rangle\langle e_j| + |e_j\rangle\langle g_j|$ and $\chi_j(\tau_j) = 1$.
- The total initial state at asymptotic past

$$|in\rangle = |g_A g_B\rangle \otimes |0_M\rangle \; . \tag{2}$$

- The final state at the asymptotic future

$$|out\rangle = Te^{iS_{int}}|in\rangle \; . \tag{3}$$

- Final density matrix of the two detectors system is

$$\rho_{AB} = Tr_\phi(|out\rangle\langle out|) \; . \tag{4}$$

Final density matrix of two detectors

- The final density matrix in the bases $\{|e_A e_B\rangle, |e_A g_B\rangle, |g_A e_B\rangle, |g_A g_B\rangle\}$ is(Reznik, Found Phys, 2003; Koga et al, PRA, 2018)

$$\rho_{AB} = \begin{pmatrix} 0 & 0 & 0 & \mathcal{E} \\ 0 & \mathcal{P}_A & \mathcal{P}_{AB} & 0 \\ 0 & \mathcal{P}_{AB}^{\star} & \mathcal{P}_B & 0 \\ \mathcal{E}^{\star} & 0 & 0 & 1-\mathcal{P}_A-\mathcal{P}_B \end{pmatrix} + O(\lambda^4)\ ; \quad (5)$$

  - Excitation probability of j^{th} detector $(\lambda_I = \lambda_j = \lambda)$

$$\mathcal{P}_j = \lambda^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_j d\tau_j'\, e^{i\Delta E(\tau_j - \tau_j')} G_W(x_j', x_j) \quad (6)$$

  - Entangling term

$$\mathcal{E} = -\lambda^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\tau_A' d\tau_B\, e^{i\Delta E(\tau_A' + \tau_B)} iG_F(x_B, x_A') \quad (7)$$

  - Shared information between the detectors $\mathcal{P}_{AB}$.

Measure of entanglement

- Positivity of the density matrix required (Koga, Kimura, and Maeda, PRA, 2018)

$$\mathcal{P}_A \mathcal{P}_B > |\mathcal{P}_{AB}|^2 \,. \tag{8}$$

- Negativity: a negative eigenvalue of partial transpose of ρ_{AB} $\left(\rho_{AB}^{T_A}\right)$ requires(Peres, PRL, 1996)

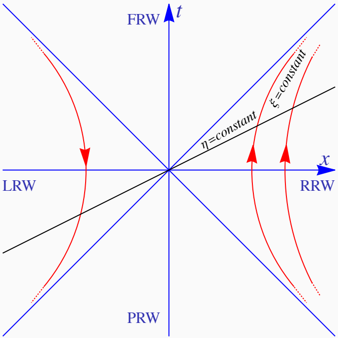
$$|\mathcal{E}|^2 > \mathcal{P}_A \mathcal{P}_B \,. \tag{9}$$

- Concurrence: a faithful quantification of entanglement (Hill, Wootters, PRL, 1997; Wootters, PRL, 1998)

$$\begin{aligned} C_J(\rho_{AB}) &= 2(|\mathcal{E}| - \sqrt{\mathcal{P}_A \mathcal{P}_B}) + O(\lambda^4) \\ \text{Concurrence } &= Max\{0, C_J\} \,. \end{aligned} \tag{10}$$

Entanglement Harvesting in Accelerated Systems

Accelerated observers



- The trajectories of uniformly accelerating detectors(Birrell, Davies, 1982)

$$t = \frac{1}{b}\sinh(b\tau),\ x = \frac{1}{b}\cosh(b\tau) \qquad \text{in RRW ;}$$
$$t' = \frac{1}{b}\sinh(b\tau'),\ x' = -\frac{1}{b}\cosh(b\tau') \qquad \text{in LRW.} \qquad (11)$$

- In terms of Rindler coordinates: $\tau = \pm e^{a\xi}\,\eta$ and $b = a\,e^{-a\xi}$. (we consider $\xi = \xi' = 0$).

Role of thermal fields

Role of thermal fields

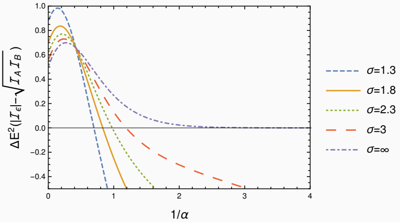
- Thermal Green's function

$$\begin{aligned} G^{\beta}_{W_{mn}}(x,x') &= Tr[e^{-\beta H}\phi_m(x)\phi_n(x')]/Tr[e^{-\beta H}]\,; \quad (12) \\ \text{with} \qquad H_k &= \omega_k(d^{\dagger}_{1,k}d_{1,k} + d^{\dagger}_{2,k}d_{2,k})\,. \end{aligned}$$

$(m,\, n) \implies$ Right (R) or Left (L) Rindler wedges.

- Using these green functions, we calculate $\mathcal{P}_j$, $\mathcal{E}$.
  – For parallel motion: $\mathcal{E} = 0 \implies$ Entanglement harvesting not possible.
  – For anti-parallel motion: $\mathcal{E} \neq 0 \implies$ Entanglement harvesting possible.
- Dimensionless quantities:
  $\sigma = \beta\Delta E$, $\alpha = a/\Delta E$ and $\Delta E^2 \mathcal{C}_J/\lambda^2\delta(0) = \Delta E^2\left(|\mathcal{I}_{\varepsilon}| - \sqrt{\mathcal{I}_A\mathcal{I}_B}\right)$.

Effect of background temperature-I: $\alpha_A = \alpha_B$



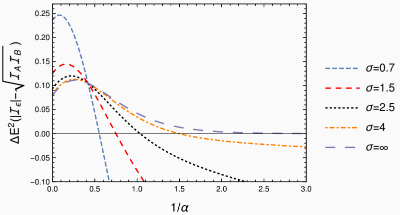


Figure 1: In $(1+1)$ and $(1+3)$ dimensionless concurrence quantity $\Delta E^2\left(|\mathcal{I}_\varepsilon| - \sqrt{\mathcal{I}_A\mathcal{I}_B}\right)$ is plotted with respect to the acceleration of the first detector α_A for different fixed inverse temperature of the thermal bath σ.
(D. Barman, S. Barman and B.R. Majhi, JHEP, 2021)

Effect of background temperature-II: fixed $\alpha_B\,(=1)$

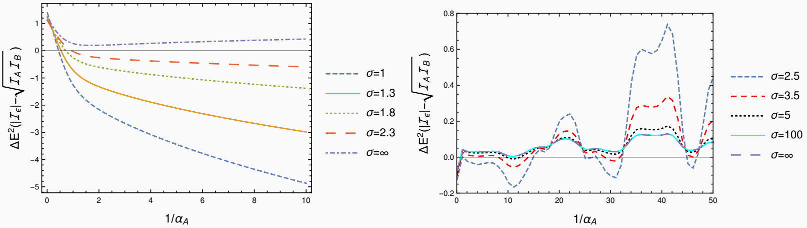
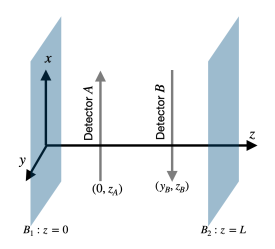


Figure 2: In $(1+1)$ and $(1+3)$ dimensionless concurrence quantity $\Delta E^2\left(|\mathcal{I}_\varepsilon| - \sqrt{\mathcal{I}_A\mathcal{I}_B}\right)$ is plotted with respect to the acceleration of the first detector α_A for different fixed inverse temperature of the thermal bath σ. The other parameters are fixed at $\alpha_B = 1$. (D. Barman, S. Barman and B.R. Majhi, JHEP, 2021)

Role of Reflecting boundaries

Systems with reflecting boundaries

- Two boundary system: B_1 at $z = 0$ and B_2 at $z = L$.
- One boundary system: B_1 at $z = 0$.
- No boundary system.



The trajectories of the detectors uniformly accelerating along the x-direction (Birrell, Davies, 1982)

$$t_A = a_A^{-1}\sinh(a_A\tau_A),\ x_A = a_A^{-1}\cosh(a_A\tau_A),\ y_A = 0,\ z_A = z_A\,;$$
$$t_B = a_B^{-1}\sinh(a_B\tau_B),\ x_B = \pm a_B^{-1}\cosh(a_B\tau_B),\ y_B = \Delta y,\ z_B = z_B. \quad (13)$$

where $0 < z_A,\, z_B < L$.

The Green's functions for two-boundary system (Birrell, Davies, 1982)

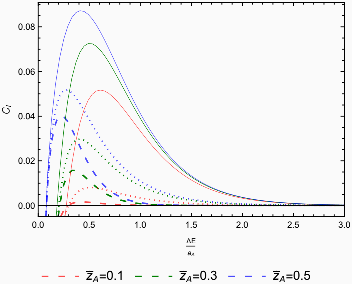
$$G_{W_{B_2}}(x,x') = -\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\Bigg(\frac{1}{(t-t'-i\epsilon)^2-(x-x')^2-(y-y')^2-(z-z'-2L\,n)^2} - \frac{1}{(t-t'-i\epsilon)^2-(x-x')^2-(y-y')^2-(z+z'-2L\,n)^2}\Bigg). \tag{14}$$

The term corresponds to $n = 0$ is the Green's function for one-boundary system, $G_{W_{B_1}}(x,x')$. (Birrell, Davies, 1982)

- Using these green functions, we calculate $\mathcal{P}_j$, $\mathcal{E}$.
  - For parallel motion: $\mathcal{E} = 0 \implies$ Entanglement harvesting not possible.
  - For anti-parallel motion: Entanglement harvesting possible.
- Dimensionless quantities:
  $\bar{z} = z\Delta E$, $\bar{L} = L\Delta E$, $a_j/\Delta E$ and $\mathcal{C}_I = \mathcal{C}_J/2\lambda^2\delta(0)$.

Results: effect of reflecting boundaries

Case-I: Both detectors are equally distanced from the $\bar{z} = \bar{L}/2$ plane.



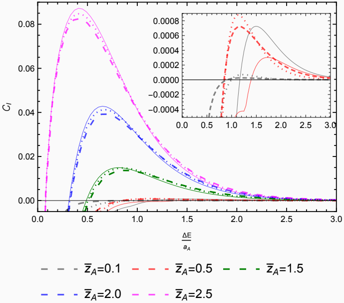


Figure 3: We plotted $\mathcal{C}_I = \mathcal{C}_J/2\lambda^2\delta(0)$ with respect to the dimensionless inverse acceleration $\Delta E/a_A$: (a) for $\bar{L} = 1.0$ and (b) for $\bar{L} = 5.0$, respectively. Different colours are used for different fixed values of $\bar{z}_A$ with $\bar{z}_B = \bar{L} - \bar{z}_A$ ($\bar{\Delta y} = 0.1$). Here we used solid, dotted and dashed lines to represent no boundary, single boundary and double boundary systems, respectively. (D. Barman and B.R. Majhi, Phys. Rev. D, 2023)

Case-II: The detectors have fixed perpendicular separation (i.e., $\bar{z}_A = \bar{z}_B$).

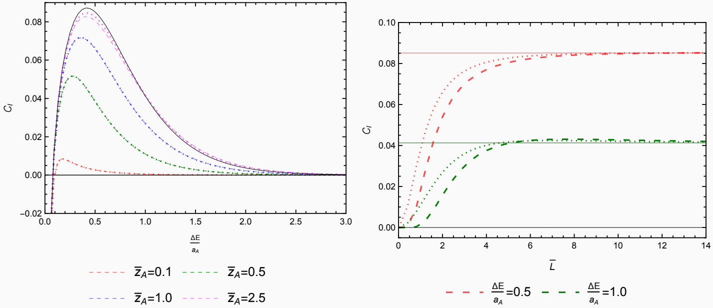


Figure 4: (a) We plotted $\mathcal{C}_I$ with respect to $\Delta E/a_A$ with $\bar{L} = 5.0$ and $\bar{z}_A = \bar{z}_B$. Different colours are used for different $\bar{z}_A$ values. (b) We plotted $\mathcal{C}_I$ with respect to $\bar{L}$ with consideration of $\bar{z}_A = \bar{z}_B = \bar{L}/2$ ($\bar{\Delta y} = 0.1$). Different colours are used for different fixed $\Delta E/a_A$ values. Here we used solid, dotted and dashed lines to represent no boundary, single boundary and double boundary systems, respectively. (D. Barman and B.R. Majhi, Phys. Rev. D, 2023)

Case-III: The detector B is fixed at $\bar{z} = 5.0\,(\bar{L} = 10.0)$ and different $\bar{z}$-positions for detector A has taken.

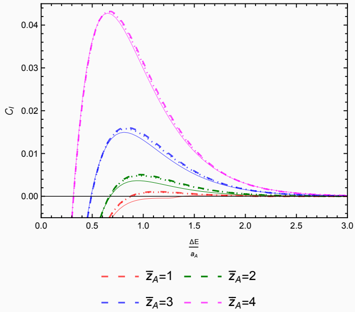


Figure 5: We plotted $\mathcal{C}_I$ with respect to $\Delta E/a_A$ and fixed values of $\bar{z}_A$. Here we used $\bar{z}_B = 5.0$ and $\bar{L} = 10.0$ ($\bar{\Delta y} = 0.1$). Different colours are used for different $\bar{z}_A$ values. Here we used solid, dotted and dashed lines to represent no boundary, single boundary and double boundary systems, respectively.
(D. Barman and B.R. Majhi, Phys. Rev. D, 2023)

Entanglement Leakage

Initially entangled system of two detectors

- Let us consider the initial state of the total system

$$|in\rangle = [\alpha|g_A g_B\rangle + \gamma|e_A e_B\rangle] \otimes |0_M\rangle \ , \tag{15}$$

with $\alpha^2 + \gamma^2 = 1$.

- The initial and final density matrices up to 2nd order in λ

$$\rho_{AB}(t_i) = \begin{pmatrix} \gamma^2 & 0 & 0 & \alpha\gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma\alpha & 0 & 0 & \alpha^2 \end{pmatrix} \ ; \ \rho_{AB}(t_f) = \begin{pmatrix} a_1 & 0 & 0 & a_2 \\ 0 & b_1 & b_2 & 0 \\ 0 & c_1 & c_2 & 0 \\ d_1 & 0 & 0 & d_2 \end{pmatrix} . \tag{16}$$

where density matrix elements(P. Chowdhury, B.R. Majhi, 2021)

$$a_1 = \gamma^2(1 - \lambda^2 P''_A - \lambda^2 P''_B) \, , \qquad b_1 = \gamma\gamma\lambda^2 P''_B \, , \qquad b_2 = \gamma\gamma\lambda^2 X^\star_{AB} = c^\star_1 \, ,$$
$$a_2 = \gamma\alpha(1 - \lambda^2 M_A - \lambda^2 M_B) = d^\star_1 \, , \qquad c_2 = \gamma\gamma\lambda^2 P''_A \, , \qquad d_2 = \alpha^2 \, . \tag{17}$$

Spontaneous entanglement leakage

$$P_j'' = \int\int d\tau_j d\tau_j' \, e^{-i\Delta E(\tau_j - \tau_j')} G_W(x_j', x_j) = \mathcal{P}_j(-\Delta E) \;;$$
$$M_j = \int\int d\tau_j d\tau_j' \, e^{i\Delta E(\tau_j - \tau_j')} \theta(\tau_j - \tau_j')(G_W(x_j', x_j) + G_W(x_j, x_j')) \;;$$
$$X_{AB} = \int\int d\tau_A d\tau_B' \, e^{i\Delta E(\tau_B' - \tau_A)} G_W(x_B', x_A) = \mathcal{P}_{AB}(-\Delta E). \qquad (18)$$

- $\underline{P_j''(= \mathcal{P}_j(-\Delta E))}$ denotes the transition probability from exited state to ground state of j^{th} detector.
  - the detectors are static→no contribution for detector's motion.
  - *spontaneous emission probability* (E.T. Akhmedov and D. Singleton, Pisma Zh. Eksp. Teor. Fiz., 2007).
- $\underline{M_j}$ contains $G_W(x_j', x_j) + G_W(x_j, x_j') = \langle 0_M | \{\phi(x_j'), \phi(x_j)\} | 0_M \rangle$.
  - depends on the field state under consideration.
  - M_j arises purely due to the *vacuum fluctuation* of field.

Expressions of the evaluated integrals

- Evaluation of the integrations yield

$$P_j'' = \frac{\delta(0)}{2c^3}\sqrt{\Delta E^2 - m^2c^4} \equiv P'' \ ;$$
$$\mathrm{Re}(M_j) = \frac{\delta(0)}{4c^3}\sqrt{\Delta E^2 - m^2c^4} \equiv M \ ;$$
$$X_{AB} = \frac{\delta(0)}{2c^3}\sqrt{\Delta E^2 - m^2c^4}\ \frac{\sin\left(\frac{d}{c}\sqrt{\Delta E^2 - m^2c^4}\right)}{\left(\frac{d}{c}\sqrt{\Delta E^2 - m^2c^4}\right)}, \qquad (19)$$

- If $\Delta E < mc^2$: density matrix is not valid.
- If $\Delta E = mc^2$: density matrix remains same.
- If $\Delta E > mc^2$: density matrix has evolved.

Negativity, a measure of entanglement

- *Negativity:* sum of all negative eigenvalues of partial transposed density matrix.
- partial transposition of ρ_{AB}:

$$\rho_{AB}^{T_B} = \begin{pmatrix} a_1 & 0 & 0 & b_2 \\ 0 & b_1 & a_2 & 0 \\ 0 & d_1 & c_2 & 0 \\ c_1 & 0 & 0 & d_2 \end{pmatrix} \tag{20}$$

- Eigenvalues:

$$\begin{aligned} \lambda_{1,2} &= \gamma^2\lambda^2 P'' \pm \alpha\gamma(1 - 2\lambda^2 M)\,, \\ \lambda_{3,4} &= \gamma^2(1 - 2\lambda^2 P''),\ \alpha^2\,. \end{aligned} \tag{21}$$

- The negativity, a measure of entanglement is given by

$$\mathcal{N} = max\left\{0,\ |\alpha\gamma| - \lambda^2(\gamma^2 P'' + 2|\alpha\gamma|M)\right\}\,. \tag{22}$$

- Entanglement between the detectors decreases with time.
(D. Barman, A. Choudhury, B. Kad, and B.R. Majhi, Phys. Rev. D, 2023)

Conclusion

Conclusion

- No influence on entanglement harvesting for parallel motion of the detectors.
- Background temperature introduces several interesting noticeable features which are absent when the temperature is zero.
- For $\alpha_A = \alpha_B$, in both $(1+1)$D and $(1+3)$D, background temperature suppress entanglement harvesting when acceleration is less than α_C. After the critical acceleration harvesting is higher for higher temperature.
- For fixed acceleration of detector B, there is single critical point in $(1+1)$D and multiple critical acceleration points in $(1+3)$D.

- In presence of mirrors, the entanglement gets suppressed if any one or both of the detectors are near the boundary or boundaries. The suppression decreases when they are away from the boundary or boundaries.
- One of the important observations is – the double boundary concurrence degrades more whenever there is a degradation. The same also holds for the enhancement of harvesting.

Conclusions

- For two initially entangled-static detectors, interacting eternally with background field, lose entanglement.
  – similar to the open quantum systems, where the environment causes decoherence for the quantum system[2].
- The leakage is unavoidable even for other type of switching function related to interaction.
- *in black hole spacetimes*
  – A Minkowski observer is equivalent to a freely falling observer in black hole spacetime.
  – two initially entangled qubits' communication fades during their free fall towards the horizon.
- Further studies may help us in understanding the black hole information paradox problem.

[2] **W.H. Zurek**; **Phys. Today 44**; **10**; **36 (1991)**; **Rev. Mod. Phys. 75**; **715 (2003)**.

References

- **Role of thermal field in entanglement harvesting between two accelerated Unruh-DeWitt detectors,** D. Barman, S. Barman and B.R. Majhi, *JHEP 07(2021)124, arXiv:2104.11269 [gr-qc].*
- **Are multiple reflecting boundaries capable of enhancing entanglement harvesting?,** D. Barman and B.R. Majhi, *Phys. Rev. D 108, 085007, arXiv:2306.09943 [gr-qc]* .
- **Spontaneous entanglement leakage of two static entangled Unruh-DeWitt detectors,** D. Barman, A. Choudhury, B. Kad, and B.R. Majhi, *Phys. Rev. D 107, 045001 (2023), arXiv:2211.00383 [quant-ph]*.

Thank You

Any Questions?

Additional Slides: Concurrence[3]

Concurrence:

$$\mathcal{C}(\rho_{AB}) = \text{Max}\{0,\, \lambda_{Max} - \lambda_2 - \lambda_3 - \lambda_4\}$$

where λ's are square-root of eigenvalues of

$$\rho_{AB}\, \tilde{\rho}_{AB}$$

with

$$\tilde{\rho}_{AB} = \sigma_y \otimes \sigma_y\, \rho^{\star}_{AB}\, \sigma_y \otimes \sigma_y$$

[3]**S. A. Hill and W. K. Wootters**; **PRL**; **1997; W. K. Wootters**; **PRL**; **1998**.

Density matrix elements: Reflecting boundaries

- The transition probability:

$$\mathcal{P}_j = \frac{\lambda^2\delta(0)}{2\left(e^{\frac{2\pi\Delta E}{a}}-1\right)}\sum_{n=-\infty}^{\infty}\left(\frac{\sin\left(\frac{2\Delta E}{a}\sinh^{-1}(|L\,a\,n|)\right)}{|L\,a\,n|\sqrt{|L\,a\,n|^2+1}}-\frac{\sin\left(\frac{2\Delta E}{a}\sinh^{-1}(|a(z_j+L\,n)|)\right)}{|a(z_j+L\,n)|\sqrt{|a(z_j+L\,n)|^2+1}}\right). \quad (23)$$

- The entangling term (anti-parallel motion):

$$\mathcal{E}(\Delta E) = -\frac{\lambda^2}{2}\frac{\delta\left(\frac{\Delta E}{a_A}-\frac{\Delta E}{a_B}\right)}{\sinh\left(\pi\frac{\Delta E}{a_A}\right)}\sum_{n=-\infty}^{\infty}\left(\frac{\sin\left(\frac{\Delta E\,\sigma_{n,-}}{a_A}\right)}{\sinh(\sigma_{n,-})}-\frac{\sin\left(\frac{\Delta E\,\sigma_{n,+}}{a_A}\right)}{\sinh(\sigma_{n,+})}\right) \quad (24)$$

with

$$\sigma_{n,\pm} = \log\left(M_{n,\pm}+\sqrt{M_{n,\pm}^2-1}\right);\; M_{n,\pm} = \frac{1}{2}\left(\frac{a_A}{a_B}+\frac{a_B}{a_A}+a_Aa_B\{\Delta y^2+(z_A\pm z_B-2Ln)^2\}\right)\;.$$

Density matrix elements: Thermal $(1+1)D$

- The transition probability:

$$\mathcal{P}_j = \lambda^2 \delta(0) \frac{\pi}{2\Delta E^j a_j} \frac{1}{\sinh \frac{\pi \Delta E^j}{a_j}} \left[\frac{e^{-\frac{\pi \Delta E^j}{a_j}}}{1 - e^{-\beta \Delta E^j}} \; + \; \frac{e^{\frac{\pi \Delta E^j}{a_j}}}{e^{\beta \Delta E^j} - 1} \right], . \quad (25)$$

- The entangling term (anti-parallel motion):

$$\mathcal{E}(\Delta E) = -\lambda^2 \frac{\delta\left(\frac{\Delta E^B - \Delta E^A}{\sqrt{a_A a_B}}\right)}{\sqrt{\sinh \frac{\pi \Delta \widetilde{E}}{a_A} \sinh \frac{\pi \Delta \widetilde{E}}{a_B}}} \frac{\pi}{\Delta \widetilde{E} \sqrt{a_A a_B}} \left[\frac{e^{\frac{\pi \Delta \widetilde{E}}{2}\left(\frac{1}{a_B} - \frac{1}{a_A}\right)}}{1 - e^{-\beta \Delta \widetilde{E}}} + \frac{e^{-\frac{\pi \Delta \widetilde{E}}{2}\left(\frac{1}{a_B} - \frac{1}{a_A}\right)}}{e^{\beta \Delta \widetilde{E}} - 1} - \sinh\left\{ \frac{\pi \Delta \widetilde{E}}{2} \left(\frac{1}{a_B} - \frac{1}{a_A} \right) \right\} \right] \quad (26)$$

Density matrix elements: Thermal $(1+3)D$

- The transition probability:

$$\mathcal{P}_j = \lambda^2 \delta(0) \frac{1}{2\pi a_j^2} \left[\frac{e^{-\frac{\pi \Delta E^j}{a_j}}}{1 - e^{-\beta \Delta E^j}} + \frac{e^{\frac{\pi \Delta E^j}{a_j}}}{e^{\beta \Delta E^j} - 1} \right] \Upsilon\left(\Delta E^j, a_j, a_j\right) \;,,. \tag{27}$$

with

$$\Upsilon\left(\bar{\varepsilon}, a_j, a_l\right) = \int_0^\infty k_p \; dk_p \; \mathcal{K}\left[\frac{i\bar{\varepsilon}}{a_j}, \frac{k_p}{a_j}\right] \mathcal{K}\left[\frac{i\bar{\varepsilon}}{a_l}, \frac{k_p}{a_l}\right]. \tag{28}$$

- The entangling term (anti-parallel motion):

$$\mathcal{E}(\Delta E) = -\lambda^2 \delta\left(\frac{\Delta E^B - \Delta E^A}{\sqrt{a_A a_B}}\right) \frac{\Upsilon\left(\Delta\widetilde{E}, a_B, a_A\right)}{\pi a_A a_B} \left[\frac{e^{\frac{\pi \Delta \widetilde{E}}{2}\left(\frac{1}{a_B} - \frac{1}{a_A}\right)}}{1 - e^{-\beta \Delta \widetilde{E}}} + \frac{e^{-\frac{\pi \Delta \widetilde{E}}{2}\left(\frac{1}{a_B} - \frac{1}{a_A}\right)}}{e^{\beta \Delta \widetilde{E}} - 1} - \sinh\left\{\frac{\pi \Delta \widetilde{E}}{2}\left(\frac{1}{a_B} - \frac{1}{a_A}\right)\right\} \right], \tag{29}$$