

Non-local operators and area law in CFT

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based on arXiv: [2007.15380](#), [2001.05129](#), [1911.11487](#), [1907.00646](#)

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Area law in physics

- Area law
 - keypoint to understand gravitational physics & holography.
 - relates geometry to physics
- Diverse area laws in physics
 - Black hole physics:

Bekenstein, Hawking, 1970'

$$S_{BH} = \frac{A}{4G_N}. \quad (1.1)$$

- Geometric entanglement entropy:

Bombelli, Koul, Lee, Sorkin, Srednicki, Callan, Wilczek, etc, 1990'

$$S_{EE} = \gamma \frac{A}{\epsilon^{d-2}} + \cdots + p \log \frac{R}{\epsilon} + \cdots \quad (1.2)$$

- Holographic description of entanglement entropy:

Ryu & Takayanagi 2006

$$S_{RT} = \frac{A}{4G_N} + \text{qc}. \quad (1.3)$$

Q: is there any other area law in physics?

- The similarity of the area laws

$$\text{Area} \sim \text{Entropy}. \quad (1.4)$$

- New area laws

$$\text{Area} \sim \text{CCF} \quad (1.5)$$

- Area law of entanglement entropy becomes a limit of the new area law.

Area law and non-local operators

- Entanglement entropy and modular Hamiltonian

$$S_A^{(n)} = \frac{1}{1-n} \log \text{tr}_A \rho_A^n = \frac{1}{1-n} \log \text{tr}_A e^{-nH_A}. \quad (1.6)$$

- Modular Hamiltonian $H_A = -\log \rho_A$.

- Non-local operator in A .
- Half plane

J.Bisognano & E.Wichmann, 1976

$$H_A = 2\pi \int_{x>0} dx d^{d-2} \vec{y} \times T_{00}. \quad (1.7)$$

- Spherical region Σ_A (CFT) Casini, Huerta & Myers, 1102.0440

$$H_A = 2\pi \int_{\Sigma_A} d^{d-1} \vec{x} \frac{R^2 - (\vec{x} - \vec{x}_0)^2}{2R} T_{00}. \quad (1.8)$$

- AdS gravity with a CFT dual Jafferis, etc, 1512.06431

$$H_A = \frac{A}{4G_N} + \mathcal{O}(G_N^0) \quad (1.9)$$

- Modular Hamiltonian is a special **OPE block** for spherical region. ≡ ↺ ↻ ↶ ↷

Definition

- Primary operators $\mathcal{O}$ with quantum number Δ, J .
- Operator product expansion (OPE)
 - two scalar primary operators

$$\begin{aligned}
 \mathcal{O}_i(x_1)\mathcal{O}_j(x_2) &= \sum_k C_{ijk}|x_{12}|^{\Delta_k-\Delta_i-\Delta_j}(\mathcal{O}_k(x_2) + \dots) \\
 &= |x_1 - x_2|^{-\Delta_i-\Delta_j} \sum_k C_{ijk} Q_k^{ij}(x_1, x_2) \quad (2.1)
 \end{aligned}$$

- OPE block: $Q_k^{ij}(x_1, x_2)$. Czech, etc, 1604.03110
 - depends on the external operators
 - depends on the insertion points
 - dimension zero & non-local operator
 - special case, $i = j$, it is independent of the external operators.

$$Q_A[\mathcal{O}_k] = Q_k^{ii}(x_1, x_2). \quad (2.2)$$

- $A \leftrightarrow (x_1, x_2)$

Timelike pair and causal diamond

- A timelike pair is in one-to-one correspondence to a causal diamond.

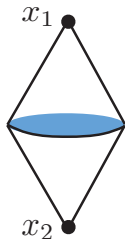


Figure: A timelike pair and causal diamond, from 1604.03310

- Diamond A is invariant under the action of conformal Killing vector

$$K^\mu = \frac{1}{2R}(R^2 - (\vec{x} - \vec{x}_0)^2, -2t(\vec{x} - \vec{x}_0)) \quad (2.3)$$

OPE block

- OPE block with equal external primary operator
de Boer, etc, 1606.03307
- In general, $\partial \cdot \mathcal{O} \neq 0 \rightarrow$ Type-O OPE block

$$Q_A[\mathcal{O}_{\mu_1 \dots \mu_J}] = \int_A d^d x K^{\mu_1} \dots K^{\mu_J} |K|^{\Delta-d-J} \mathcal{O}_{\mu_1 \dots \mu_J}, \quad (2.4)$$

- conserved current $\partial \cdot \mathcal{J} = 0 \rightarrow$ Type-J OPE block

$$Q_A[\mathcal{J}_{\mu_1 \dots \mu_J}] = \int_{\Sigma_A} d^{d-1} \vec{x} (K^0)^{J-1} \mathcal{J}_{0 \dots 0}. \quad (2.5)$$

- Modular Hamiltonian is a special Type-J OPE block for spherical region
- generated by stress tensor, $\partial_\mu T^{\mu\nu} = 0$.

Deformed reduced density matrix

- Reduced density matrix is the exponential operator of modular Hamiltonian $\rho_A = e^{-H_A}$.
- Replace modular Hamiltonian by a general OPE block $\rightarrow$ deformed reduced density matrix

$$\rho_A = e^{-\mu Q_A}. \quad (3.1)$$

- Q_A could be a linear superposition of OPE blocks.
- when $[H_A, Q_A] = 0$, μ could be interpreted as the chemical potential which is dual to Q_A .
- not always well defined if Q_A has no lower bound
- a formal generator of connected correlation function (CCF)

$$T_A(\mu) = \log \langle e^{-\mu Q_A} \rangle \quad (3.2)$$

- (m) -type CCF

$$\langle Q_A^m \rangle_c = (-1)^m \partial_\mu^m T_A(\mu)|_{\mu=0}. \quad (3.3)$$

Connected correlation function (CCF)

- several spacelike separated regions $A, B, C, \dots$
- m_1 OPE blocks in A , m_2 OPE blocks in B , etc.
- symmetry between separated regions, $m_1 \geq m_2 \geq \dots \geq m_n \geq 1$.
- $Y = (m_1, m_2, \dots, m_n)$
- Y -type CCF

$$\langle Q_A^{m_1} Q_B^{m_2} \dots \rangle_c \quad (3.4)$$

- conformal symmetry constrains $(m, 1)$ -type CCF is always a conformal block

$$\langle Q_A[\mathcal{O}_1] \dots Q_A[\mathcal{O}_m] Q_B[\mathcal{O}] \rangle_c = D[\mathcal{O}_1, \dots, \mathcal{O}_m, \mathcal{O}] G_{\Delta, J}(z). \quad (3.5)$$

- $G_{\Delta, J}$ is the conformal block associated with primary operator $\mathcal{O}$.
- z denotes the cross ratio related to two diamonds A and B .

(m) -type CCF of modular Hamiltonian (I)

- Rényi entanglement entropy is the generator of (m) -type CCF for modular Hamiltonian.

$$\langle H_A^m \rangle_c = (-1)^m \partial_n^m (1-n) S_A^{(n)}|_{n \rightarrow 1} \quad (3.6)$$

- Area law of Rényi EE

$$S_A^{(n)} = \gamma(n) \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + p(n) \log \frac{R}{\epsilon} + \cdots . \quad (3.7)$$

- (m) -type CCF of modular Hamiltonian should also obey area law

$$\langle H_A^m \rangle_c = \tilde{\gamma} \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + \tilde{p} \log \frac{R}{\epsilon} + \cdots . \quad (3.8)$$

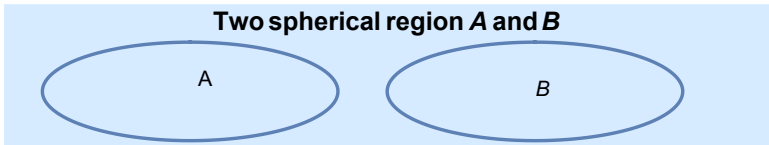
(m) -type CCF of modular Hamiltonian (II)

- An argument without reference to Rényi EE
- $(m-1, 1)$ -type CCF is always conformal block, especially for modular Hamiltonian

$$\langle H_A^{m-1} H_B \rangle_c = D[T_{\mu_1 \nu_1}, \dots, T_{\mu_m \nu_m}] G_{d,2}(z). \quad (3.9)$$

- Choose A and B as follows

$$A = \{(0, \vec{x}) | (\vec{x} - \vec{x}_A)^2 \leq R^2\}, \quad B = \{(0, \vec{x}) | \vec{x}^2 \leq R'^2\} \quad (3.10)$$



- A and B are spacelike separated, the cross ratio $0 < z < 1$.

$$z = \frac{4RR'}{x_A^2 - (R - R')^2} \quad (3.11)$$

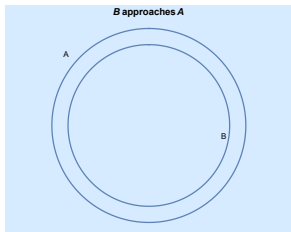
(m) -type CCF of modular Hamiltonian (III)

- consider the limit $B \rightarrow A$,

$$\langle H_A^{m-1} H_B \rangle_c \rightarrow \langle H_A^m \rangle_c, \quad B \rightarrow A \quad (3.12)$$

- need a way to continue conformal block
- $x_A = 0$, then $R' \rightarrow R$ through

$$R' = R - \epsilon, \quad z = \frac{4R(R - \epsilon)}{-\epsilon^2} \sim -\frac{R^2}{\epsilon^2} \rightarrow -\infty \quad (3.13)$$



- continue conformal block $G_{d,2}(z)$ to the region $z \rightarrow -\infty$

(m) -type CCF of modular Hamiltonian (IV)

- area law from the continuation of conformal block

$$\begin{aligned}
 \langle H_A^m \rangle_c &= \lim_{B \rightarrow A} \langle H_A^{m-1} H_B \rangle_c = \lim_{z \rightarrow -\frac{R^2}{\epsilon^2}} D[T_{\mu_1 \nu_1}, \dots, T_{\mu_m \nu_m}] G_{d,2}(z) \\
 &= \gamma \frac{R^2}{\epsilon^2} + \dots + p_1^e \log \frac{R}{\epsilon} + \dots, \quad d = 4.
 \end{aligned} \tag{3.14}$$

with

$$p_1^e = -120D. \tag{3.15}$$

- we obtain area law from continuation of $(m-1, 1)$ -type CCF.
- D : leading behaviour when A and B are far away (IR)
- p_1^e : cutoff independent coefficient when A and B are the same (UV).
- -120 is from the continuation of conformal block which is fixed by conformal symmetry.
- $p \sim E \times D$, a typical UV/IR relation

(m) -type CCF of OPE block (I)

- Modular Hamiltonian H_A is a type-J OPE block for spherical region
- (m) -type CCF of modular Hamiltonian obeys area law
- More general (m) -type CCF

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_m] \rangle_c \quad (4.1)$$

where $Q_A[\mathcal{O}_i]$ belong to the same type of OPE block.

- Consider $(m-1, 1)$ -type CCF

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_{m-1}] Q_B[\mathcal{O}_m] \rangle_c = D[\mathcal{O}_1, \cdots, \mathcal{O}_m] G_{\Delta_m, J_m}(z). \quad (4.2)$$

- Continuation for general conformal block

(m) -type CCF of OPE block (II)

- We obtain the following behaviour

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_m] \rangle_c = \gamma \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + p_q[\mathcal{O}_1, \cdots, \mathcal{O}_m] \log^q \frac{R}{\epsilon} + \cdots \quad (4.3)$$

- If the coefficient D is **finite**,
 - Leading term obeys area law
 - q : **maximal** power of the logarithmic term, **degree** of the (m) -type CCF.

$$q = \begin{cases} 1, & \text{type-J \& d even.} \\ 2, & \text{type-O \& d even.} \\ 0, & \text{type-J \& d odd.} \\ 1, & \text{type-O \& d odd.} \end{cases} \quad (4.4)$$

- q is fixed by the conformal block associated with $\mathcal{O}_m$.
- coefficient p_q is cutoff independent.
- UV/IR** relation

$$p_q[\mathcal{O}_1, \cdots, \mathcal{O}_m] = E[\mathcal{O}_m] D[\mathcal{O}_1, \cdots, \mathcal{O}_m] \quad (4.5)$$

UV/IR relation

- $p = E \times D$.
- D: leading behaviour of $(m-1, 1)$ -type CCF when A and B are far apart (IR aspect)
- p: cutoff independent coefficient in the subleading term of (m) -type CCF, A and B should be the same (UV aspect)
- E : encodes kinematic information
 - E can be obtained from analytic continuation of conformal block. For example, for type-J OPE block in four dimensions

$$E[\mathcal{O}] = \begin{cases} 12, & \Delta = 3, J = 1. \\ -120, & \Delta = 4, J = 2. \\ \dots \end{cases} \quad (4.6)$$

UV/IR relation: $p = E \times D$

- For type-O OPE block in four dimensions

$$E[\mathcal{O}] = \begin{cases} -\frac{2^{2\Delta-1}\Gamma(\frac{\Delta-1}{2})\Gamma(\frac{\Delta+1}{2})}{\pi\Gamma(\frac{\Delta-2}{2})^2}, & \Delta > 1, \quad J = 0. \\ \frac{2^{2\Delta-1}\Gamma(\frac{\Delta}{2})\Gamma(\frac{\Delta+2}{2})}{\pi\Gamma(\frac{\Delta-3}{2})\Gamma(\frac{\Delta+1}{2})}, & \Delta > 3, \quad J = 1. \\ -\frac{4^{\Delta-1}(\Delta-2)\Gamma(\frac{\Delta-3}{2})\Gamma(\frac{\Delta+3}{2})}{\pi\Gamma(\frac{\Delta-4}{2})\Gamma(\frac{\Delta+2}{2})}, & \Delta > 4, \quad J = 2. \\ \dots & \end{cases} \quad (4.7)$$

- Unitary bound** of scalar operator in four dimensions $\Delta \geq 1$
 - Note: $\Delta \rightarrow 1$, E is divergent.
- Unitary bound of a non-conserved primary current in four dimensions $\Delta > J + 2$. [S.Minwalla, 9712074](#)
 - limiting behaviour: $\Delta \rightarrow J + 2$, $E[\mathcal{O}] \rightarrow 0$, $p_2 \rightarrow 0$ for finite D . The degree $q = 2$ becomes $q = 1$.

UV/IR relation: $p = E \times D$

- The relation is 'asymmetric' since E just depends on operator $\mathcal{O}_m$.

$$p_q[\mathcal{O}_1, \dots, \mathcal{O}_m] = E[\mathcal{O}_m] D[\mathcal{O}_1, \dots, \mathcal{O}_m] \quad (4.8)$$

- There should be m different ways to move one OPE block to region B , for example

$$\langle Q_A[\mathcal{O}_2] \cdots Q_A[\mathcal{O}_m] Q_B[\mathcal{O}_1] \rangle_c \quad (4.9)$$

leads to another UV/IR relation

$$p_q[\mathcal{O}_2, \dots, \mathcal{O}_m, \mathcal{O}_1] = E[\mathcal{O}_1] D[\mathcal{O}_2, \dots, \mathcal{O}_m, \mathcal{O}_1]. \quad (4.10)$$

- p_q is cutoff independent
- Cyclic identity** ($m = 3$ as an example)

$$p_q[\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_1] = p_q[\mathcal{O}_3, \mathcal{O}_1, \mathcal{O}_2] = p_q[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3]. \quad (4.11)$$

- The leading term coefficient γ is cutoff dependent, it doesn't satisfy cyclic identity

$$\gamma[\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_1] \neq \gamma[\mathcal{O}_3, \mathcal{O}_1, \mathcal{O}_2] \neq \gamma[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3]. \quad (4.12)$$

Example 1: Chiral operators in CFT_2 (I)

- In two dimensions, the leading term of (m) -type CCF is already the logarithmic term.
- A chiral operator $\mathcal{O}$ in CFT_2 only depends on (anti-)holomorphic coordinate

$$\bar{\partial}\mathcal{O}(z) = 0 \quad (5.1)$$

- The ball in one spatial dimension is an interval, we assume the length is 2 and the center is 0.
- The corresponding OPE block is type-J.

$$Q_A[\mathcal{O}] = \int_{-1}^1 dz \left(\frac{1-z^2}{2} \right)^{h-1} \mathcal{O}(z). \quad (5.2)$$

- degree $q = 1$.

Chiral operators in CFT_2 (II)

- (2)-type, $\sqrt{\cdot}$; UV/IR relation $\sqrt{\cdot}$

$$p_1[\mathcal{O}, \mathcal{O}] = \frac{(-1)^{-h} \sqrt{\pi} \Gamma(h)}{\Gamma(h + \frac{1}{2})} N_{\mathcal{O}}.. \quad (5.3)$$

- (3)-type, using UV/IR relation

$$p_1[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3] = \frac{\pi^{3/2} 2^{3-h_1-h_2-h_3} (-1)^{\frac{h_1+h_2+h_3}{2}} \Gamma(h_1) \Gamma(h_2) \Gamma(h_3) \kappa C_{123}}{\Gamma(\frac{1+h_1+h_2-h_3}{2}) \Gamma(\frac{1+h_1+h_3-h_2}{2}) \Gamma(\frac{1+h_2+h_3-h_1}{2}) \Gamma(\frac{h_1+h_2+h_3}{2})} \quad (5.4)$$

with $\kappa = \frac{1+(-1)^{h_1+h_2+h_3}}{2}$.

- $\sqrt{\cdot}$ for h_i are integers and no larger than 6.
- cyclic identity $\sqrt{\cdot}$

$$p_1[\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_1] = p_1[\mathcal{O}_3, \mathcal{O}_1, \mathcal{O}_2] = p_1[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3] \quad (5.5)$$

- (4)-type, free scalar theory and theory with $\mathcal{W}$ symmetry $\sqrt{\cdot}$

Example 2: Non-conserved primary operator in CFT_4 (I)

- For a primary operator which is non-conserved, $Q_A[\mathcal{O}]$ is type-O.
- Degree $q = 2$.
- (2)-type

$$p_2[\mathcal{O}, \mathcal{O}] = \begin{cases} -\frac{4\pi^2(\Delta-1)\Gamma(\Delta-2)^2\Gamma(\frac{\Delta}{2})^4}{\Gamma(\Delta)^2\Gamma(\Delta-1)^2} N_{\mathcal{O}}, & J=0, \Delta \geq 1. \\ -\frac{4^{1-\Delta}\pi^3\Delta\Gamma(\frac{\Delta-3}{2})\Gamma(\frac{\Delta+1}{2})}{\Gamma(\frac{\Delta}{2}+1)^2} N_{\mathcal{O}}, & J=1, \Delta > 3. \\ -\frac{3\pi^4(\Delta-2)\Delta^2\Gamma(\frac{\Delta}{2}-2)^2\Gamma(\frac{\Delta}{2}-1)^2}{64\Gamma(\Delta-4)\Gamma(\Delta+2)} N_{\mathcal{O}}, & J=2, \Delta > 4. \\ \dots & \end{cases} \quad (5.6)$$

- They are obtained from UV/IR relation
- They can be checked by computing the integral directly with regularization for specific Δ .

Non-conserved primary operator in CFT₄ (II)

- (3)-type, scalar primary operator

$$p_2[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3] = C \int_{\mathbb{D}^2} d^2\mu_0 \int_{\mathbb{D}^2} d^2\mu'_0 \int_0^\pi d\theta \frac{\sin \theta}{(a + b \cos \theta)^{\frac{\Delta_{12,3}}{2}}},$$

where $C = -2^{4-\Delta_1-\Delta_2-\Delta_3} \pi^3 C_{123}$ and

$$\begin{aligned} d^2\mu_0 &= d\zeta d\bar{\zeta} (\zeta + \bar{\zeta})^2 (1 - \zeta^2)^{\frac{\Delta_1-4}{2}} (1 - \bar{\zeta}^2)^{\frac{\Delta_1-4}{2}}, \\ d^2\mu'_0 &= d\zeta' d\bar{\zeta}' (\zeta' + \bar{\zeta}')^2 (1 - \zeta'^2)^{\frac{\Delta_2-4}{2}} (1 - \bar{\zeta}'^2)^{\frac{\Delta_2-4}{2}} \end{aligned} \quad (5.7)$$

$$a = \zeta\bar{\zeta} + \zeta'\bar{\zeta}' + \frac{1}{2}(\zeta - \bar{\zeta})(\zeta' - \bar{\zeta}'), \quad b = -\frac{1}{2}(\zeta + \bar{\zeta})(\zeta' + \bar{\zeta}') \quad (5.8)$$

$$\Delta_{12,3} = \Delta_1 + \Delta_2 - \Delta_3 \quad (5.9)$$

- cyclic identity

$$p_2[4, 6, 8] = p_2[4, 8, 6] = p_2[8, 4, 6] = -\frac{\pi^3}{1728} C_{123}. \quad (5.10)$$

Example 3: Conserved currents in CFT₄ (I)

- OPE block is type-J, degree $q = 1$.
- (2)-type

$$p_1[\mathcal{J}, \mathcal{J}] = \begin{cases} -\frac{\pi^2}{3} C_J, & J = 1, \\ -\frac{\pi^2}{40} C_T, & J = 2, \\ \dots \end{cases} \quad (5.11)$$

- regularize the integral
- UV/IR relation
- For $J = 2$, stress tensor, this is consistent with universal results of modular Hamiltonian

E. Perlmutter, 1308.1083

$$\langle H_A^2 \rangle_c = -\frac{1}{2\pi^2} S'_{q=1} = -\frac{\pi^2}{40} C_T. \quad (5.12)$$

Conserved currents in CFT₄ (II)

- (3)-type

$$p_1[\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3] = \begin{cases} -\frac{\pi^3}{2} C_{T\mathcal{J}\mathcal{J}}, & J_1 = J_2 = 1, J_3 = 2. \\ \frac{\pi^3}{12} C_{TTT}, & J_1 = J_2 = J_3 = 2. \\ \dots \end{cases} \quad (5.13)$$

- Three point function for conserved currents

H.Osborn & A.C.Petkou, 9307010 J.Erdmenger & H.Osborn, 9605009

- Spin 1-1-2. Only two independent structures, a, b

$$C_{T\mathcal{J}\mathcal{J}} = \frac{3b - 4a}{8}. \quad (5.14)$$

- Spin 2-2-2. Only three independent structures, $\mathcal{A}, \mathcal{B}, \mathcal{C}$

$$C_{TTT} = \frac{-2(4 - 5d + 2d^2)\mathcal{A} + d\mathcal{B} + 2(5d - 4)\mathcal{C}}{4d^2}, \quad d = 4. \quad (5.15)$$

- $J_1 = J_2 = J_3$, universal results of modular Hamiltonian

J.Lee, A.Lewkowycz, E.Perlmutter & B.R.Safdi, 1407.7816

$$\langle Q_A[\mathcal{O}] \cdots Q_A[\mathcal{J}] \rangle_c$$

- In previous discussion, OPE blocks in CCF belong to the same type.
- Type-O & type-J
- A severe puzzle

$$\begin{aligned} \langle Q_A[\mathcal{O}] \cdots Q_A[\mathcal{J}] \rangle_c &\rightarrow \langle Q_A[\mathcal{O}] Q_A[\tilde{\mathcal{O}}] Q_B[\mathcal{J}] \rangle_c = D[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}] G_{\Delta, J}(z), \\ \langle Q_A[\mathcal{O}] \cdots Q_A[\mathcal{J}] \rangle_c &\rightarrow \langle Q_A[\tilde{\mathcal{O}}] Q_A[\mathcal{J}] Q_B[\mathcal{O}] \rangle_c = D[\tilde{\mathcal{O}}, \mathcal{J}, \mathcal{O}] G_{\Delta', J'}(z) \end{aligned}$$

- UV/IR relation, rather different degree q .

$$\begin{aligned} p_1[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}] &= E[\mathcal{J}] D[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}], \\ p_2[\tilde{\mathcal{O}}, \mathcal{O}, \mathcal{J}] &= E[\mathcal{O}] D[\tilde{\mathcal{O}}, \mathcal{J}, \mathcal{O}]. \end{aligned} \tag{6.1}$$

- $D[\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{J}]$ is **divergent**.

An example

- A simplest nontrivial CCF, spin 2-0-0.

$$\langle Q_A[T_{\mu\nu}]Q_A[\mathcal{O}]Q_A[\mathcal{O}] \rangle_c \quad (6.2)$$

- From $\langle Q_A[T_{\mu\nu}]Q_A[\mathcal{O}]Q_B[\mathcal{O}] \rangle_c$

$$D[T_{\mu\nu}, \mathcal{O}, \mathcal{O}] = -\frac{\pi^5 4^{3-2\Delta} \Gamma(\frac{\Delta}{2} - 1)^4}{\Delta(\Delta - 2) \Gamma(\frac{\Delta-3}{2}) \Gamma(\frac{\Delta-1}{2})^2 \Gamma(\frac{\Delta+1}{2})} a.$$

$$\Rightarrow p_2[T_{\mu\nu}, \mathcal{O}, \mathcal{O}] = \frac{2^{5-2\Delta} \pi^4 \Gamma(\frac{\Delta}{2} - 1)^2}{\Delta(\Delta - 2) \Gamma(\frac{\Delta-3}{2}) \Gamma(\frac{\Delta-1}{2})} a. \quad (6.3)$$

- From $\langle Q_A[\mathcal{O}]Q_A[\mathcal{O}]Q_B[T_{\mu\nu}] \rangle_c$

$$D[\mathcal{O}, \mathcal{O}, T_{\mu\nu}] = -\frac{\pi^3}{3840} a \log \frac{R}{\epsilon} + \dots \quad \text{for } \Delta = 4.$$

$$\Rightarrow p_2 = \frac{\pi^3}{32} a \quad (6.4)$$

- logarithmic divergence of D increases the degree by 1.

Divergent D

- The puzzle is from the assumption that D is always **finite**.
- The coefficient $D[\mathcal{O}, \mathcal{O}, T_{\mu\nu}]$ presents logarithmic divergence behaviour

$$\langle Q_A[\mathcal{O}]^2 Q_B[T_{\mu\nu}] \rangle_c \sim D_{\log}[\mathcal{O}, \mathcal{O}, T_{\mu\nu}] \log \frac{R}{\epsilon} G_{4,2}(z). \quad (6.5)$$

- D is finite when the OPE blocks belong to the same type
- $D[\mathcal{O}, \dots, \mathcal{J}]$ should present logarithmic divergence behaviour to cure the puzzle.
- The degree $q = 2$ rather than 1.
- UV/IR relation becomes

$$p_2[\mathcal{O}, \mathcal{O}, T_{\mu\nu}] = E[T_{\mu\nu}] D_{\log}[\mathcal{O}, \mathcal{O}, T_{\mu\nu}]. \quad (6.6)$$

Summary

- We find a new area law

$$\langle Q_A[\mathcal{O}_1] \cdots Q_A[\mathcal{O}_m] \rangle_c = \gamma \frac{R^{d-2}}{\epsilon^{d-2}} + \cdots + p_q[\mathcal{O}_1, \cdots, \mathcal{O}_m] \log^q \frac{R}{\epsilon} + \cdots \quad (7.1)$$

- We obtain UV/IR relation

$$p = E \times D. \quad (7.2)$$

- We check **cyclic identity** of p for $m = 3$.

$$p[\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3] = p[\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_1] = p[\mathcal{O}_3, \mathcal{O}_1, \mathcal{O}_2]. \quad (7.3)$$

Outlook

- Area law
 - $m \geq 4$?
 - CCF including both type-J and type-O OPE blocks ?
- Logarithmic behaviour
 - Degree $0 \leq q \leq 2$?
 - Logarithmic divergence of $(m, 1)$ -type CCF

$$\langle Q_A[\mathcal{O}] \cdots Q_B[\mathcal{J}] \rangle_c \stackrel{?}{=} (D_{\log}[\mathcal{O}, \cdots, \mathcal{J}] \log \frac{R}{\epsilon} + \cdots) G_{J+d-2,J}(z) \quad (7.4)$$

- UV/IR relation
 - Proof?
 - Modified version when D is divergent

$$p = E \times D_{\log}. \quad (7.5)$$

- Cyclic identity of p for general $m \geq 4$.

$$p[\mathcal{O}_1, \cdots, \mathcal{O}_m] = \cdots = p[\mathcal{O}_2, \cdots, \mathcal{O}_m, \mathcal{O}_1]. \quad (7.6)$$

- It is only checked using several examples in free scalar theory for $d = 2$.