7TH INTERNATIONAL CONFERENCE ON HOLOGRAPHY AND STRING THEORY IN DA NANG

Comments on Quantum Chaos and Complexity

Keun-Young Kim

2024.08. 22

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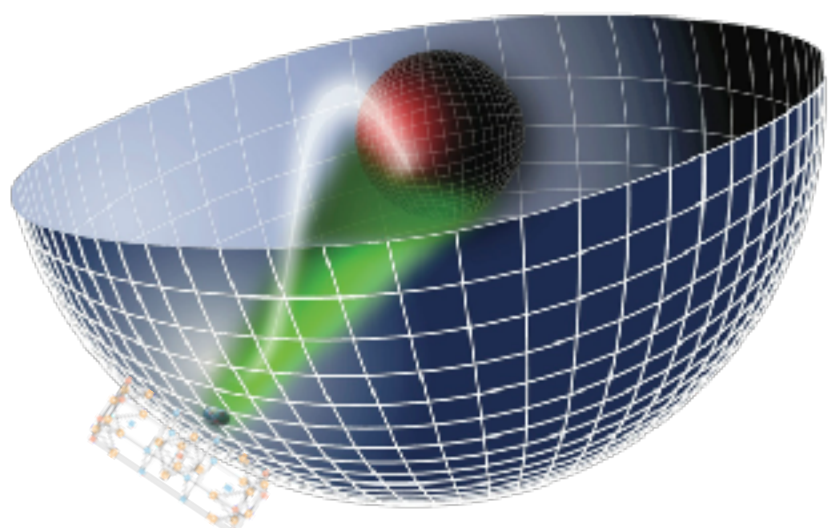
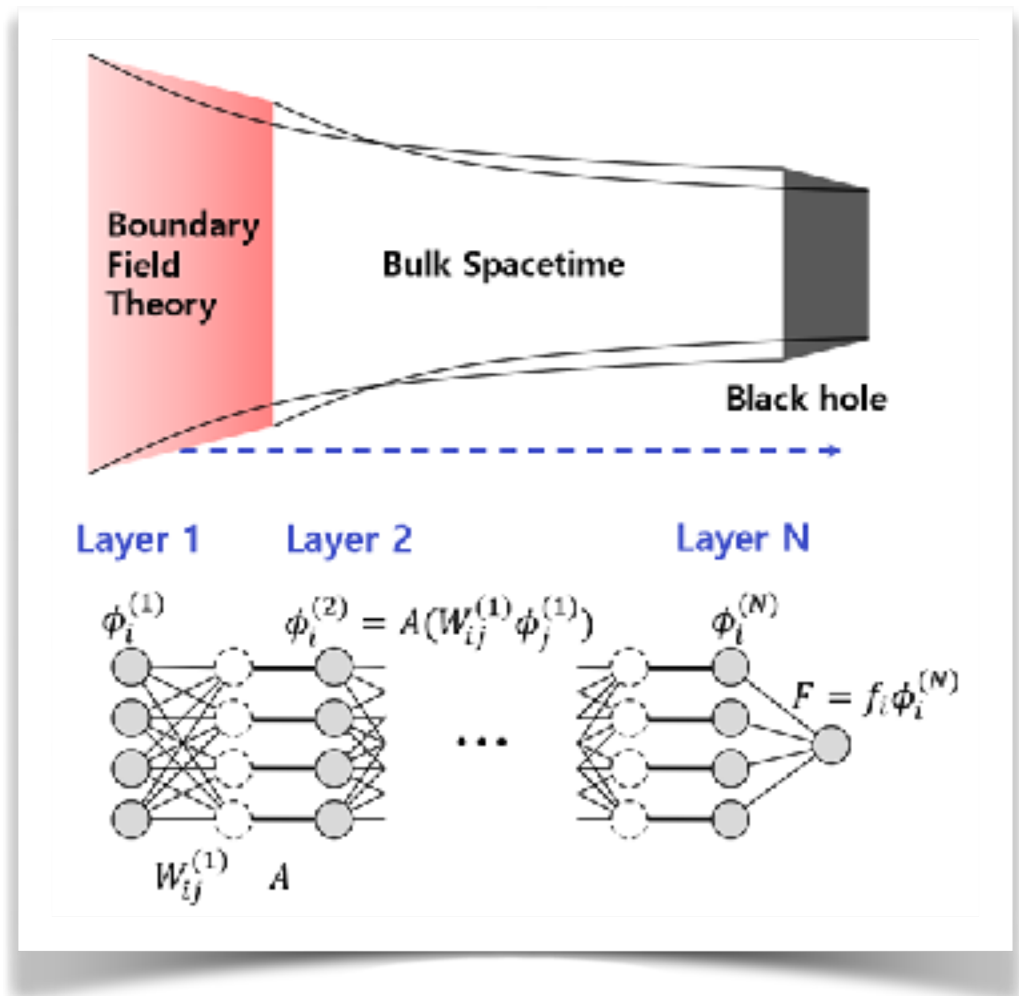
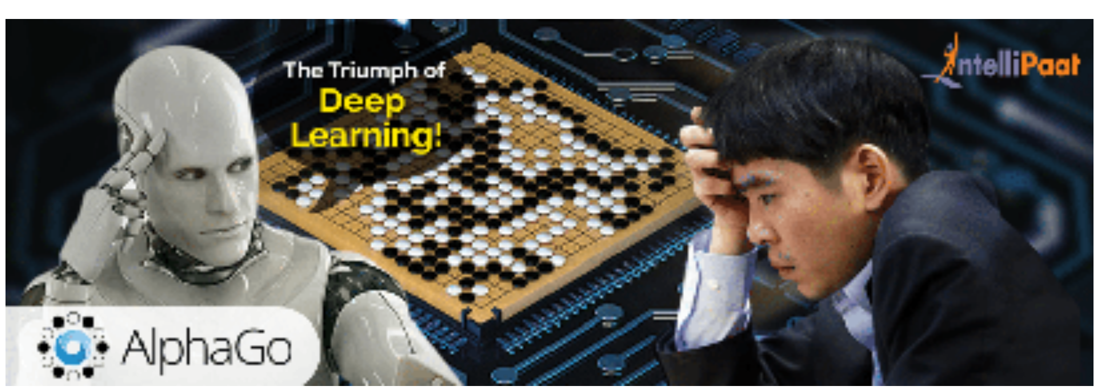






For fun,

Let's ask AI (Artificial Intelligence)

AI-painter "MidJourney" wins 1st prize at Colorado State Fair

"Space Opera Theater" by Midjourney

Quantum chaos and complexity

Midjourney

Quantum Chaos

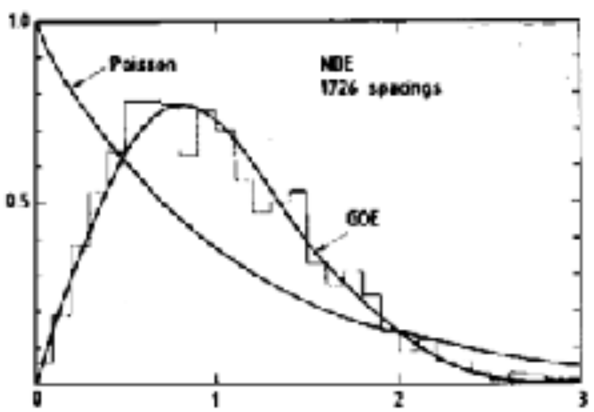
$$|\{q^i(t), p^j(0)\}_{PB} = \left|\frac{\partial q^i(t)}{\partial q^j(0)}\right| \sim e^{\lambda t}$$

$$-\langle [q^i(t), p^j(0)]^2 \rangle_\beta,$$

$$-\langle [V(t), W(0)]^2 \rangle_\beta \quad \sim e^{\lambda t}$$

Out-of-time-order correlator (OTOC)

Level spacing statistics



Random Matrix Theory

Thermalization
(ETH, Quantum device)
Quantum black holes
Quantum gravity

FO What is quantum chaos?

ChatGPT

Quantum chaos is a field of physics that studies the behavior of quantum systems that are classically chaotic. Classically chaotic systems are those that exhibit sensitive dependence on initial conditions, meaning small changes in the initial conditions lead to vastly different outcomes over time.

In quantum mechanics, chaotic behavior can manifest in the statistical properties of the system's energy spectrum or in the time evolution of its wave function. The study of quantum chaos is concerned with understanding how the behavior of the system changes as classical chaos is introduced or increased, and how this behavior can be observed and measured experimentally.

Quantum chaos has applications in various fields, including solid-state physics, quantum computing, and quantum information theory. It has also led to new insights into the fundamental nature of quantum mechanics and the connections between classical and quantum physics.

Comments on Quantum Chaos and Complexity

Now

Let's get more serious

arXiv > hep-th > arXiv:2405.11254

High Energy Physics – Theory

[Submitted on 18 May 2024 (v1), last revised 3 Jun 2024 (this version, v2)]

Spread and Spectral Complexity in Quantum Spin Chains: from Integrability to Chaos

Hugo A. Camargo, Kyoung-Bum Huh, Viktor Jahnke, Hyun-Sik Jeong, Keun-Young Kim, Mitsuhiro Nishida

arXiv > hep-th > arXiv:2407.17054

High Energy Physics – Theory

[Submitted on 24 Jul 2024 (v1), last revised 26 Jul 2024 (this version, v2)]

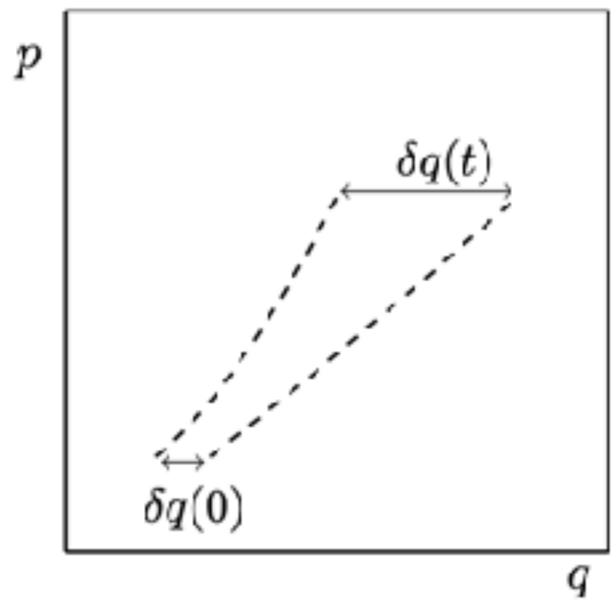
Krylov complexity as an order parameter for quantum chaotic-integrable transitions

Matteo Baggioli, Kyoung-Bum Huh, Hyun-Sik Jeong, Keun-Young Kim, Juan F. Pedraza

[Reminder]
Quantum Chaos
Why?

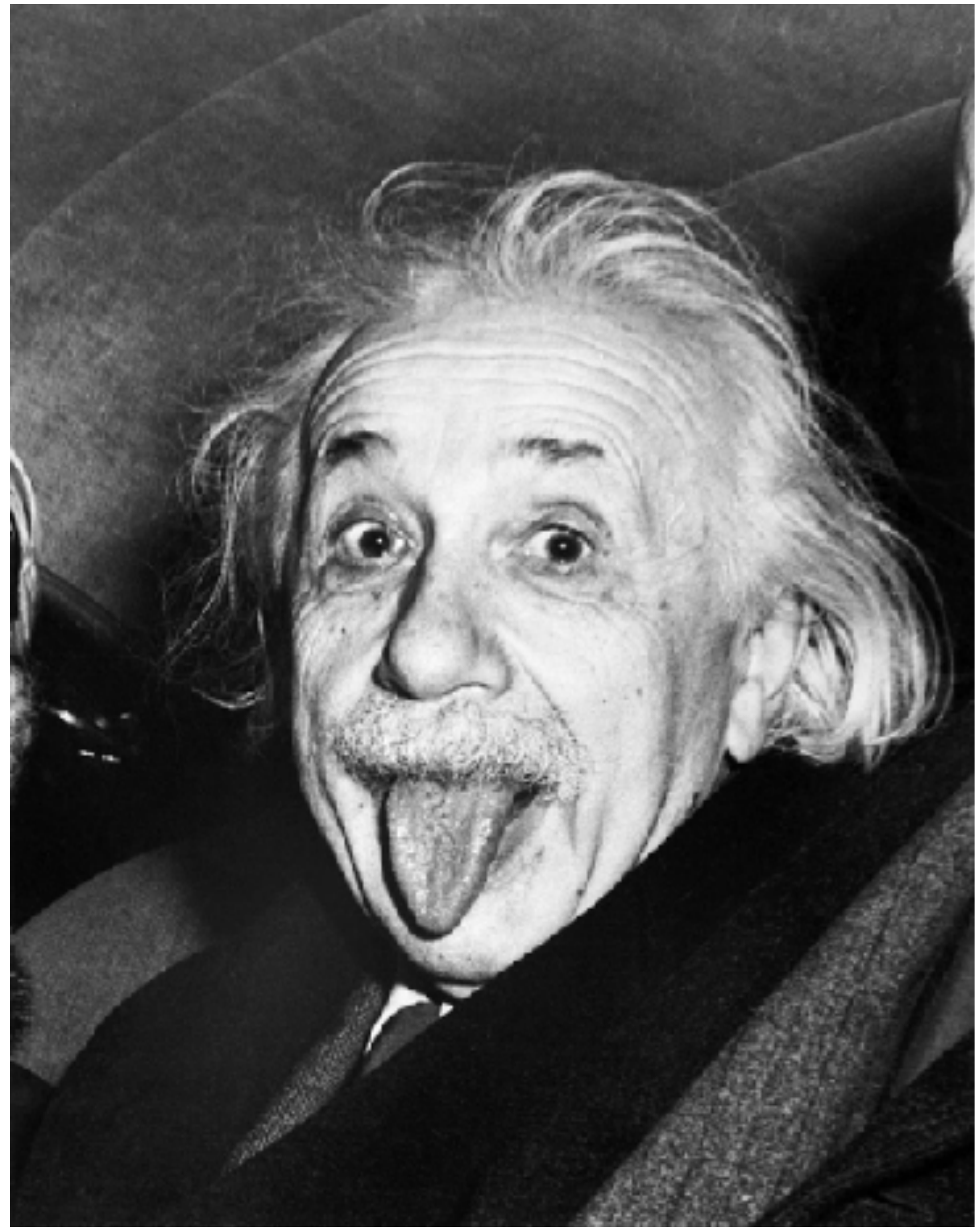
[Reminder]
Quantum Chaos

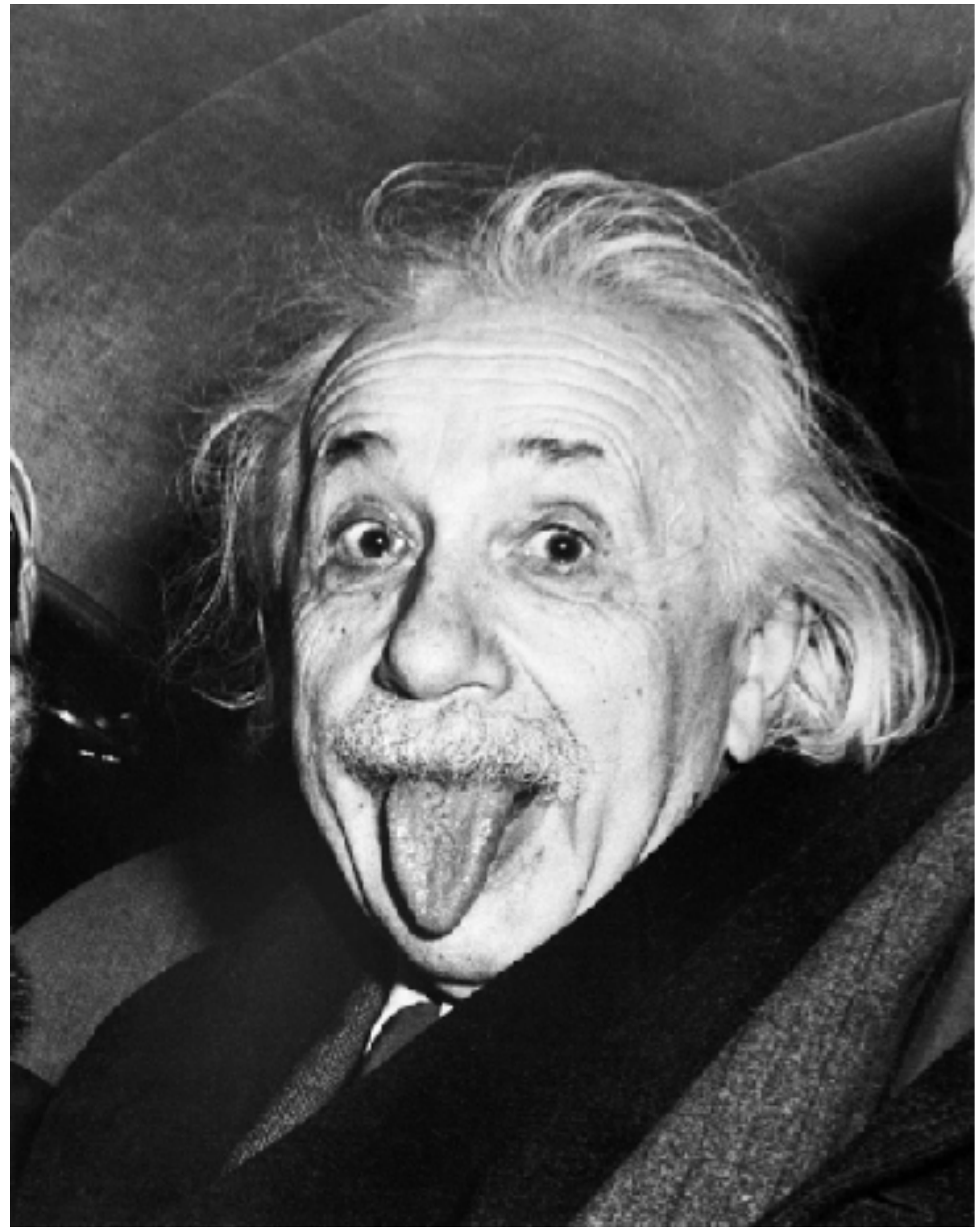
Classical Chaos $\xleftarrow{\hbar \to 0}$ Quantum chaos



- “Quantum” is more fundamental. “Classical” is approximation.
- Quantum chaos may exist even without classical counter part.
- We do not need to stick to classical concept.

Who is the first raising the issue of quantum chaos
In physics literature?

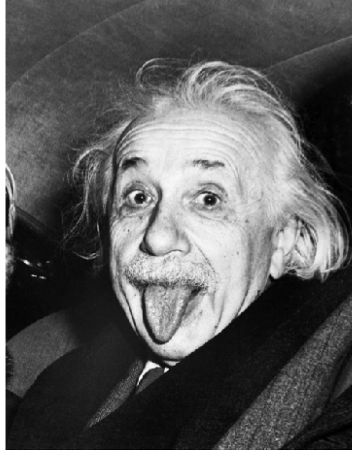




I have thought a hundred times as much about the quantum problems as I have about general relativity.

- 1905: Photon concept
- 1916: Quantum theory of radiation
- 1917: Quantum chaos
- 1925: Bose-Einstein condensation
- 1935: EPR paradox

First identification of the problem of quantizing chaotic motion



A. Einstein, *Zum quantensatz von sommerfeld und epstein*, *Deutsche physikalische Gesellschaft, Verhandlungen* **19** (1917) 82–92.

Before Schrödinger equation (1926)

434 DOC. 45 QUANTUM THEOREM

Doc. 45

[p. 82] **On the Quantum Theorem of Sommerfeld and Epstein**

by A. Einstein

(Presented at the session of May 11)

(cf. above, p. 79)

§1. Previous Formulation. There is hardly any more doubt that the quantum condition for periodic mechanical systems with one degree of freedom is (after SOMMERFELD [1] and DEBYE)

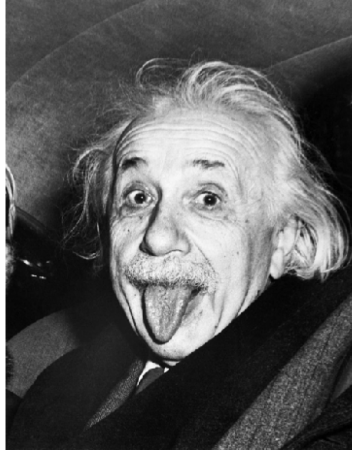
$$\int p dq = \int p \frac{dq}{dt} dt = nh. \tag{1}$$

Non-integrable (chaotic) $\longrightarrow$ How to quantize?

Non- chaotic $\longrightarrow$ How to thermalize in quantum system?

Not ergodic?

First identification of the problem of quantizing chaotic motion



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Forgotten for 55 years
And
Rediscovered (independently) in 1971

M. C. Gutzwiller, *Periodic orbits and classical quantization conditions, Journal of Mathematical Physics* **12** (1971) 343.

Regular and irregular spectra

I C Percival

Joint Institute for Laboratory Astrophysics, University of Colorado, Boulder, Colorado, 80302†
Physics Department, University of Stirling, Stirling, Scotland‡

Received 6 August 1973

$$I_k = (n_k + \text{constant})\hbar.$$

Apart from the phase space formulation and the possibility of a non-zero constant (Keller 1958) this result is due to Einstein (1917).

History of quantum chaos

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Michael Berry

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Berry Tabor conjecture

M.V. Berry and M. Tabor, Level clustering in the regular spectrum, *Proc. Roy. Soc.* A **356** (1977) 375-394.

Michael Berry

M.V. Berry, Quantizing a classically ergodic system: Sinai's billiard and the KKR method, *Ann. Phys.* **131** (1981) 163-216.

BGS conjecture

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Look at the energy level instead of the classical path

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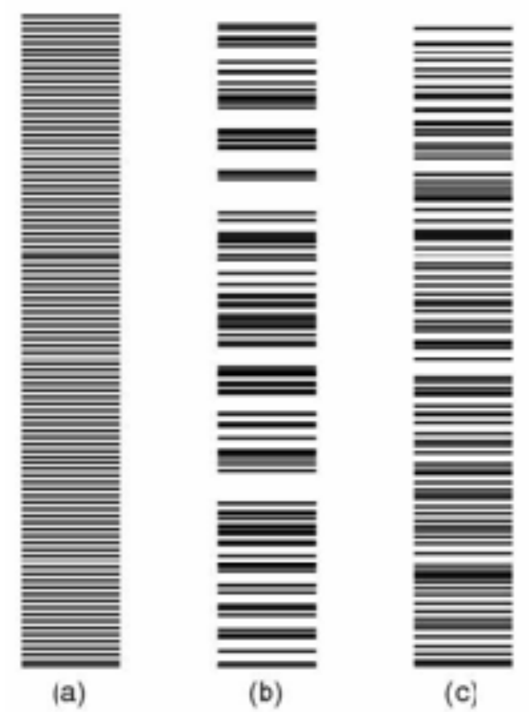
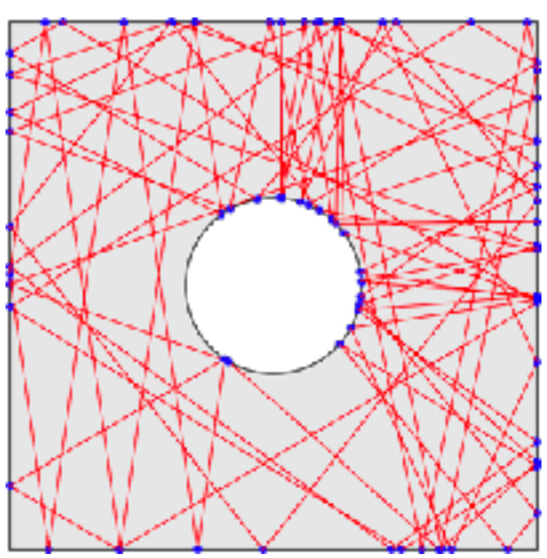
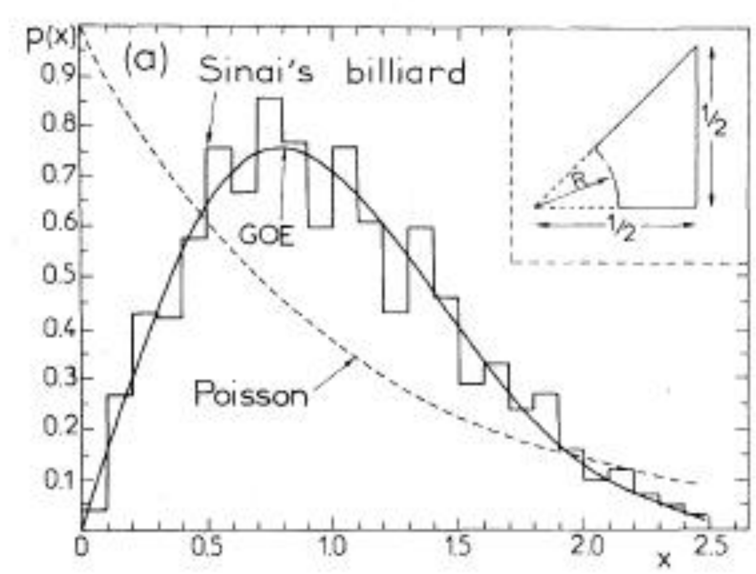


Sinai Billiard (classical)



Sinai Billiard (quantum)



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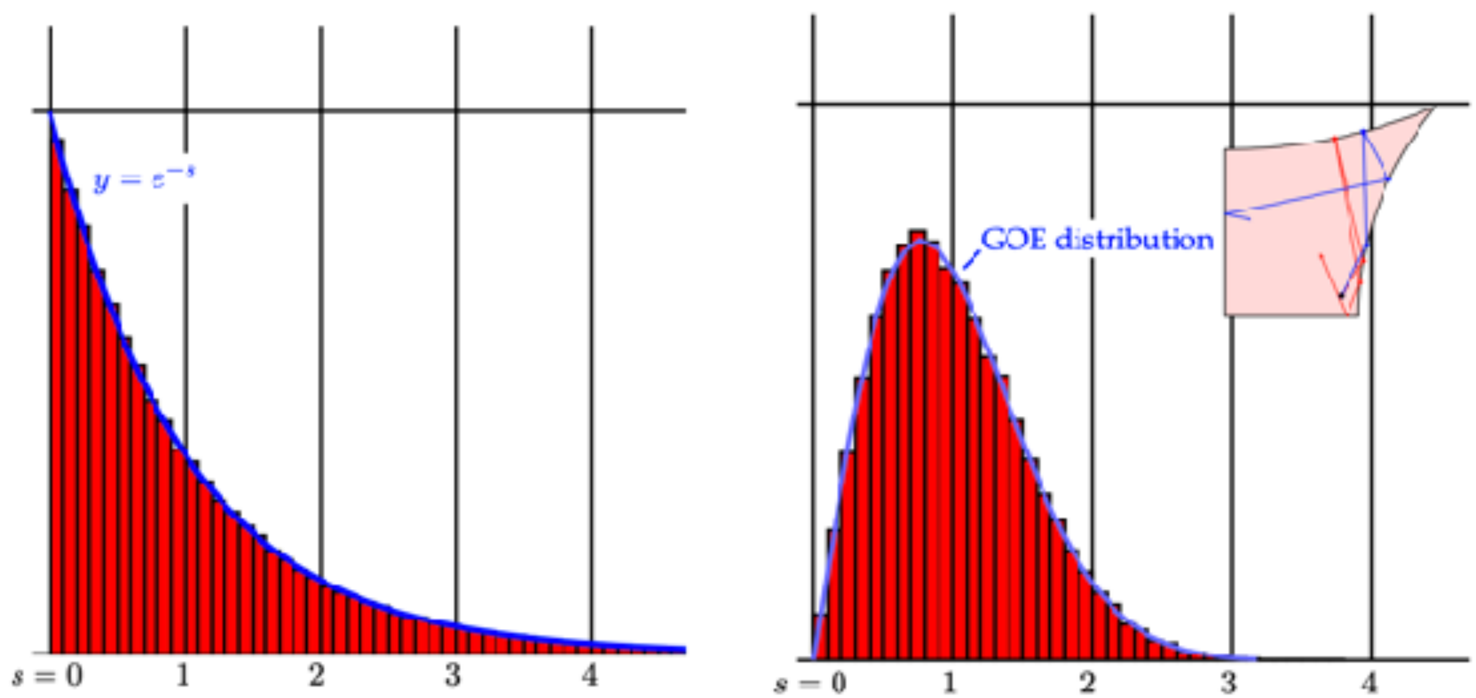
Look at the energy level instead of the classical path

Single particle in a cavity



Z. Rudnick, Notices of the AMS 55 (2008), p. 32.

History of quantum chaos

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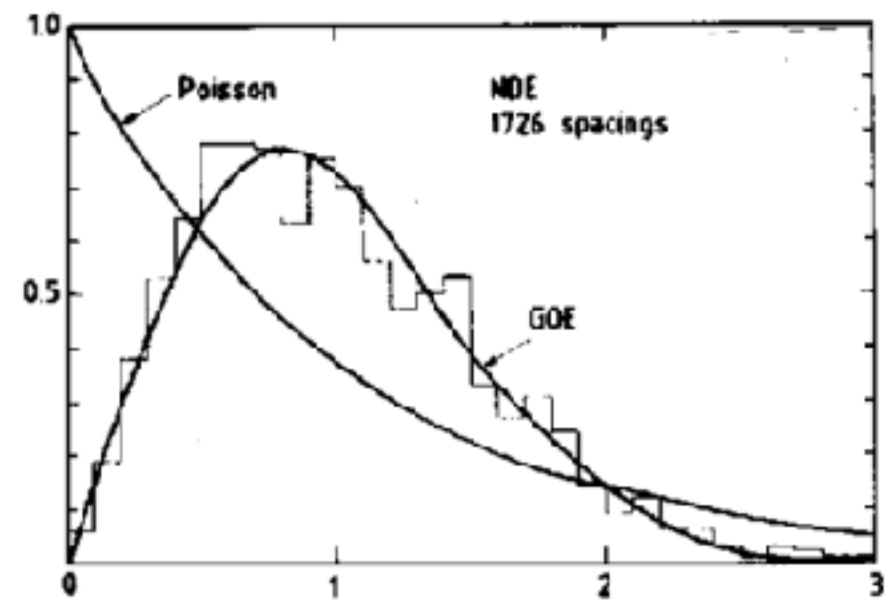
Heavy nuclei

Wigner E. Wigner, Ann. of Math. 62 (1955), pp. 548–564

- Hopeless to predict the exact energy levels of complex systems such as large nuclei
- Focus on statistical property
- Property of random matrix

chaotic?



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Random matrix theory

Random matrix

$$\hat{H} \doteq \begin{bmatrix} \varepsilon_1 & \frac{V}{\sqrt{2}} \\ \frac{V^*}{\sqrt{2}} & \varepsilon_2 \end{bmatrix}$$

ε_1, ε_2, and V from a Gaussian distribution with zero mean and variance σ

Eigenvalues

$$E_{1,2} = \frac{\varepsilon_1 + \varepsilon_2}{2} \pm \frac{1}{2}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + 2|V|^2}.$$

Statistics of level separation

$$P(E_1 - E_2 = \omega) \equiv P(\omega)$$

$$P(\omega) = \frac{1}{(2\pi)^{3/2}\sigma^3} \int d\varepsilon_1 \int d\varepsilon_2 \int dV\, \delta\left(\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + 2V^2} - \omega\right) \exp\left(-\frac{\varepsilon_1^2 + \varepsilon_2^2 + V^2}{2\sigma^2}\right)$$

$$= \frac{\omega}{2\sigma^2} \exp\left[-\frac{\omega^2}{4\sigma^2}\right]$$

$H_{ij} = H_{ji}$

GOE (Gaussian orthogonal ensemble), time reversal symm.

$$= \frac{\omega^2}{2\sqrt{\pi}\,(\sigma^2)^{3/2}} \exp\left[-\frac{\omega^2}{4\sigma^2}\right]$$

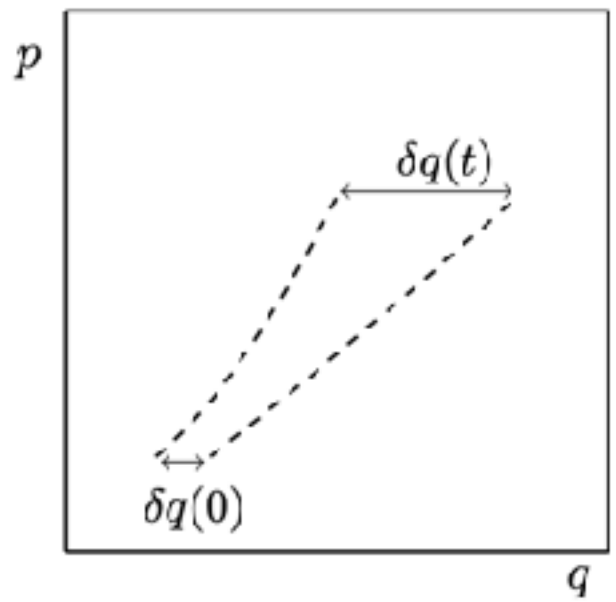
$H_{ij} = H_{ji}^*$

GUE (Gaussian unitary ensemble), no time reversal symm.

Wigner Surmise Wigner-Dyson distributions

[Reminder]
Quantum Chaos

Classical Chaos $\underset{\not\longrightarrow}{\xleftarrow{\hbar \to 0}}$ Quantum chaos



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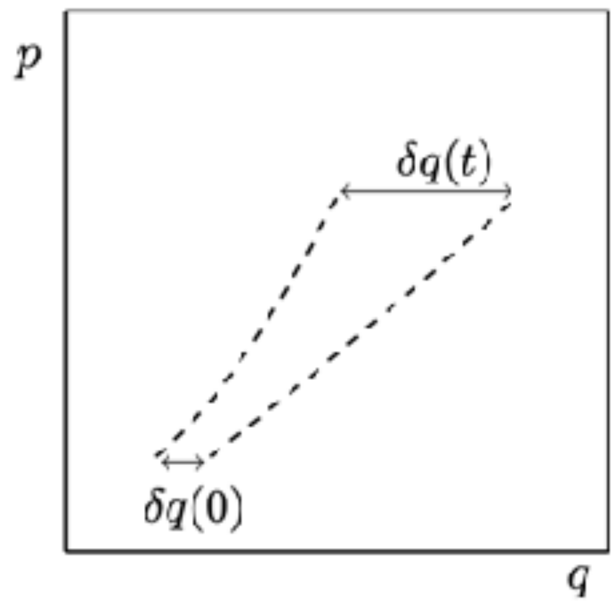
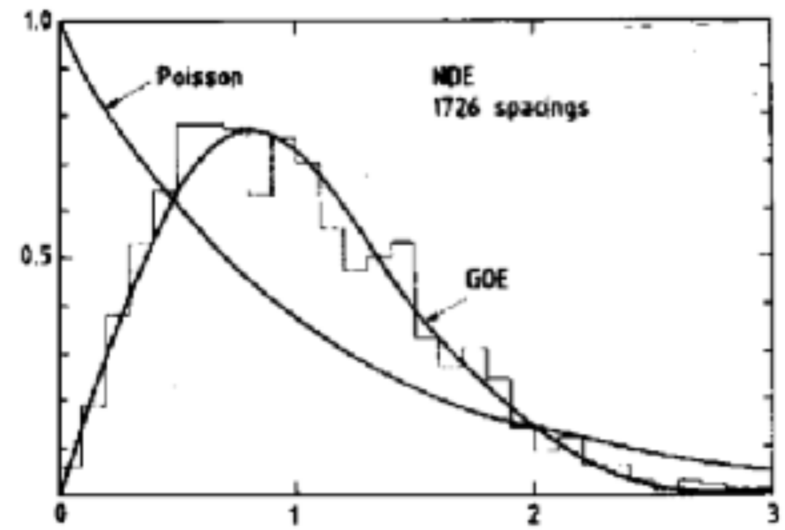
[Reminder]
Quantum Chaos

$$\hbar \to 0$$

Classical Chaos ⟵ Quantum chaos





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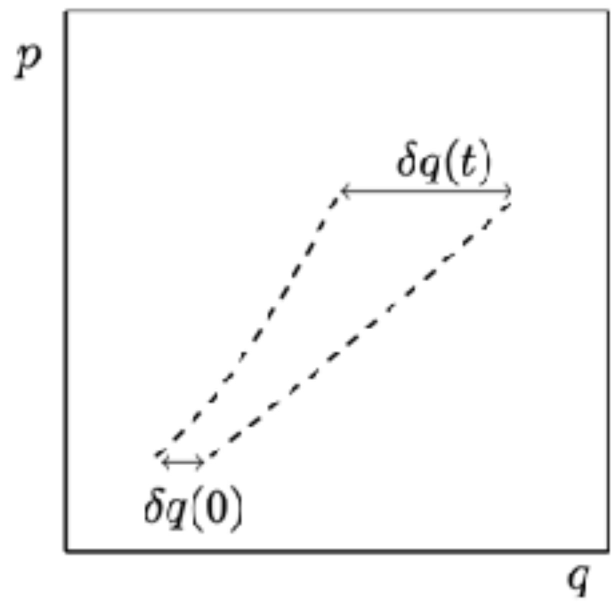
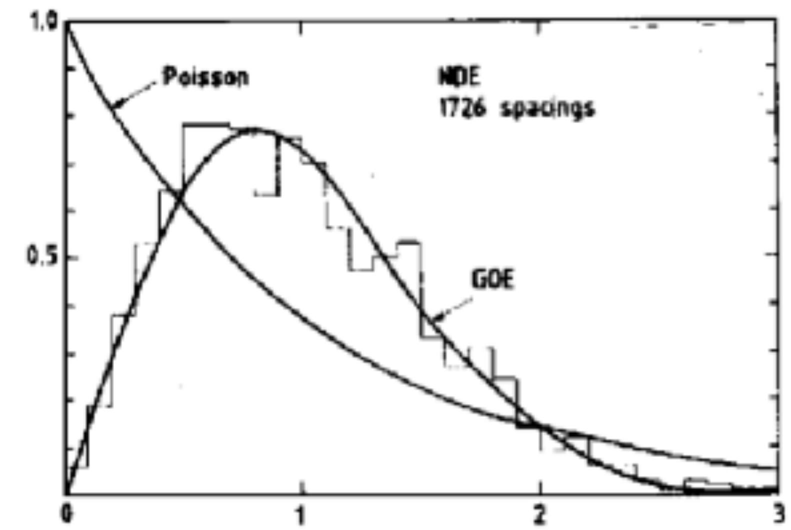




- No time? Can we see the time-dynamics?
- Whose time dynamics in what sense?

  - “How fast” do the operator or state spread (get scrambled) in the “Krylov space”?

Holography

Infinitely strong interaction ~ universality ~ black hole

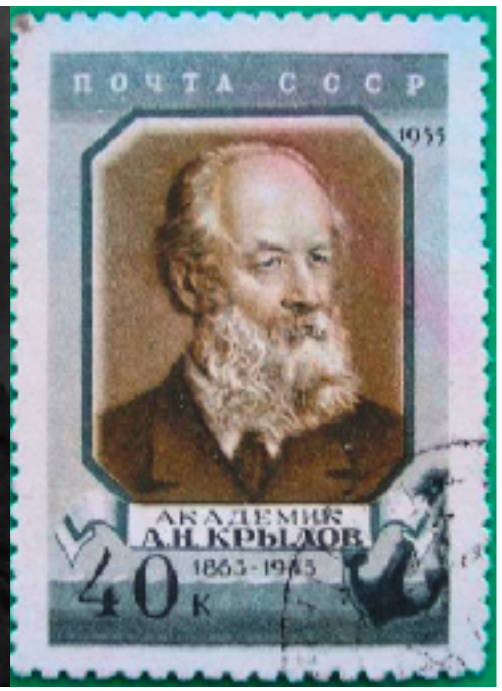
Maximally chaotic ~ universality ~ black hole

~ random (matrix)

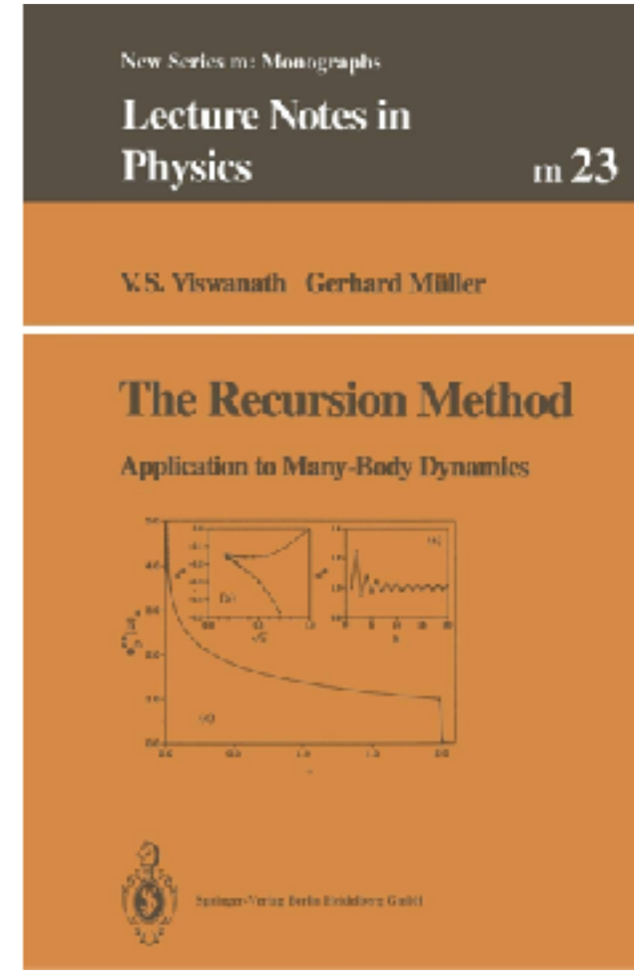
Krylov Complexity

Contents

Aleksey Nikolaevich Krylov (1863 –1945)
a Russian naval engineer, applied mathematician



- Short Review on Krylov Complexity
  - Operator growth
  - Krylov space
  - Lanczos coefficient
  - Krylov complexity
- Success in lattice systems
- Towards field theory
  - Too good to be true
  - How to extract info from the power spectrum (IR/UV cutoff effect)



1994

Cornelius (Cornel) Lanczos (1893-1974):
a Hungarian-American and later Hungarian-Irish mathematician and physicist.

Operator growth

1812.08657: Parker et al.

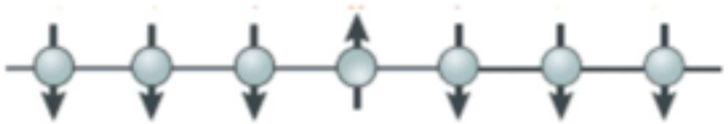
The time evolution of an operator O by a time independent Hamiltonian H

$$\partial_t \mathcal{O}(t) = i\,[H, \mathcal{O}(t)]$$

$$\mathcal{O}(t) = e^{itH}\,\mathcal{O}(0)\,e^{-itH}$$

Baker-Campbell-Hausdorff (BCH) formula $e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{\mathcal{L}_X^n Y}{n!}$

$$\mathcal{O}(t) = \mathcal{O}_0 + it[H,\mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!}[H,[H,[H,\mathcal{O}]]] + \cdots \quad .$$

ex) 1D spin chain 

$$H = -\sum (Z_i Z_{i+1} + gX_i + hZ_i)$$

$$Z_1(t) = Z_1 + it[H,Z_1] - \frac{t^2}{2!}[H,[H,Z_1]] - \frac{it^3}{3!}[H,[H,[H,Z_1]]] + \ldots$$

$$\begin{aligned}
[H,Z_1] &\sim Y_1 \\
[H,[H,Z_1]] &\sim Y_1 + X_1 Z_2 \\
[H,[H,[H,Z_1]]] &\sim Y_1 + Y_2 X_1 + Y_1 Z_2 \\
[H,[H,[H,[H,Z_1]]]] &\sim X_1 + Y_1 + Z_1 + X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 + X_1 Z_2 + \\
&\quad + Z_3 Y_1 + Y_1 Z_2 Y_2 + Z_1 X_2 X_1 + X_2 Z_3 X_1
\end{aligned}$$

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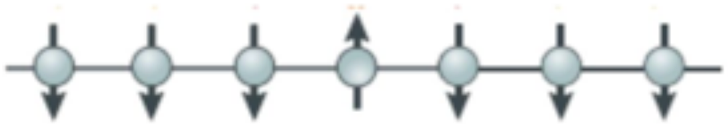
ex) 1D spin chain 

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$$[H, Z_1] \sim Y_1$$

$$[H,[H,Z_1]] \sim Y_1 + X_1 Z_2$$

$$[H,[H,[H,Z_1]]] \sim Y_1 + Y_2 X_1 + Y_1 Z_2$$

$$[H,[H,[H,[H,Z_1]]]] \sim X_1 + Y_1 + Z_1 + X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 + X_1 Z_2 + Z_3 Y_1 + Y_1 Z_2 Y_2 + Z_1 X_2 X_1 + X_2 Z_3 X_1$$

Operator growth

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- The set of operators $\{\tilde{\mathcal{O}}_n\}$ defines a basis of the so-called *Krylov space* associated to the operator $\mathcal{O}$
- Regard the operator as a state $\mathcal{O} \to |\mathcal{O})$ in the Hilbert space of operators

Inner product: Wightman inner product

$$(A|B) := \langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B\rangle_\beta = \frac{1}{\mathcal{Z}_\beta}\mathrm{Tr}(e^{-\beta H/2} A^\dagger e^{-\beta H/2} B) \qquad \mathcal{Z}_\beta := \mathrm{Tr}(e^{-\beta H})$$

Krylov basis

$(\mathcal{O}_m|\mathcal{O}_n) = \delta_{mn}$ (Lanczos algorithm: Gram–Schmidt procedure)

$\{b_n\}$: Lanczos coefficients

$$|\mathcal{O}_0) := |\tilde{\mathcal{O}}_0) := |\mathcal{O}(0))$$

$$|\mathcal{O}_1) := b_1^{-1}\mathcal{L}|\tilde{\mathcal{O}}_0) \qquad b_1 := (\tilde{\mathcal{O}}_0\mathcal{L}|\mathcal{L}\tilde{\mathcal{O}}_0)^{1/2}$$

$$|\mathcal{O}_n) := b_n^{-1}|A_n) \qquad b_n := (A_n|A_n)^{1/2}$$

$$|A_n) := \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$$

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$$\partial_t |\mathcal{O}(t)) = i\mathcal{L}|\mathcal{O}(t))$$

$$\mathcal{O}(t) = e^{i\mathcal{L}t}\,\mathcal{O}(0)$$

$$|\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \varphi_n(t)|\mathcal{O}_n) \qquad \sum_{n=0}^{\infty} |\varphi_n(t)|^2 = 1$$

"probability amplitudes"

$$\mathcal{O}(t) = \mathcal{O}_0 + it[H,\mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!}[H,[H,[H,\mathcal{O}]]] + \cdots .$$

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$$(A|B) := \langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B\rangle_\beta = \frac{1}{\mathcal{Z}_\beta}\mathrm{Tr}(e^{-\beta H/2} A^\dagger e^{-\beta H/2} B) \qquad \mathcal{Z}_\beta := \mathrm{Tr}(e^{-\beta H})$$

Krylov basis

$(\mathcal{O}_m|\mathcal{O}_n) = \delta_{mn}$ (Lanczos algorithm: Gram–Schmidt procedure)

$$|\mathcal{O}_0) := |\tilde{\mathcal{O}}_0) := |\mathcal{O}(0))$$

$\{b_n\}$: Lanczos coefficients

$$|\mathcal{O}_1) := b_1^{-1}\mathcal{L}|\tilde{\mathcal{O}}_0) \qquad b_1 := (\tilde{\mathcal{O}}_0\mathcal{L}|\mathcal{L}\tilde{\mathcal{O}}_0)^{1/2}$$

$$|\mathcal{O}_n) := b_n^{-1}|A_n) \qquad b_n := (A_n|A_n)^{1/2}$$

$$|A_n) := \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$$

Discrete "Schrodinger equation"

$$\partial_t \mathcal{O}(t) = i\,[H,\mathcal{O}(t)]$$

$$\downarrow$$

$$\partial_t |\mathcal{O}(t)) = i\mathcal{L}|\mathcal{O}(t)) \qquad |\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \varphi_n(t) |\mathcal{O}_n) \qquad \varphi_n(t) := i^{-n}(\mathcal{O}_n|\mathcal{O}(t))$$

"probability amplitudes" $\sum_{n=0}^{\infty} |\varphi_n(t)|^2 = 1$

$$\begin{pmatrix} \ell^D_{00} & \ell^D_{01} & \ell^D_{02} & \cdots \\ \ell^D_{10} & \ell^D_{11} & \ell^D_{12} & \cdots \\ \ell^D_{20} & \ell^D_{21} & \ell^D_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$L_{nm} := (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} = b_n\delta_{m,n-1} + b_{n+1}\delta_{m,n+1}$$

$$\dot\varphi_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$

$$\dot\varphi_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$

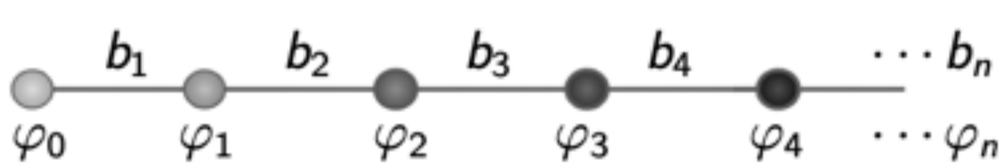
$$\vdots$$

$$\dot\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$\frac{\mathrm{d}\,\varphi_n(t)}{\mathrm{d}t} = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t) \qquad \varphi_n(0) = \delta_{n,0} \qquad \varphi_{-1}(t) \equiv 0 \equiv b_0$$

a quantum-mechanical particle on a 1- dimensional chain.

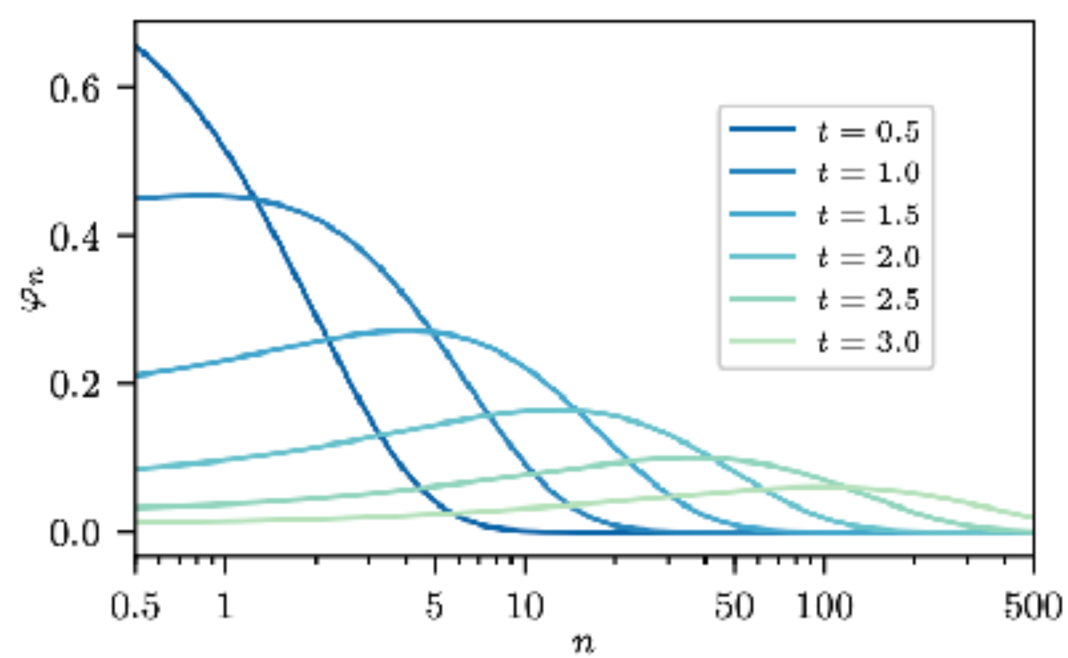
b_n = hopping amplitudes



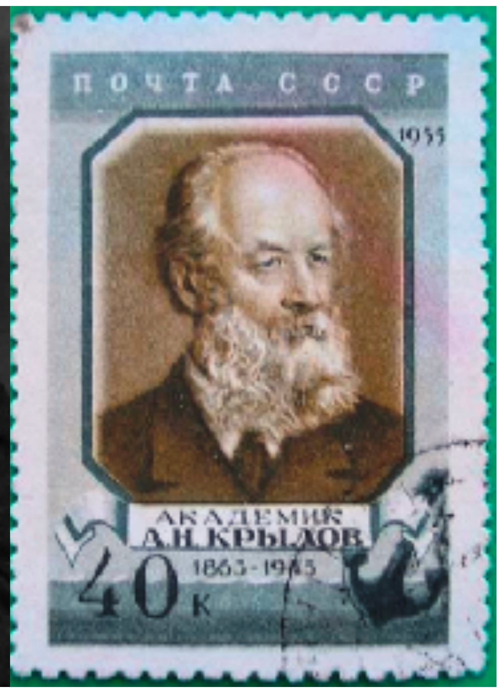
Krylov complexity

average position over the chain

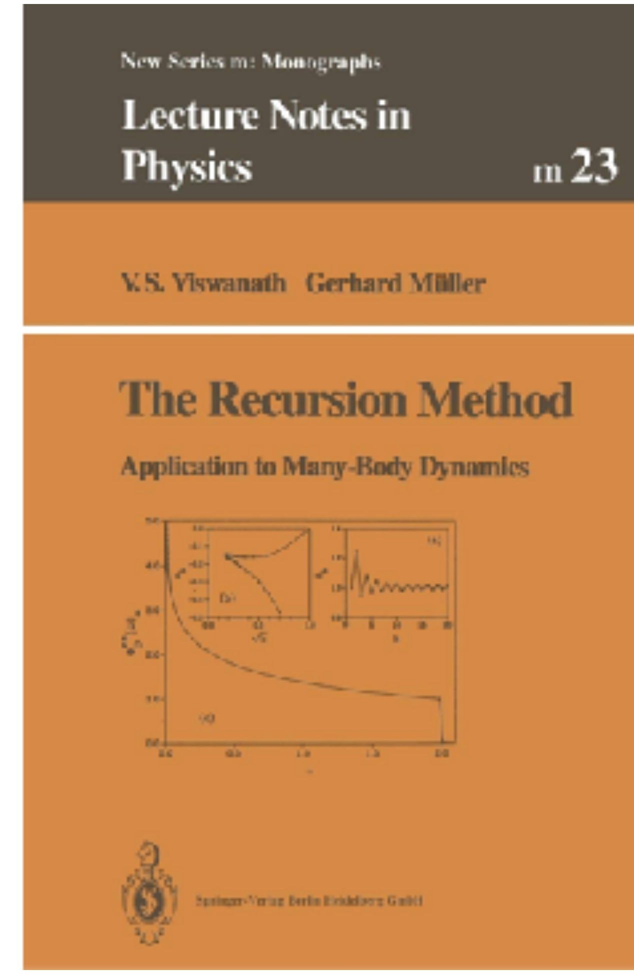
$$K_{\mathcal{O}}(t) := (\mathcal{O}(t)|n|\mathcal{O}(t)) = \sum_{n=0}^{\infty} n|\varphi_n(t)|^2$$



Aleksey Nikolaevich Krylov (1863 –1945)
a Russian naval engineer, applied mathematician



- Short Review on Krylov Complexity
  - Operator growth
  - Krylov space
  - Lanczos coefficient
  - Krylov complexity
- Success in lattice systems
- Towards field theory
  - Too good to be true
  - How to extract info from the power spectrum (IR/UV cutoff effect)



1994

Cornelius (Cornel) Lanczos (1893-1974):
a Hungarian-American and later Hungarian-Irish mathematician and physicist.

K-complexity from the auto-correlation function

1812.08657: Parker et al.

Auto-correlation function

$C(t) = \Pi^W(t) = \varphi_0(t)$

$$\begin{aligned} C(t) &:= (\mathcal{O}(t)|\mathcal{O}(0)) = \varphi_0(t) \\ &= \langle e^{i(t-i\beta/2)H}\mathcal{O}^\dagger(0)e^{-i(t-i\beta/2)H}\mathcal{O}(0)\rangle_\beta \\ &= \langle \mathcal{O}^\dagger(t-i\beta/2)\mathcal{O}(0)\rangle_\beta =: \Pi^W(t) \, . \end{aligned}$$

$$\langle \cdots \rangle_\beta = \mathrm{Tr}(e^{-\beta H}\cdots)/\mathrm{Tr}(e^{-\beta H})$$

Power spectrum

$f^W(\omega)$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{-i\omega t} f^W(\omega)$$

Moments

μ_{2n}

$$\Pi^W(t) := \sum_{n=0}^{\infty} \mu_{2n}\frac{(it)^{2n}}{(2n)!} \qquad \mu_{2n} := \frac{1}{i^{2n}}\frac{\mathrm{d}^{2n}\Pi^W(t)}{\mathrm{d}t^{2n}}\bigg|_{t=0}$$

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega)$$

Lanczos coefficients from moments

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$$\mu_2 = b_1^2\,, \quad \mu_4 = b_1^4 + b_1^2 b_2^2\,, \quad \cdots\,.$$

$$b_n = \sqrt{M_{2n}^{(n)}}\,, \qquad M_{2l}^{(j)} = \frac{M_{2l}^{(j-1)}}{b_{j-1}^2} - \frac{M_{2l-2}^{(j-2)}}{b_{j-2}^2} \quad \text{with} \quad l = j,\ldots,n\,,$$

$$M_{2l}^{(0)} = \mu_{2l} \quad , \quad b_{-1} \equiv b_0 := 1 \quad , \quad M_{2l}^{(-1)} = 0\,.$$

Lanczos coefficients

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\color{red}{f^W(\omega)} \arrow[r, "\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega)"] \arrow[d, swap, "\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{-i\omega t} f^W(\omega)"] & \mu_{2n} \arrow[d, "b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}"] \\
C(t) = \Pi^W(t) = \varphi_0(t) & b_n
\end{tikzcd}
$$

K-complexity

$$\Big\downarrow$$

$$
\begin{aligned}
\dot{\varphi}_0(t) &= b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t) \\
\dot{\varphi}_1(t) &= b_1\varphi_0(t) - b_2\varphi_2(t) \\
&\;\vdots \\
\dot{\varphi}_n(t) &= b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)
\end{aligned}
$$

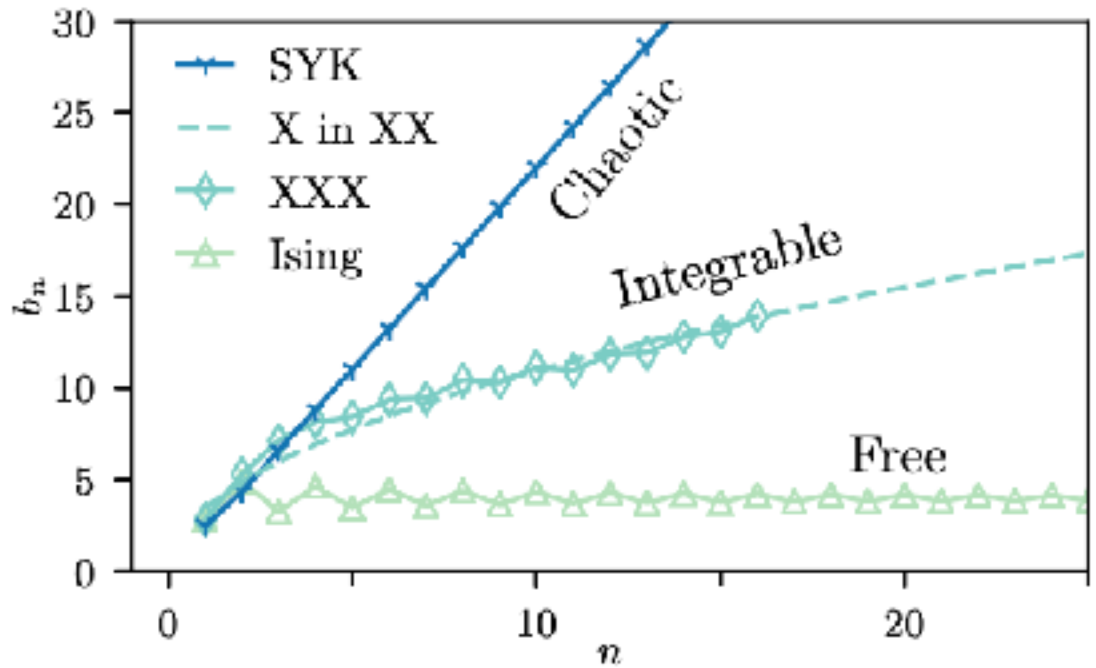
$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2\,, \quad n_{\max} = 200.$$

Success in lattice systems

Universal operator growth hypothesis

1812.08657: Parker et al.

$$b_n \sim n^\delta \Longleftrightarrow f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$
$$\delta \leq 1$$



D. S. Lubinsky, "A survey of general orthogonal polynomials for weights on finite and infinite intervals," Acta Applicandae Mathematica **10**, 237–296 (1987).

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Signatures of chaos in time series generated by many-spin systems at high temperatures

Tarek A. Elsayed, Benjamin Hess, and Boris V. Fine
Phys. Rev. E **90**, 022910 – Published 20 August 2014

$f^W(\omega) \sim e^{-\frac{\omega}{\omega_0}}$ Is a signature of classical chaos

Universal operator growth hypothesis

In a chaotic quantum system

Lanczos coefficients $\{b_n\}$ grow as fast as possible

$$b_n \sim \alpha n$$

the slowest possible decay of the power spectrum

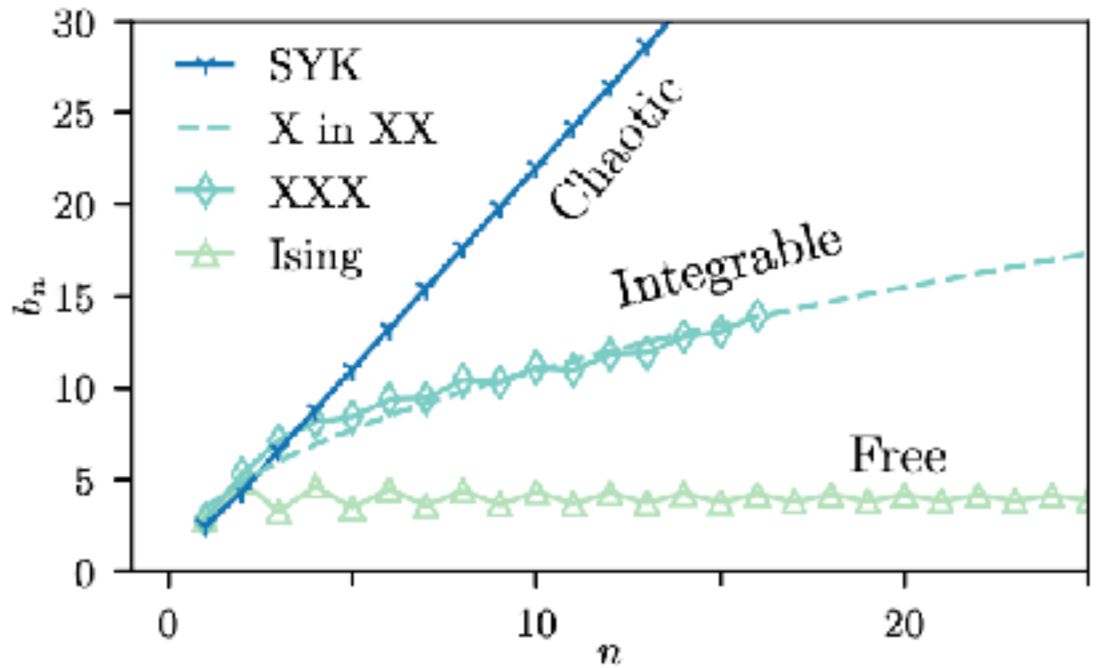
$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

Krylov complexity grows exponentially

$$K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \quad \Longleftrightarrow \quad b_n \sim \alpha n \quad \Longleftrightarrow \quad K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$b_n \sim n^\delta \Longleftrightarrow f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$
$$\delta \leq 1$$



$$\left|\{q^i(t), p^j(0)\}_{PB} = \left|\frac{\partial q^i(t)}{\partial q^j(0)}\right| \sim e^{\lambda t}\right.$$
$$-\langle [q^i(t), p^j(0)]^2 \rangle_\beta,$$
$$-\langle [V(t), W(0)]^2 \rangle_\beta \quad \sim e^{\lambda t}$$

Universal operator growth hypothesis

In a chaotic quantum system

Lanczos coefficients $\{b_n\}$ grow as fast as possible

$$b_n \sim \alpha n$$

the slowest possible decay of the power spectrum

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

Krylov complexity grows exponentially

$$K_O(t) \sim e^{2\alpha t}$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \quad \Longleftrightarrow \quad b_n \sim \alpha n \quad \Longleftrightarrow \quad K_O(t) \sim e^{2\alpha t}$$

Complexity of state

Complexity of a state

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \qquad |\psi(t)\rangle = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}|\psi_n\rangle \qquad |\psi_n\rangle = H^n|\psi(0)\rangle.$$

$$|A_{n+1}\rangle = (H-a_n)|K_n\rangle - b_n|K_{n-1}\rangle$$

$$a_n = \langle K_n|H|K_n\rangle\,, \quad b_n = \langle A_n|A_n\rangle^{1/2}$$

$$|K_0\rangle = |\psi(0)\rangle\,, \quad |K_n\rangle = b_n^{-1}|A_n\rangle\,, \quad b_0 \equiv 0\,.$$

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle$$

$$|\psi(t)\rangle = \sum_n \psi_n(t)|K_n\rangle$$

$$H = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & \cdots & 0 \\ 0 & b_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & b_{L-1} \\ 0 & 0 & 0 & b_{L-1} & a_{L-1} \end{pmatrix}$$

Spread complexity

$$C(t) \equiv \sum_n n p_n(t) = \sum_n n|\psi_n(t)|^2\,.$$

$$C_{|\psi\rangle}(t) := \min_{B}\left[C_B(t)\right]$$

$$\begin{pmatrix} h^B_{00} & h^B_{01} & h^B_{02} & \cdots \\ h^B_{10} & h^B_{11} & h^B_{12} & \cdots \\ h^B_{20} & h^B_{21} & h^B_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Spread complexity

Complexity of a state

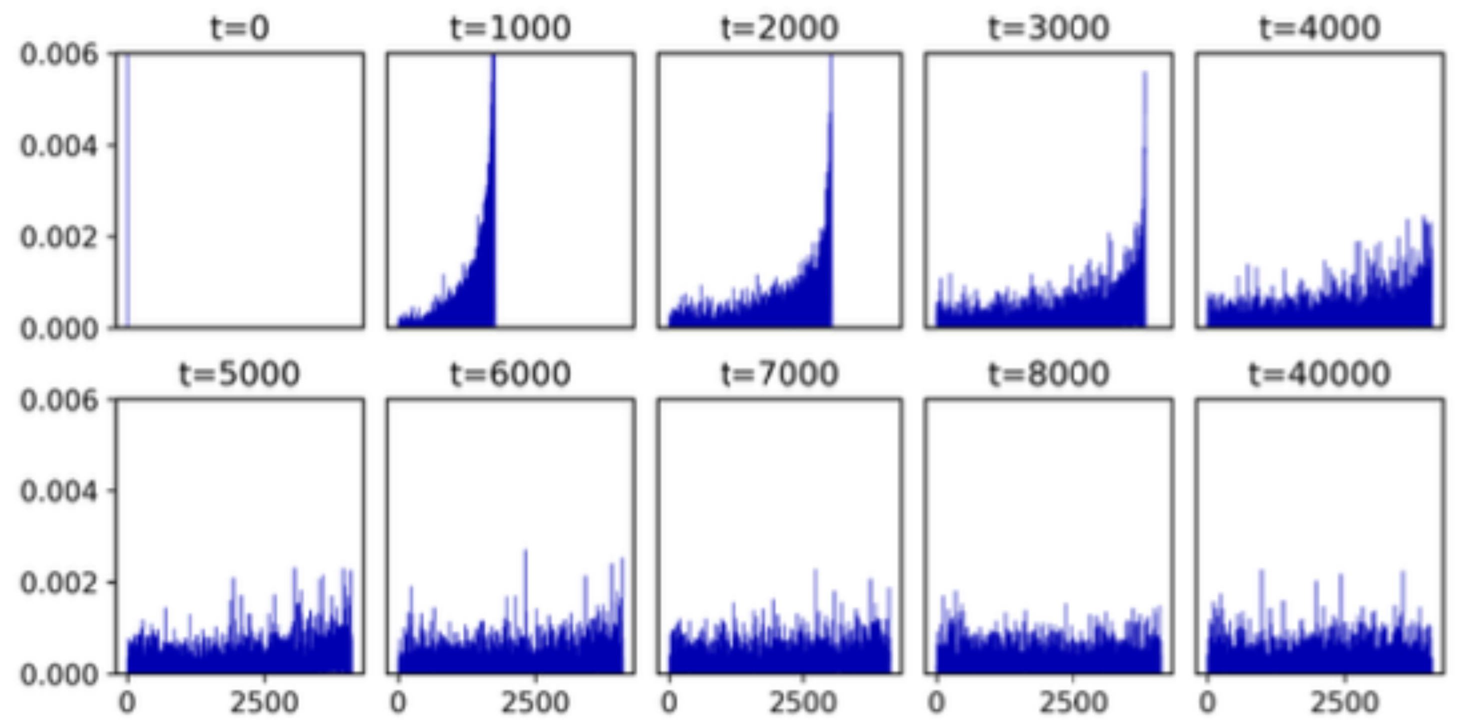
$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle$$

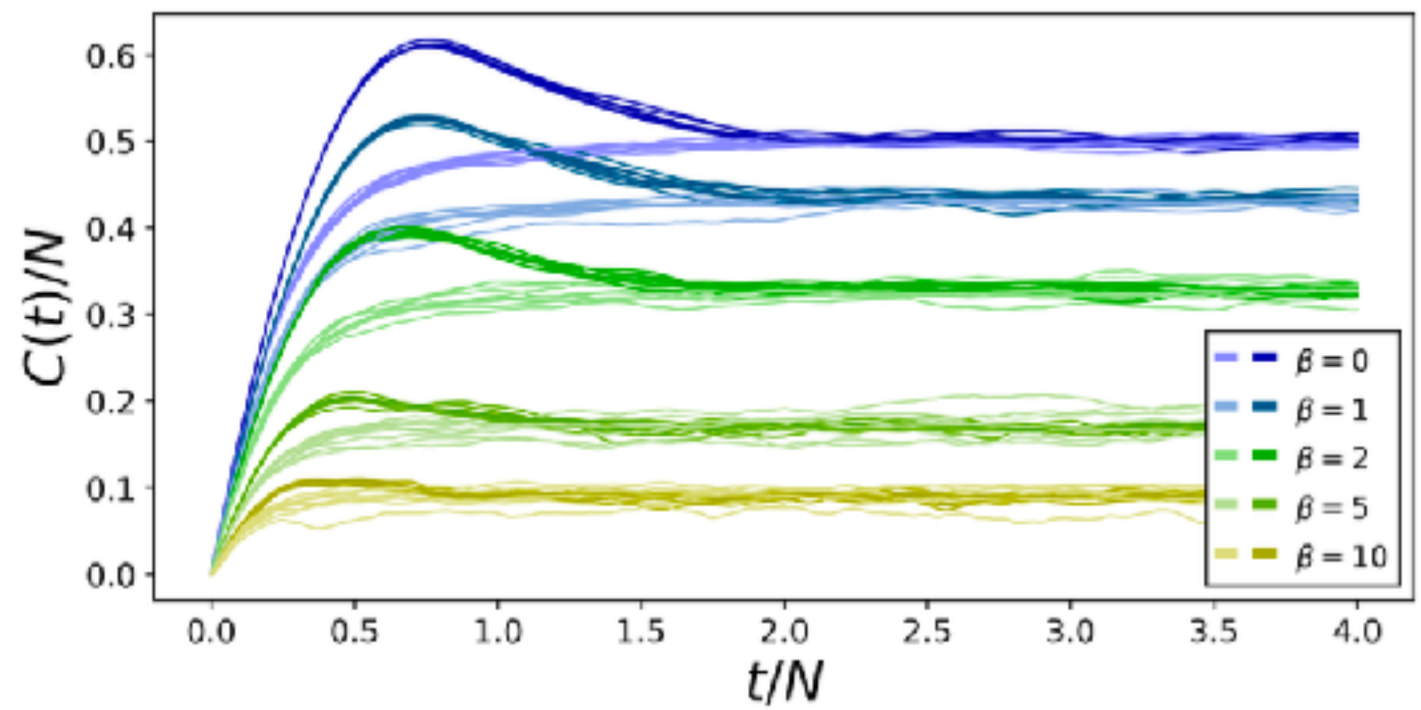
$$H = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & \cdots & 0 \\ 0 & b_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & b_{L-1} \\ 0 & 0 & 0 & b_{L-1} & a_{L-1} \end{pmatrix}$$

$$|\psi(t)\rangle = \sum_n \psi_n(t)|K_n\rangle$$

Spread complexity

$$C(t) \equiv \sum_n n p_n(t) = \sum_n n|\psi_n(t)|^2 .$$





Complexity of a state

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle$$

$$H = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & \cdots & 0 \\ 0 & b_2 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & b_{L-1} \\ 0 & 0 & 0 & b_{L-1} & a_{L-1} \end{pmatrix}$$

$$|\psi(t)\rangle = \sum_n \psi_n(t)|K_n\rangle$$

Spread complexity

$$C(t) \equiv \sum_n n p_n(t) = \sum_n n|\psi_n(t)|^2 .$$

Survival amplitude

$$S(t) = \langle\psi(t)|\psi(0)\rangle = \langle\psi(0)|e^{iHt}|\psi(0)\rangle$$

$$\begin{aligned}\mu_n &= \frac{d^n}{dt^n}S(t)\Big|_{t=0} = \langle\psi(0)|\frac{d^n}{dt^n}e^{iHt}|\psi(0)\rangle\Big|_{t=0} \\ &= \langle K_0|(iH)^n|K_0\rangle .\end{aligned}$$

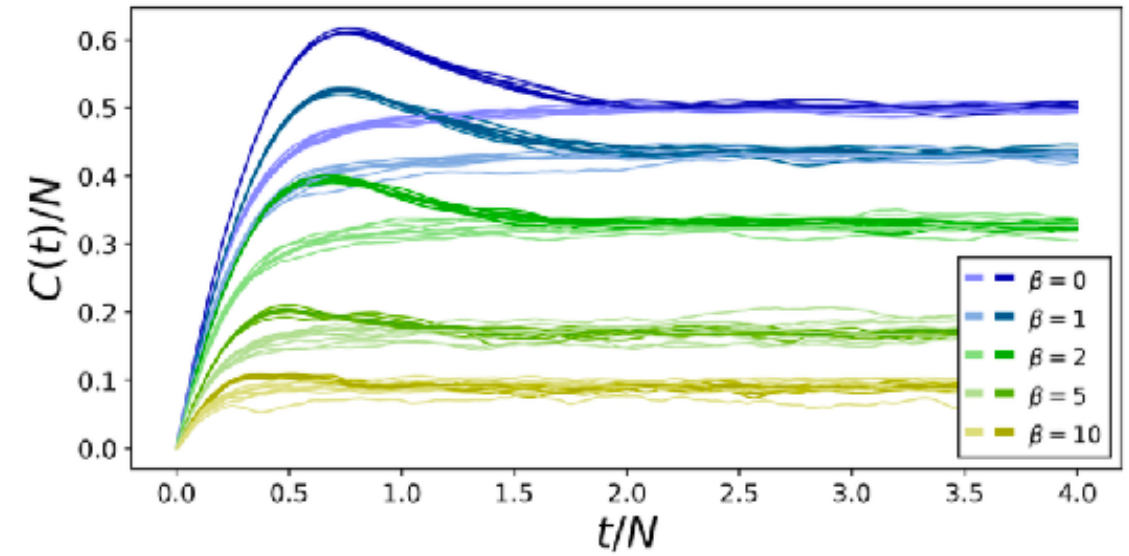
Thermo Field Double (TFD) state

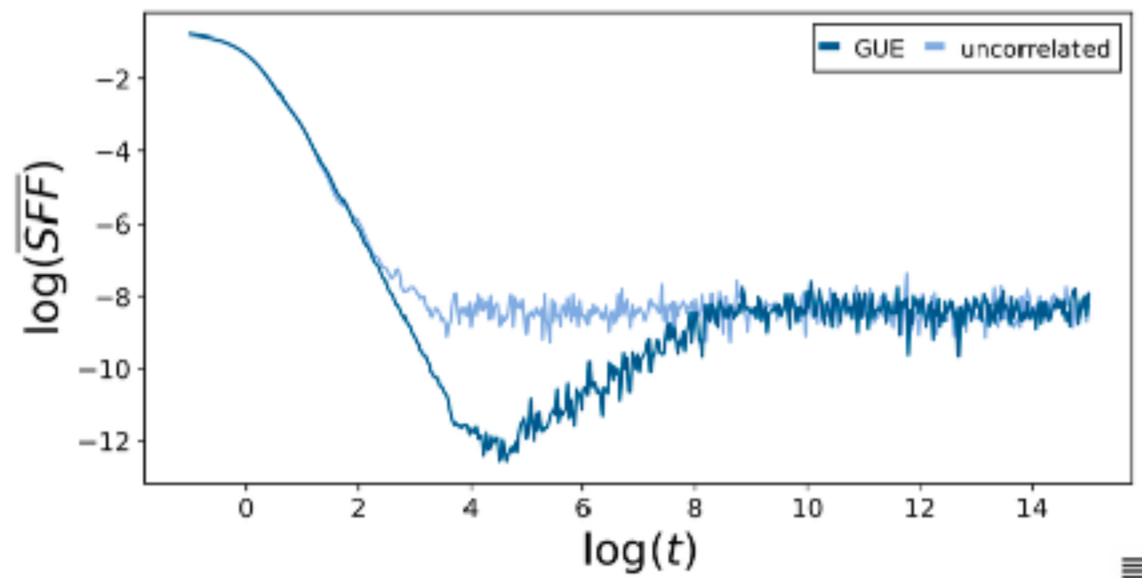
$$|\psi_\beta\rangle \equiv \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{\beta E_n}{2}} |n, n\rangle$$

$$|\psi_\beta(t)\rangle = e^{-iHt}|\psi_\beta\rangle = |\psi_{\beta+2it}\rangle$$

$$S(t) = \langle\psi_{\beta+2it}|\psi_\beta\rangle = \frac{Z_{\beta-it}}{Z_\beta}$$

$$SFF_{\beta-it} = \frac{|Z_{\beta-it}|^2}{|Z_\beta|^2}.$$





Survival amplitude

$$S(t) = \langle\psi(t)|\psi(0)\rangle = \langle\psi(0)|e^{iHt}|\psi(0)\rangle$$

$$\mu_n = \frac{d^n}{dt^n} S(t)\Big|_{t=0} = \langle\psi(0)|\frac{d^n}{dt^n} e^{iHt}|\psi(0)\rangle\Big|_{t=0}$$
$$= \langle K_0|(iH)^n|K_0\rangle\,.$$

Observations for RMT, SYK

Universal for Maximal chaos? Why?

What if not TFD