

Linear Stability of Einstein and de Sitter Universes in the Quadratic Theory of Modified Gravity



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4th INTERNATIONAL
CONFERENCE
on HOLOGRAPHY, STRING
THEORY and DISCRETE
APPROACH
HANOI, VIETNAM

Nash theory of gravity and its stability

We have studied the Bianchi type-I universe in the context of Nash gravity using the Noether symmetry approach.

Here, we study its stability in the context of some models

Content:

- Nash Theory of gravity.
- Noether Symmetry.
- Bianchi-type I model
- Point like Lagrangian and Noether symmetry.
- Cosmological implication:
 - Einstein static solutions and their stabilities
 - Stability of the de-Sitter of the de Universe
 - Conclusions

ArXiv list:

- 1807.07406
- 1812.01990
- 1808.06483



Noether symmetry in the Nash theory of gravity

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Received: 13 March 2018 / Accepted: 11 July 2018 / Published online: 20 July 2018
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Abstract This paper deals with the study of Bianchi type-I universe in the context of Nash gravity by using the Noether symmetry approach. We shortly revisit the Nash theory of gravity. We make a short recap of the Noether symmetry approach and consider the geometry for Bianchi-type I model. We obtain the exact general solutions of the theory inherently exhibited by the Noether symmetry. We also examine the cosmological implications of the model by discussing the two cases of viable scenarios. Surprisingly, we find that the predictions are nicely compatible with those of the Λ CDM model.

1 Introduction

Several cosmological observations show that the observable universe is undergoing a phase of accelerated expansion [1–5]. Regarding the late-time cosmic acceleration, there are at least two promising explanations, to date. The first one is to introduce the dark energy component in the universe [6]. However, the dark energy sector of the universe remains still unknown. Conversely, the second popular approach is to interpret this phenomenon by using a purely geometrical picture. The latter is well known as the modified gravity. Modified theories of gravity have received more attention lately due to numerous motivations ranging from high-energy physics, cosmology and astrophysics [7–9].

The modified theories of gravity can be in principle achieved from different contexts. One of the earlier modifications to Einstein's general relativity was known as the Brans–Dicke gravity. In addition to the gravitational sector, this theory introduced a dynamical scalar field to represent a variable gravitational constant [10]. Later it was found that the authors of Ref. [11] studied a scalar-tensor theory of grav-

ity in which the metric is coupled to a scalar field. Regarding the work of Ref. [11], a 'missing-mass problem' can be successfully described. Moreover, this approach can be applied to the Bianchi cosmological models.

Another simplest modification to the standard general relativity is the $f(R)$ theories of gravity in which the Lagrangian density f is an arbitrary function of the scalar curvature R [12,13]. Among numerous alternatives, these theories include higher order curvature invariants, see reviews on $f(R)$ theories [14,15]. In recent years, a new stimulus for this study leads to a number of interesting results. Notice that the model with $f(R) = R + \alpha R^2$ ($\alpha > 0$) can lead to the accelerated expansion of the Universe because of the presence of the αR^2 term. This particular case is the first model that can describe cosmic inflation proposed by Starobinsky [16]. There exists another different class of the modified gravity theory, called MOG, which can alternatively explain the flat rotation curve of galaxies without invoking cold dark matter particles [17] (see also recent examination [18]). Likewise, John Nash has developed an alternative theory to the Einstein's theory. The theory has been proven to be formally divergence free and considered to be of interest in constructing theories of quantum gravity.

In order to quantify the exact solutions, it has been proven that the Noether symmetry technique proved to be very useful not only to fix physically viable cosmological models with respect to the conserved quantities, but also to reduce dynamics and achieve exact solutions. Moreover, the existence of Noether symmetries plays crucial roles when studying quantum cosmology [19]. In addition, the Noether symmetry approach has been employed to various cosmological scenarios so far including the $f(T)$ gravity [20], the $f(R)$ gravity [36], the alpha-attractors [30], and others cosmological scenarios, e.g. [21–29]. Moreover, the Noether symmetry approach has been also utilized to study the Bianchi models [37,38] in order to obtain the exact solutions for potential functions, scalar field and the scale factors.

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Nash's theory for gravity

Nash theory of gravity marked one of the alternative theories of gravity.

The theory can be viewed as a modification of GR and has been considered to be of interest in attempting to develop theories of quantum gravity.

- Action of Nash gravity: $\mathcal{S} = \int d^4x \mathcal{L} = \int d^4x \sqrt{-g} (2R^{\mu\nu} R_{\mu\nu} - R^2).$
- Taking into account the action, the gravitational field can be obtained as:

$$\square G^{\mu\nu} + G^{\alpha\beta} (2R_{\alpha}^{\mu} R_{\beta}^{\nu} - \frac{1}{2} g^{\mu\nu} R_{\alpha\beta}) = 0,$$

- where $\square$ is the d'Alembertian operator, and

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

Motivation

- One of the alternative theories to Einstein's general theory developed as $f(R, R_{\mu\nu}R^{\mu\nu})$.
- Divergence-free theory with $f(R, R_{\mu\nu}R^{\mu\nu}) = 2R_{\mu\nu}R^{\mu\nu} - R^2$ although it doesn't have classical Einstein limits.
- Found to have some applications in quantum gravity.

We first derive the equations of motion in the flat FLRW spacetime and examine the behaviors of the solutions by invoking specific forms of the Hubble parameter.

Noether symmetry-Recap

- According to the [Noether theorem](#), if there exists a vector field, $\mathbf{X}$, for which the [Lie derivative of a given Lagrangian \$\mathcal{L}\$, vanishes](#), $L_{\mathbf{X}}\mathcal{L} = \mathbf{X}\mathcal{L} = 0$, [the Lagrangian admits a Noether symmetry](#) and thus yields a conserved current.

$$\mathbf{X} = \alpha^i(\mathbf{q}) \frac{\partial}{\partial q^i} + \left(\frac{d}{d\lambda} \alpha^i(\mathbf{q}) \right) \frac{\partial}{\partial \dot{q}^i},$$

- where λ is an affine parameter. The dot indicates the differentiation with respect to time, t .

- Any function $F(q, \dot{q})$ is invariant under the transformation $\mathbf{X}$ (being the symmetry for the dynamics derived by the $\mathcal{L}$) if

$$L_{\mathbf{X}}F = \alpha^i(\mathbf{q})\frac{\partial F}{\partial q^i} + \left(\frac{d}{d\lambda}\alpha^i(q)\right)\frac{\partial F}{\partial \dot{q}^i} = 0$$

Then if $L_{\mathbf{X}} = 0$ then $\mathbf{X}$ is a symmetry for the dynamics derived by $\mathcal{L}$

- The *Noether Theorem* states that if $L_{\mathbf{x}}\mathcal{L} = 0$, then the constant of motion function:

$$\Sigma_0 = \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$$

Bianchi-I Universe

- The **Bianchi Type I** line-element in coordinates $x^\mu = (t, x, y, z)$ is

$$ds^2 = - dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2$$

- A, B, C are scale factor-functions of t .

- For **homogeneous Isotropic Universe** $A=B=C$. The expansion factor $a(t)$ of the total volume is:

$$a(t) = (ABC)^{1/3} = V^{1/3}$$

- The mean of the three Hubble parameters is

$$H = \frac{1}{3} \sum H_i \text{ where } H_i = \frac{d \ln(A_i)}{dt}, A_i = \{A, B, C\}$$

- For **anisotropic Universe** (discussed her):

$$A \neq B \neq C$$

Point-like Lagrangian & Noether equations

- We can write the Lagrangian as: $\mathcal{L} = \mathcal{L}(A, B, C, H_1, H_2, H_3, \dot{H}_1, \dot{H}_2, \dot{H}_3)$
- We define: $H_1 = \frac{\dot{A}}{A} = \frac{d\ln(A(t))}{dt}, H_2 = \frac{\dot{B}}{B} = \frac{d\ln(B(t))}{dt}, H_3 = \frac{\dot{C}}{C} = \frac{d\ln(C(t))}{dt}.$
- The generator symmetry is: $\mathbf{X} = f \frac{\partial}{\partial H_1} + g \frac{\partial}{\partial H_2} + h \frac{\partial}{\partial H_3} + \dot{f} \frac{\partial}{\partial \dot{H}_1} + \dot{g} \frac{\partial}{\partial \dot{H}_2} + \dot{h} \frac{\partial}{\partial \dot{H}_3}$
- Generalized phase space for our system spans on a six dimensional manifold with coordinates (H_1, H_2, H_3) where $1 \leq a \leq 3$ and f, g, h are functions of $(H_a, \dot{H}_a)$ and

$$\dot{f} = \sum_{a=1}^3 \dot{H}_a \frac{\partial f}{\partial H_a}$$

Point-like Lagrangian, which enables us to investigate the symmetry properties of the system

$$\begin{aligned}
 \mathcal{L} = \frac{-4}{ABC} & \left(H_1^3 H_2 + H_1^3 H_3 + H_1^2 H_2^2 + 3 H_1^2 H_2 H_3 + H_1^2 H_3^2 + H_1 H_2^3 \right. \\
 & + 3 H_1 H_2^2 H_3 + 3 H_1 H_2 H_3^2 + H_1 H_3^3 + H_2^3 H_3 + H_2^2 H_3^2 \\
 & + H_2 H_3^3 + H_1^2 \dot{H}_2 + H_1^2 \dot{H}_3 + H_1 H_2 \dot{H}_1 + H_1 H_2 \dot{H}_2 + 2 H_1 H_2 \dot{H}_3 \\
 & + H_1 H_3 \dot{H}_1 + 2 H_1 H_3 \dot{H}_2 + H_1 H_3 \dot{H}_3 + H_2^2 \dot{H}_1 + H_2^2 \dot{H}_3 + 2 H_2 H_3 \dot{H}_1 \\
 & \left. + H_2 H_3 \dot{H}_2 + H_2 H_3 \dot{H}_3 + H_3^2 \dot{H}_1 + H_3^2 \dot{H}_2 + \dot{H}_2 \dot{H}_1 + \dot{H}_3 \dot{H}_1 + \dot{H}_3 \dot{H}_2 \right).
 \end{aligned}$$

Testing the stability of Nash theory:

- EINSTEIN STATIC SOLUTIONS AND THEIR (IN)STABILITY
- STABILITY OF THE DE SITTER UNIVERSE

General Solution: Model-I

- Einstein static solutions Universe:

- $-H_1^3 H_2 A' BC - H_3 H_1^3 A' BC - H_2^2 H_1^2 A' BC - \dots + AB' C \dot{H}_2^2 + AB' C \dot{H}_2 \dot{H}_3 - \dot{H}_3 \dot{H}_2 A' BC - \ddot{H}_2 ABC - \ddot{H}_3 ABC = 0,$

- $-H_1^3 H_2 B' AC - H_3 H_1^3 B' AC - H_2^2 H_1^2 B' AC - \dots + A' BC \dot{H}_1^2 + A' BC \dot{H}_1 \dot{H}_3 - \ddot{H}_1 ABC - \ddot{H}_3 ABC = 0,$

- $-H_1^3 H_2 C' AB - H_3 H_1^3 C' AB - H_2^2 H_1^2 C' AB - 3 H_3 H_1^2 H_2 C' AB - \dots + AB' C \dot{H}_2^2 + A' BC \dot{H}_1^2 + A' BC \dot{H}_1 \dot{H}_2 - \ddot{H}_1 ABC - \ddot{H}_2 ABC = 0,$

where

$$A' = \frac{dA(t)}{dH_1}, B' = \frac{dB(t)}{dH_2}, C' = \frac{dC(t)}{dH_3}$$

Since the Einstein static universe is a closed universe and the scale factors $A(t), B(t), C(t)$ are constant:

$$H_i = \dot{H}_i = 0, \quad i = 1, 2, 3$$

Introducing perturbation into the Hubble constant which only depends on time:

$$H_i \rightarrow \delta H_i, \quad \dot{H}_i \rightarrow \delta \dot{H}_i, \quad \ddot{H}_i \rightarrow \delta \ddot{H}_i.$$

Substituting this into the previous equations:

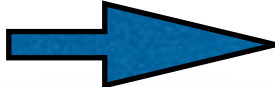
$$\delta \ddot{H}_2 + \delta \ddot{H}_3 = 0, \quad \delta \ddot{H}_1 + \delta \ddot{H}_3 = 0, \quad \delta \ddot{H}_1 + \delta \ddot{H}_2 = 0.$$

We obtain:

$$\delta H_i = H_i^{(0)} \left(\frac{t}{t_0} \right) + H_i^{(1)},$$

where $H_i^{(0)}, H_i^{(1)}$ are constants of integrations.

From the above equation:

No oscillation 

Einstein static universe is unstable
against the perturbations

General Solution: Model II

- Stability of the de Sitter Universe
- Anisotropic metric  anisotropic expansion of the flat FLRW metric.
- Directional Hubble parameters $H_i = h_i$ are stationary values at the beginning, when the spacetime metric is stationary.
- Considering: $A = ae^{h_1 t}$, $B = be^{h_2 t}$, $C = ce^{h_3 t}$
- Then, the solution is:

We get

- $-h_1^3 h_2 A' BC - h_3 h_1^3 A' BC - h_2^2 h_1^2 A' BC - \dots + h_2^3 ABC + 3 h_3 h_2^2 ABC + h_3^3 ABC = 0,$
- $-h_1^3 h_2 B' AC - \dots + 6 h_3 h_1 h_2 ABC + 3 h_3^2 h_1 ABC + 3 h_3 h_2^2 ABC + 2 h_3^2 h_2 ABC + h_3^3 ABC = 0,$
- $-h_1^3 h_2 C' AB - \dots + 6 h_3 h_1 h_2 ABC + 3 h_3^2 h_1 ABC + h_2^3 ABC + 2 h_3 h_2^2 ABC + 3 h_3^2 h_2 ABC = 0,$

Testing the stability of de Sitter Model

Introducing the perturbation $H_i \rightarrow h_i + \delta\xi_i(t)$:

$$\begin{aligned} &-(h_1^3 h_2 + 3h_1^2 h_2 \delta\xi_1 + h_1^3 \delta\xi_2)A' BC - (h_3 h_1^3 + 3h_3 h_1^2 \delta\xi_1 + h_1^3 \delta\xi_3)A' BC - \dots - (\delta\ddot{\xi}_2)ABC - (\delta\ddot{\xi}_3)ABC = 0, \\ &-(h_1^3 h_2 + 3h_1^2 h_2 \delta\xi_1 + h_1^3 \delta\xi_2)B' AC - (h_3 h_1^3 + \dots - h_3 \delta\dot{\xi}_3)ABC - \delta\ddot{\xi}_1 ABC - \delta\ddot{\xi}_3 ABC = 0, \\ &-(h_1^3 h_2 + 3h_1^2 h_2 \delta\xi_1 + h_1^3 \delta\xi_2)C' AB - (h_3 h_1^3 + 3h_3 h_1^2 \delta\xi_1 + \dots + h_3 \delta\dot{\xi}_1)ABC + h_3 \delta\dot{\xi}_2 ABC - \delta\ddot{\xi}_1 ABC - \delta\ddot{\xi}_2 ABC = 0, \end{aligned}$$

Substituting the equations into each other:

$$\begin{aligned}
& -(3h_1^2h_2\delta\xi_1 + h_1^3\delta\xi_2)A'BC - (3h_3h_1^2\delta\xi_1 + h_1^3\delta\xi_3)A'BC - \dots - (h_3\delta\xi_3)ABC - (\delta\xi_2)ABC - (\delta\xi_3)ABC = 0, \\
& -(3h_1^2h_2\delta\xi_1 + h_1^3\delta\xi_2)B'AC - (3h_3h_1^2\delta\xi_1 + \dots - h_1\delta\xi_1ABC + h_2\delta\xi_1ABC + h_2\delta\xi_3ABC - h_3\delta\xi_3ABC - \delta\xi_1ABC - \delta\xi_3ABC) = 0, \\
& -(3h_1^2h_2\delta\xi_1 + h_1^3\delta\xi_2)C'AB - (3h_3h_1^2\delta\xi_1 + \dots + h_3\delta\xi_1ABC + h_3\delta\xi_2ABC - \delta\xi_1ABC - \delta\xi_2ABC) = 0,
\end{aligned}$$

Taking the derivatives of A, B, C:

$$\begin{aligned}
& (- (3h_1^3h_2\delta\xi_1 + h_1^4\delta\xi_2) - (3h_3h_1^3\delta\xi_1 + \dots - (h_2\delta\xi_3) - 2(h_2\delta\xi_3) - (h_3\delta\xi_3) - (\delta\xi_2) - (\delta\xi_3))(abc) e^{(h_1+h_2+h_3)t} = 0, \\
& (- (3h_1^2h_2h_3\delta\xi_1 + h_1^3h_3\delta\xi_2) - (3h_3^2h_1^2\delta\xi_1 + h_1^3h_3\delta\xi_3) - \dots - h_2\delta\xi_2 + h_3\delta\xi_1 + h_3\delta\xi_2 - \delta\xi_1 - \delta\xi_2)(abc) e^{(h_1+h_2+h_3)t} = 0 \\
& (- (3h_1^2h_2^2\delta\xi_1 + h_1^3h_2\delta\xi_2) - (3h_3h_1^2h_2\delta\xi_1 + \dots - h_3\delta\xi_3 - \delta\xi_1 - \delta\xi_3)(abc) e^{(h_1+h_2+h_3)t} = 0,
\end{aligned}$$

Collecting perturbation terms together:

$$\begin{aligned}
& (-\delta\xi_2 - \delta\xi_3 + \delta\xi_2(-h_1^3 - h_2h_1^2 - 2h_3h_1^2 - \dots + 6h_2h_1 - 2h_2^2h_3h_1 + 4h_3h_1 + 3h_2^2 + 3h_3^2))(abc) e^{(h_1+h_2+h_3)t} = 0 \\
& (-\delta\xi_1 - \delta\xi_3 + \delta\xi_1(h_1^3 + h_2h_1^2 + 2h_3h_1^2 - h_2^2h_1 + h_3^2h_1 - \dots - 2h_2^2h_3) + \dots + 6h_1h_3))(abc) e^{(h_1+h_2+h_3)t} = 0 \\
& (-\delta\xi_1 - \delta\xi_2 + \delta\xi_1(h_1^3 + 2h_2h_1^2 + \dots - 3hh_2^2 - 2h_2h_3 + h_3) + \dots + 6h_2h_3))(abc) e^{(h_1+h_2+h_3)t} = 0
\end{aligned}$$

These can be reduced to a single equation:

$$\begin{aligned}
& (-h_2^4 - 3h_1h_2^3 - 2h_3h_2^3 - 5h_1^2h_2^2 - 7h_1h_3h_2^2 + 2h_2^2 - 3h_1^3h_2 + 2h_3^3h_2 - 6h_1^2h_3h_2 \\
& + 6h_3h_2 + h_1(h_3^2 + 4)h_2 + h_3^4 + h_1h_3^3 + h_1^2h_3^2 + 2h_3^2 - 3h_1^3h_3 + 8h_1h_3)\xi_1 \\
& + (-h_1^4 - 3h_2h_1^3 - 2h_3h_1^3 - 5h_2^2h_1^2 - 7h_2h_3h_1^2 + 2h_1^2 - 3h_2^3h_1 \\
& + 2h_3^3h_1 - 6h_2^2h_3h_1 + 6h_3h_1 + h_2(h_3^2 + 4)h_1 + h_3^4 + h_2h_3^3 + h_2^2h_3^2 + 2h_3^2 - 3h_2^3h_3 + 8h_2h_3)\xi_2 \\
& + (-h_1^4 - 2h_3^2h_1^3 - 4h_2h_1^3 - h_3h_1^3 - 6h_2^2h_1^2 + 2h_3^2h_1^2 - 5h_2h_3h_1^2 + 4h_1^2 - 4h_2^3h_1 + 2h_3^3h_1 + 9hh_2^2h_1 + h_2h_3^2h_1 + 6h_2h_1 \\
& - 8h_2^2h_3h_1 + 4h_3h_1 - h_2^4 + 3h_2h_3^3 + 4h_2^2 - h_2^2h_3^2 + 2h_3^2 - h_2^3h_3 - 2h_2h_3)\xi_3 \\
& + (-h_2^3 + 3hh_2^2 - 2h_1h_2^2 - 2h_3h_2^2 - h_1^2h_2 - h_3^2h_2 + 2h_3h_2 + h_2 + h_3^3 + 2h_1h_3^2 + h_1^2h_3 - h_3)\dot{\xi}_1 \\
& + (-h_1^3 - h_2h_1^2 - h_2h_3h_1^2 - 2h_3h_1^2 - h_2^2h_1 + 2h_3^2h_1 + h_2h_3h_1 + h_1 + h_3^3 + 2h_2h_3^2 + h_2 + h_2^2h_3 - h_3)\dot{\xi}_2 \\
& + (-h_1^3 - 3h_2h_1^2 - 2h_2^2h_1 + 2h_3^2h_1 + h_1 - h_2^3 + 2h_3^3 - h_2^2 + 3h_2h_3^2 - 2h_2 - 2h_3)\dot{\xi}_3 = 0
\end{aligned}$$

After solving the *long* differential equation we obtain:

$$\delta\xi_1 = c_1 \exp\left(\frac{1}{2}At\right) + c_2 \exp\left(\frac{1}{2}Bt\right)$$

$$A = -A_1 - \sqrt{A_1^2 - A_2}$$

$$B = -A_1 + \sqrt{A_1^2 - A_2}$$

$$A_1 = \frac{1}{2} (2h_1^3 + 3h_2h_1^2 + 3h_3h_1^2 - 2h_1 - h_2^3 - h_3^3 - 3hh_2^2 - h_2h_3^2 + h_2 - 2h_2^2h_3 - 2h_2h_3 + h_3)$$

$$A_2 = 2(h_2^4 + h_1h_2^3 + 4h_3h_2^3 + h_1^2h_2^2 + 6h_3^2h_2^2 + 5h_1h_3h_2^2 - 4h_2^2 - 3h_1^3h_2 + 4h_3^3h_2 + 5h_1h_3^2h_2 - 4h_1h_2 - 6h_3h_2 + h_3^4 + h_1h_3^3 - 6h_1^2 + h_1^2h_3^2 - 4h_3^2 - 3h_1^3h_3 - 4h_1h_3)$$

$$\delta\xi_2 = c_3 \exp\left(\frac{1}{2}Ct\right) + c_4 \exp\left(\frac{1}{2}Dt\right)$$

$$C = -C_1 - \sqrt{C_1^2 - C_2}$$

$$D = -C_1 + \sqrt{C_1^2 - C_2}$$

$$C_1 = \frac{1}{2} \left(-h_1^3 - h_2 h_1^2 - h_2 h_3 h_1^2 - 2h_3 h_1^2 + 3h_2^2 h_1 - 2h_3^2 h_1 + h_2 h_3 h_1 + h_1 + 2h_2^3 - h_3^3 - h_2 + 3h_2^2 h_3 + h_3 \right)$$

$$C_2 = 2(h_1^4 + h_2 h_1^3 + 4h_3 h_1^3 + h_2^2 h_1^2 + 6h_3^2 h_1^2 + 5h_2 h_3 h_1^2 - 4h_1^2 - 3h_2^3 h_1 + 4h_3^3 h_1 \\ + 5h_2 h_3^2 h_1 - 4h_2 h_1 - 6h_3 h_1 + h_3^4 + h_2 h_3^3 - 6h_2^2 + h_2^2 h_3^2 + 2h_3^2 - 3h_2^3 h_3 - 4h_2 h_3)$$

$$\delta\xi_3 = c_5 \exp\left(\frac{1}{2}Et\right) + c_6 \exp\left(\frac{1}{2}Ft\right)$$

$$E = -E_1 - \sqrt{E_1^2 - E_2}$$

$$F = -E_1 + \sqrt{E_1^2 - E_2}$$

$$E_1 = \frac{1}{2} (h_1^3 + 3h_2h_1^2 + 2h_2^2h_1 - 2h_3^2h_1 - h_1 + h_2^3 - 2h_3^3 + h_2^2 - 3h_2h_3^2 + 2h_2 + 2h_3)$$

$$E_2 = 2(-h_1^4 - 2h_3^2h_1^3 - 4h_2h_1^3 - h_3h_1^3 - 6h_2^2h_1^2 + 2h_3^2h_1^2 - 5h_2h_3h_1^2 + 4h_1^2 - 4h_2^3h_1 + 2h_3^3h_1 + 9hh_2^2h_1 \\ + h_2h_3^2h_1 + 6h_2h_1 - 8h_2^2h_3h_1 + 4h_3h_1 - h_2^4 + 3h_2h_3^3 + 4h_2^2 - h_2^2h_3^2 + 2h_3^2 - h_2^3h_3 - 2h_2h_3)$$

Cosmological implications

- In the above equations, provided that $A_{1,2}, C_{1,2}, E_{1,2} > 0$, if we substitute very large numbers for t ; (i.e $t \rightarrow \infty$) we find out that $\delta\xi_i \rightarrow 0$ for any values of h_1, h_2, h_3 , since the power of the exponent is negative..
- Consequently, the de Sitter universe is stable against the perturbations.
- Therefore, ***the de Sitter universe is stable in the context of the Nash gravity.***

Extension of the Model

we are investigating a soft modification of the Nash action by adding an IR completeness term αR to the action as following:

$$S_{EH-Nash} = \frac{1}{2} \int d^4x \sqrt{-g} \left(\alpha R + 2R_{\mu\nu}R^{\mu\nu} - R^2 \right).$$

The action is an **Einstein-Hilbert corrected Nash gravity**.

EoM in the metric formalism,

$$\begin{aligned}
 (\alpha - 2R)G_{\mu\nu} = & \frac{1}{2}g_{\mu\nu}(2R_{\mu\nu}R^{\mu\nu} + R^2) - 2(g_{\mu\nu})\square R - \nabla_\mu \nabla_\nu R - 4R^\gamma_\mu R_{\gamma\nu} \\
 & - 2g_{\mu\nu} \nabla_\gamma \nabla_\theta R^{\gamma\theta} - 2\square R_{\mu\nu} + 4\nabla_\gamma \nabla_\theta R^\gamma_{(\mu} R^\theta_{\nu)}.
 \end{aligned}$$

where the trace is given as:

$$\square \phi = \frac{1}{9} \left(\alpha R - 4R_{\mu\nu}R^{\mu\nu} + R^2 \right)$$

$$\phi \equiv \alpha - \frac{2}{3}R$$

$$\square \delta\phi = \frac{\alpha}{4}\delta\phi = m_s^2\delta\phi.$$

For metric perturbations, we write

$$\alpha(\delta R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta R) = (\eta_{\mu\nu}k^2 - k_\mu k_\nu)(\delta\phi - \frac{4}{3}\delta R)2\eta_{\mu\nu}k_\gamma k_\theta \delta R^{\gamma\theta} + 2k^2\delta R_{\mu\nu} - 4k_\gamma k_\theta \delta R_{(\mu}^{\gamma} R_{\theta)}^{\theta}.$$

Using the same gauge freedom $D_\mu \bar{h}^{\mu\nu} = 0$ and $\bar{h} = 0$ we used in the previous section:

$$\partial_\mu \bar{h}^{\mu\nu} = \partial_\mu (h^{\mu\nu} - \frac{\bar{h}}{2}\eta^{\mu\nu} + \eta^{\mu\nu}h_f) = 0 \quad \bar{h} = 0, \quad h = 4h_f,$$

we simplify it and find:

$$\left(k^2 + \frac{k^4}{m_{spin2}^2}\right)\bar{h}_{\mu\nu} = 0,$$

$$m_{spin2}^2 = -\frac{\alpha}{2}$$

$$\square h_f = m_s^2 h_f$$

In case of negative α , we constrain a graviton mass using a uniform prior probability on the graviton mass $m_g \in [10^{-26}, 10^{-16}]$ to obtain $|\alpha| \in [10^{-52}, 10^{-32}]$.

Conclusions

- Nash theory of gravity marked one of the alternative theories of gravity.
- We study of Bianchi type-I universe in the context of Nash gravity by using the Noether symmetry approach.
- We obtain the exact general solutions of the theory inherently exhibited by the Noether symmetry.
- We also examine the stability of the model in context of Einstein Universe and de-Sitter Universe.
- We find that the Nash gravity model is not stable in the context of Λ Einstein Universe.
- Alternatively the Nash model of gravity is stable in the context of de-Sitter model.

Work Extension

- To compare the results with data.
- To modify the theory by adding R to it and study its stability.
- To test the stability with other theories.
- Since the Nash theory is similar to Gauss-Bonnet we can study its black hole and cylindrical solution.
- ...