Physics of the Split Property

Masataka Watanabe (Nagoya University)

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Today's Goal

- How do you define the entanglement entropy (and the reduced density matrix) in CFTs?
- Physics way: $H = H_A \otimes_\epsilon H_B$ with a UV cutoff $\epsilon \approx 1/\Lambda$.
- Mathematics way: The split property of the von Neumann algebra. One can always "approximate" the algebra of operators using the matrix algebra.
- I will bridge the two understandings.

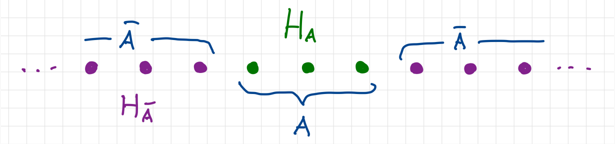
What is EE in Physics?

What is EE in Mathematics?

What is EE in Physics?

EE in spin systems

- The total state $|\psi\rangle$ lives in $\mathcal{H} \simeq \mathbb{C}^{2N}$. The density matrix is defined as $\rho = |\psi\rangle\langle\psi|$.
- The Hilbert space tensor factorises: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$.



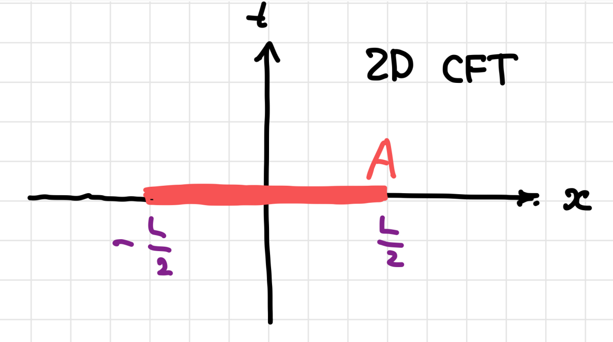
- The reduced density matrix is $\rho_A \equiv \mathrm{Tr}_{\bar{A}}\, \rho$.
- The Renyi entropy is defined as

$$S_n \equiv \frac{1}{1-n} \log \frac{\mathrm{Tr}_A\, \rho_A^n}{(\mathrm{Tr}_A\, \rho_A)^n}. \tag{1}$$

- ρ_A has a discrete spectrum with integer multiplicities.

EE in 2D CFT

- The total state $|\psi\rangle$ lives in $\mathcal{H}$ which is (countably) infinitely dimensional (indentify the two ends $x = \pm\infty$).



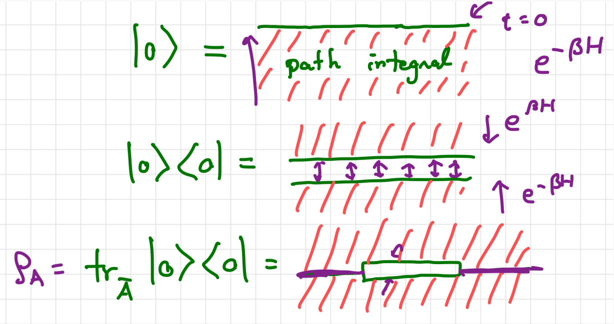
- IF the Hilbert space tensor factorises: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$,

$$|\psi\rangle \equiv \sum c_{nm} |n\rangle \otimes |m\rangle \,.$$

- But we say there is no such a thing. Why?

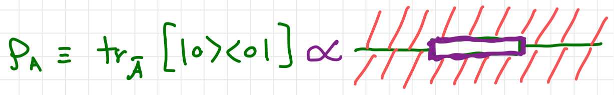
EE in 2D CFT

- We can still compute the EE in 2D CFTs. We do this with a "regulator". What does this mean?
- For example the reduced density matrix for $|0\rangle$:

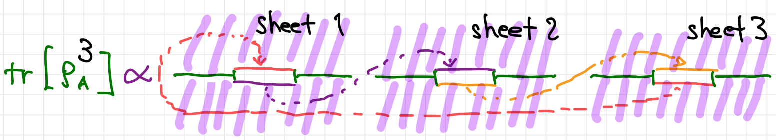


- The states are not normalised. For example $\langle 0|0\rangle = Z = O(\log \Lambda)$.

The replica trick



- The Renyi entropy is given by the log of $\mathrm{Tr}\,\rho_A^n$. This is given by the path integral on an n-sheeted Riemann surface.



- This can be computed as a two-point function of the twist operator

$$\mathrm{Tr}\,\rho_A^n \propto \langle \sigma_{-n}(-L/2)\sigma_n(L/2)\rangle$$

The UV scale

- The (exp of the) n-th Renyi EE is

$$\operatorname{Tr} \rho_A^n \propto \langle \sigma_{-n}(-L/2)\sigma_n(L/2)\rangle = L^{-\frac{c}{12}(n+\frac{1}{n})}$$

- The LHS is dimensionless while the RHS is dimensionful. The overall normalisation has a scale in it because of dimensional transmutation (*e.g.*, $Z_{S^2} = O(\log \Lambda)$).
- We fix it by introducing the UV (length) scale ϵ:

$$\operatorname{Tr} \rho_A^n \equiv \left(\frac{L}{\epsilon}\right)^{-\frac{c}{12}(n+\frac{1}{n})}$$

- You need to regularise to define the EE in QFT.

Confusions

- But. Does this really make sense?
- Quantum information theory wants the (reduced) density matrix to have discrete spectrum with integer multiplicities after regulating it.
- This means, after diagonalisation (ρ_A is always unitary),

$$\mathrm{Tr}\,\rho_A^n = a_0^n \times \left(1 + \left(\frac{a_1}{a_0}\right)^n + \cdots\right)$$

- But if we expand at large-n (the replica number)

$$\mathrm{Tr}\,\rho_A^n \overset{???}{\equiv} \left(\frac{L}{\epsilon}\right)^{-\frac{c}{12}(n+\frac{1}{n})} = \left(\frac{L}{\epsilon}\right)^{-\frac{c}{12}n} \times \left(1 - \frac{1}{n}\frac{c}{12}\log\frac{L}{\epsilon} + \cdots\right)$$

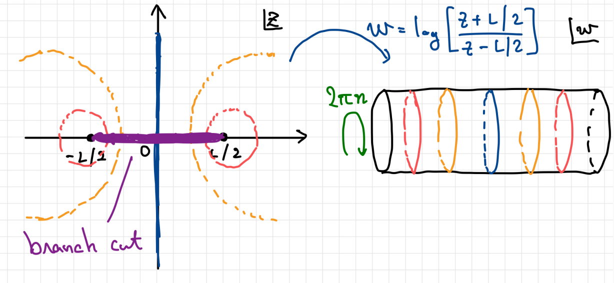
- ρ_A, as defined right now, has continuous spectrum even after regularisation. This is problematic in terms of QuIT.

The correct UV regularisation?

- It seems like one cannot take any regularisation to define EE. We need ones that are consistent with QuIT. What are they?
- And, will they affect the physics (even though they are just UV cutoffs)?
- Any questions so far?

Another conformal frame

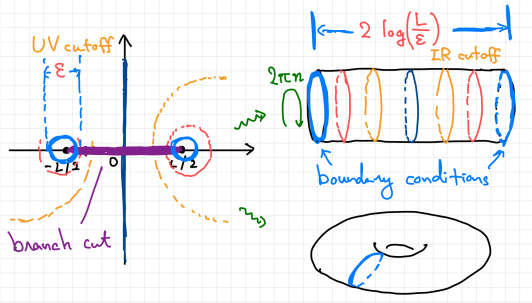
- Let us compute the EE again in a different way.
- We needed to compute the partition function with an n-th branch cut in the z-frame.
- Now a conformal transformation $w = \log\left(\frac{z+L/2}{z-L/2}\right)$:



- The geometry is an infinitely long cylinder. It is now clear why the EE is divergent without a cutoff. There is an IR divergence (with a continuous entanglement spectrum).

Another conformal frame

- There are two ways to regulate it, by chopping the two ends off with boundary conditions, or identifying the two.
- You get a finite-length cylinder or a torus after regularisation.

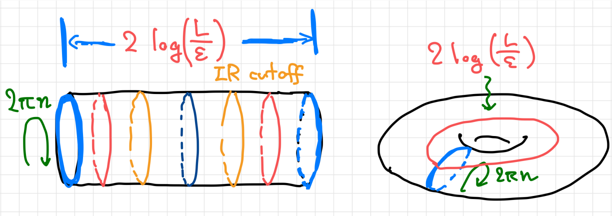


- In the z-frame, the cutoffs are holes around the entangling points, which are UV.

Another conformal frame

- It is clear that the cutoff is compatible QuIT. For example,

$$\mathrm{Tr}\,\rho_A^n = \mathrm{Tr}\,e^{-2\pi n H_{\text{interval}}} \quad \text{or} \quad \mathrm{Tr}\,e^{-2\pi n H_{\text{circle}}}$$



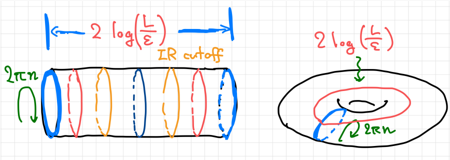
- In other words, the modular Hamiltonian is the Hamiltonian on an interval or a circle. This has discrete spectra.
- The reduced density matrix depends on the way you regularise. It's not just a simple ϵ-dependence.

Recovering the familiar result

- Is it actually useful in computing the EE? Yes.
- The partition function on a cylinder/torus can be computed in the modular conjugated frame.

$$\text{Tr}\,\rho_A^n = \langle B_1 | e^{-\frac{\ell}{n} H_{S^1}} | B_2 \rangle\,, \quad \text{or} \quad \text{Tr}\,\rho_A^n \quad = \sum_{i,j} \langle i | e^{-\frac{\ell}{n} H_{S^1}} | j \rangle$$

with $\ell \equiv 2\log\left(\frac{L}{\epsilon}\right) \gg 1$.

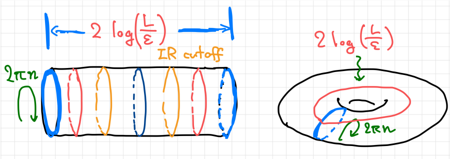


Recovering the familiar result

- We have

$$\operatorname{Tr}\rho_A^n = \langle B_1|0\rangle\langle 0|B_2\rangle\, e^{\frac{\ell}{n}\frac{c}{12}} \times \left(1 + O(e^{-\frac{\ell}{n}\Delta_{\mathrm{gap}}})\right)$$

$$\text{or} \quad = e^{\frac{\ell}{n}\frac{c}{12}} \times \left(1 + O(e^{-\frac{\ell}{n}\Delta_{\mathrm{gap}}})\right)$$



- The Renyi EE is

$$S_n = \log \frac{\operatorname{Tr}\rho_A^n}{(\operatorname{Tr}\rho_A)^n} = \frac{c}{6}\log\frac{L}{\epsilon} + (\log g_1 g_2) + O(e^{-\frac{\ell}{n}\Delta_{\mathrm{gap}}})$$

The expansion is valid when $n \ll \log L/\epsilon$.

Summary

- The "bare" entanglement spectrum of a QFT is continuous.
- Only certain regularisations are compatible with quantum information theory as the entanglement spectrum needs to be discrete.
- The "regularised" entanglement spectrum depends crucially on regularisation.
- We cannot use our usual intuition about EE in the large replica number region.
- Any questions?

What is EE in Mathematics?

Hilbert space tensor factorisation in Maths

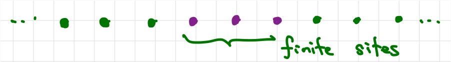
- We still need to impose the UV cutoff to define the entanglement entropy, even in mathematics.
- But they are usually phrased in terms of operator algebras than quantum states.
- I will explain how mathematicians define EE and then explain how we should understand it in terms of physics.

Very quick vN algebra recap

- In the following our spacetime is always compact. I want the Hilbert space to be separable (so it needs to have a countable dimension).
- We are only interested in hyperfinite algebras (only the ones which are limits of finite-demsional matrix algebras).
- They are classified into Type I, II and III.
- The split property concerns the relation between Type I and III. No Type II today.

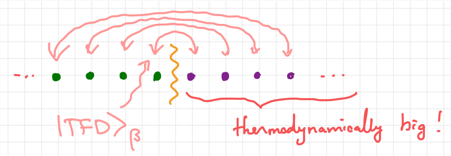
Type I algebra

- Type I algebras are just matrix algebras.
- Example 1: Algebra of all bounded operators of a QFT on a compact space, $\mathcal{B}(\mathcal{H})$.
- Just span the Hilbert space $\mathcal{H}$ with energy eigenstates. Bounded operators acting on them are just nice matrices with well-defined traces.
- The density matrix ρ is a member of $\mathcal{B}(\mathcal{H})$ and we can define the entropy properly: $S_n = \operatorname{Tr} \rho^n$.
- Example 2: Algebra of local operators of lattice systems. Sum of local operators are still just matrices.
- The reduced density matrix of a spin system is a sum of local operators. So one can define the entanglement entropy without regularisation: $S_n = \operatorname{Tr} \rho_A^n$.



Type III algebra

- It is known that one can construct Type III algebras in the following way. (Think about it as a definition during the talk.)
- Imagine a spin system with N qubits. Our subregion A has $N/2$ qubits.



- One can vary $0 < \beta < \infty$ among different qubits.
- The EE is infinite because of $N \to \infty$.

The split property & EE

- No hope for mathematically defining EE? Not quite.

- Split property: Known to hold in all normal CFTs.
  - discrete spectrum
  - finite species
  - ⋮

$\exists\ \mathcal{M}$ (Type I) s.t. $A \subset \mathcal{M} \subset B$ (A, B: Type III)

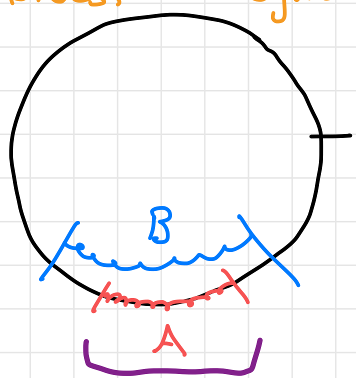
(∞ choices, ∞ regularizations)

$\Downarrow$ $\exists$ partial trace; $\mathcal{M} \equiv \mathcal{B}(\mathcal{H}_\mathcal{M})$

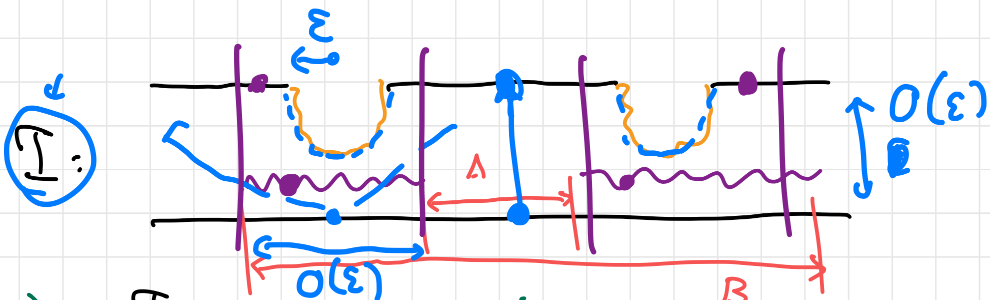
$$\mathcal{H} = \mathcal{H}_\mathcal{M} \otimes \mathcal{H}_{\bar{\mathcal{M}}}$$

$$\left(\rho_\mathcal{M} \equiv \mathrm{tr}_{\bar{\mathcal{M}}}[\rho] \quad \text{etc.}\right)$$

- One can compute EE of $\mathcal{M}$!



Euclidean path-integrals



$*)$ $H \equiv \sum_n f_n L_n$

and maps a finite energy state to itself, when $f_n \sim \frac{1}{n^3}$ at large n.

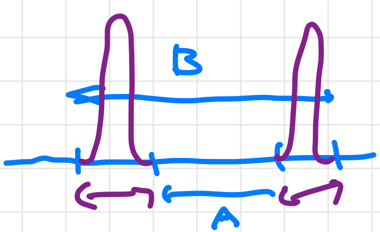
$f(x) \neq 0$ only on $\bar{A} \cup B$

1) I is not unitary

2) $I \, O \, I^\dagger \neq O$ for $O \in A \cup \bar{B}$

($\exists$ inverse)

Easy fix: 1) $U \equiv I / \sqrt{I^\dagger I}$



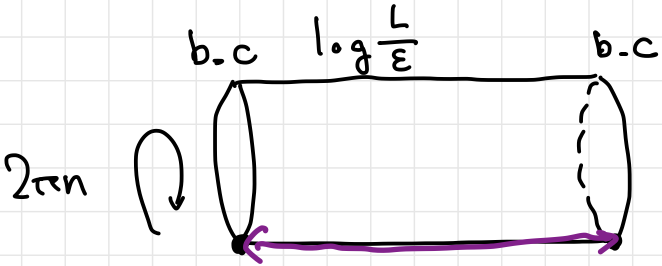
2) The path integral evolves with

$$H = \int dx \, g(x) \, T_{00}(x)$$

- You can do whatever you like to the boundary. (inequivalently)

① Place a boundary condition

The modular Hamiltonian = open string Hamiltonian



= This has discrete spectrum

= Type I

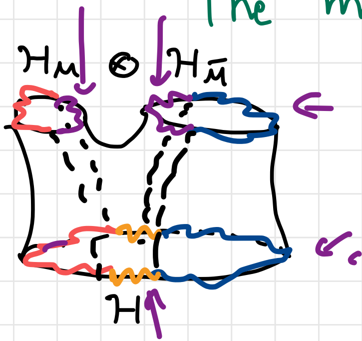
This means that

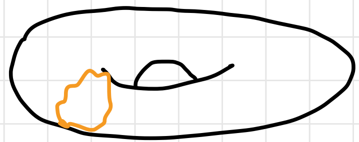
$$A \subset B(H_{BCFT}) \subset B$$

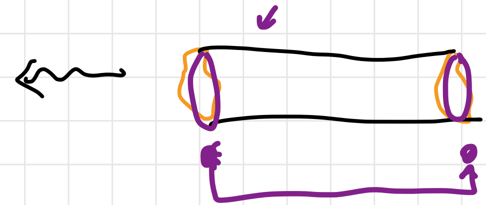
We have given an intermediate Type I factor !!

② Place a periodic b.c.

The modular Hamiltonian = Hamiltonian on S^1







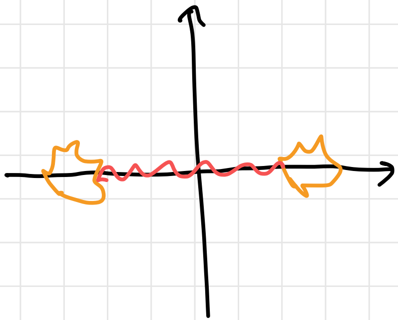
$$A \subset B(\mathcal{H}_{S^1}) \subset B$$

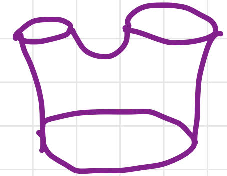
These two are the physics 'example of the intermediate algebra.

*) But the modular Hamiltonian becomes slightly non-local. Similar to the canonical Type I.

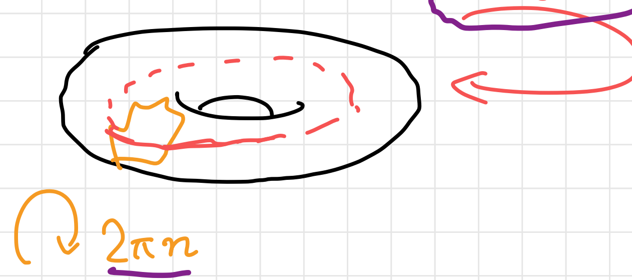
- What is EE in Physics ?
- What is EE in Mathematics ?
- Comparison between the two.
- Examples ← New !

AdS/CFT



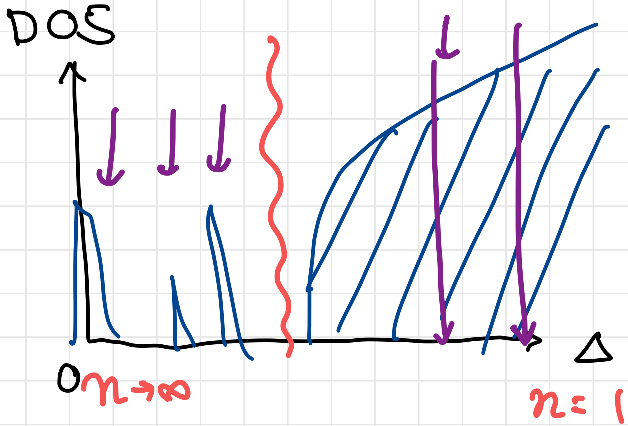


$$\ell = \log \frac{L}{\varepsilon}$$



Take large-N; torus partition function = n-th Renyi undergoes a Hawking-Page transition.

$$\beta \equiv \frac{2\pi n}{\ell} \lessgtr \beta_{\text{Hawking-Page}} \sim 2\pi$$



Type I – Type III transition of EE

$n_{th} = O\left(\log \frac{L}{\varepsilon}\right)$

If the Renyi index n is big enough,

$$\frac{2\pi n}{\log \frac{L}{\varepsilon}} \gg \beta_{\text{Hawking-Page}} \sim 2\pi$$

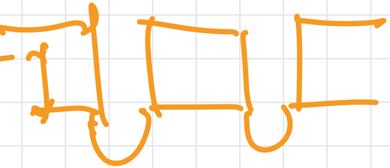
but if the Renyi index n is small, the entanglement spectrum becoms continuous $\Longrightarrow$ algebra is Type III

even though we have UV cut off

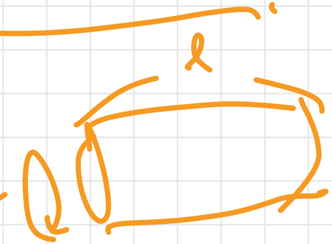
Summary

- Reviewed the split property in Type III vN alg.
- Provided a physics construction of the intermediate algebra (also a physics proof that the split property holds)
- Even if you introduce a UV cutoff, the algebra can be Type II at large-N.

$$\frac{\sum - t\,(\log \sum \ell N)}{\ell} = O(1)$$

δ 

$$\frac{\quad}{\ell}$$

$2\pi n$ 

$\ell \quad \forall n$

$$\overline{\log \ell_0}$$

$$\frac{N}{\ell} = O(1)$$

n

$$\frac{2\pi n}{\ell} N \leq O(1)$$

$$\frac{\ell}{2\pi n} \geq N$$

$$e^{-\frac{2\pi n}{\ell} N \Delta}$$

$$\ell \ll 2\pi n$$