Go with the flow, hydrodynamic theory of electrically driven non-equilibrium steady states

Matteo Baggioli

Jiao-Tong University Shanghai

Based on

arXiv > cond-mat > arXiv:2404.05568

Condensed Matter > Statistical Mechanics

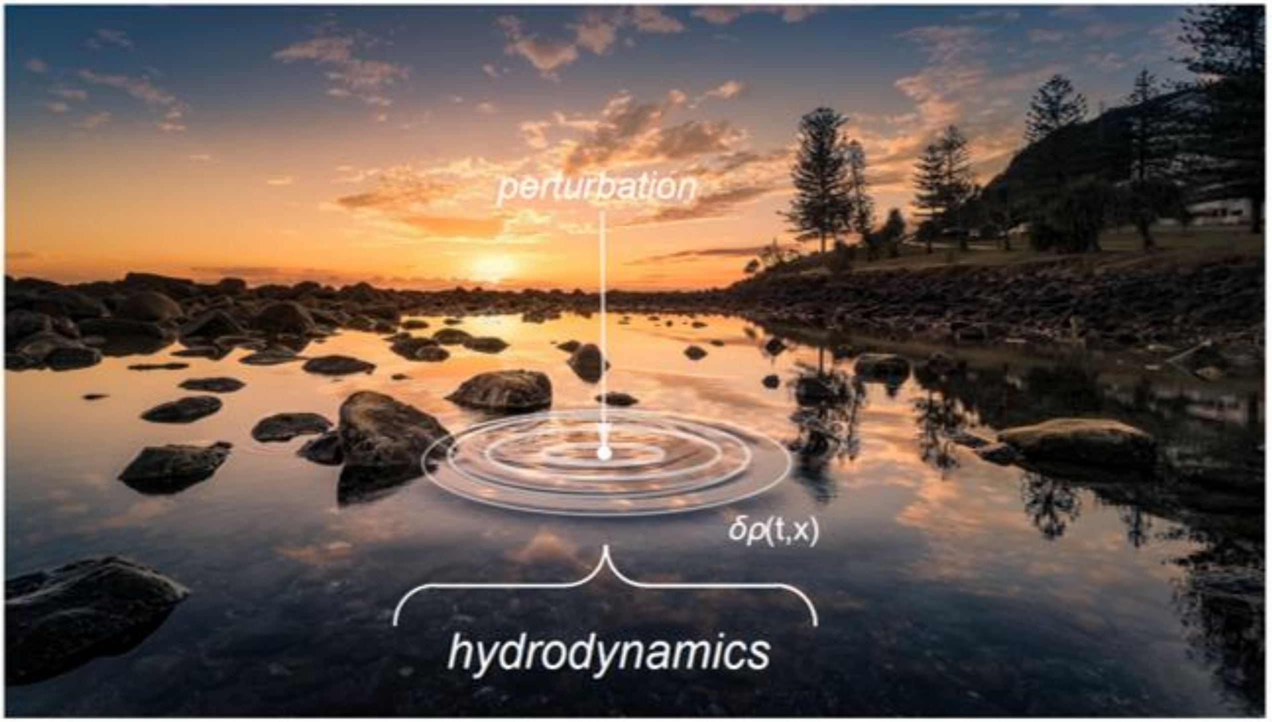
[Submitted on 8 Apr 2024]

Relaxed hydrodynamic theory of electrically driven non-equilibrium steady states

Daniel K. Brattan, Masataka Matsumoto, Matteo Baggioli, Andrea Amoretti

What this is about in one slide

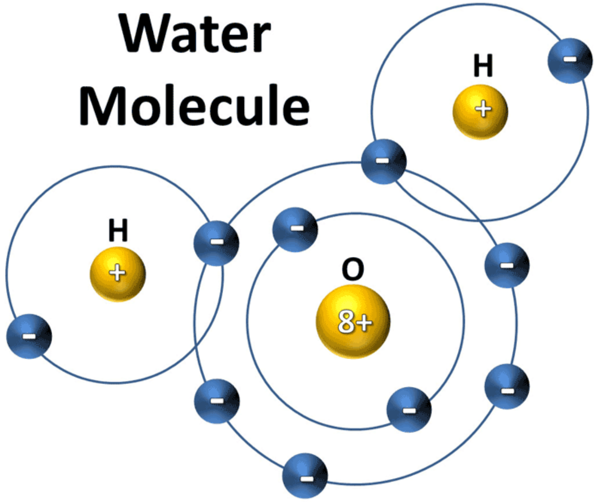
thermal equilibrium state

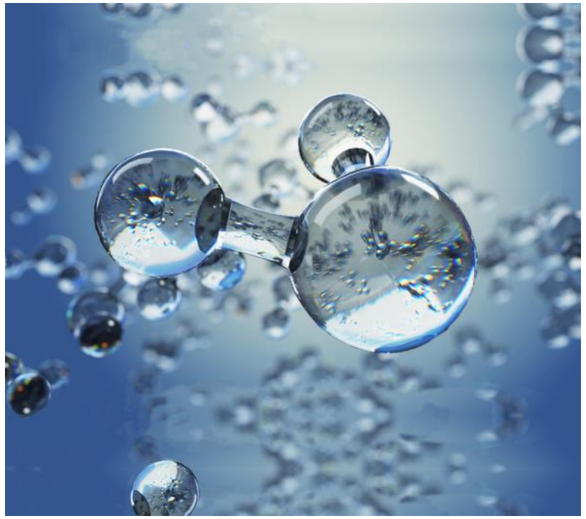


non–equilibrium steady state

Answer: yes, <u>but</u> …

What is hydrodynamics ? *[conservative view]*





Effective description of long-wavelength and late-time dynamics around thermal equilibrium

Long-wavelength and late time

Characteristic length scale of the problem is much larger than the mean free path

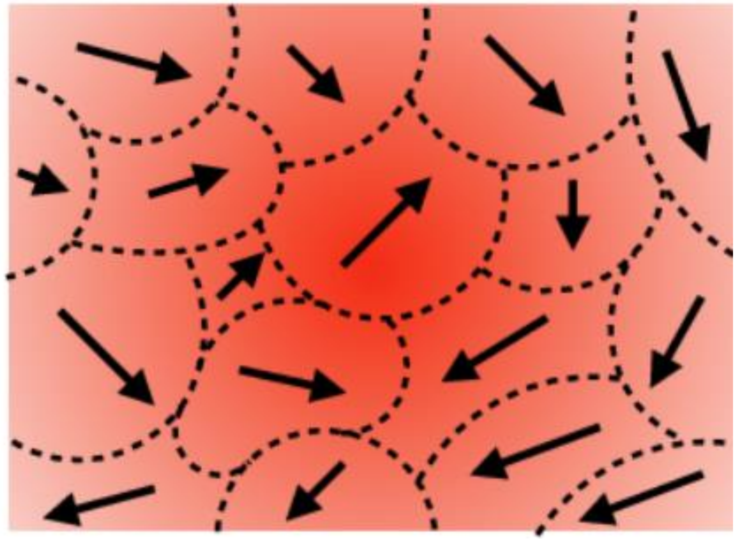
Characteristic time scale of the problem is much larger than the microscopic relaxation time

Knudsen number

$$Kn = \frac{l_{\rm mfp}}{L}$$

must be small!

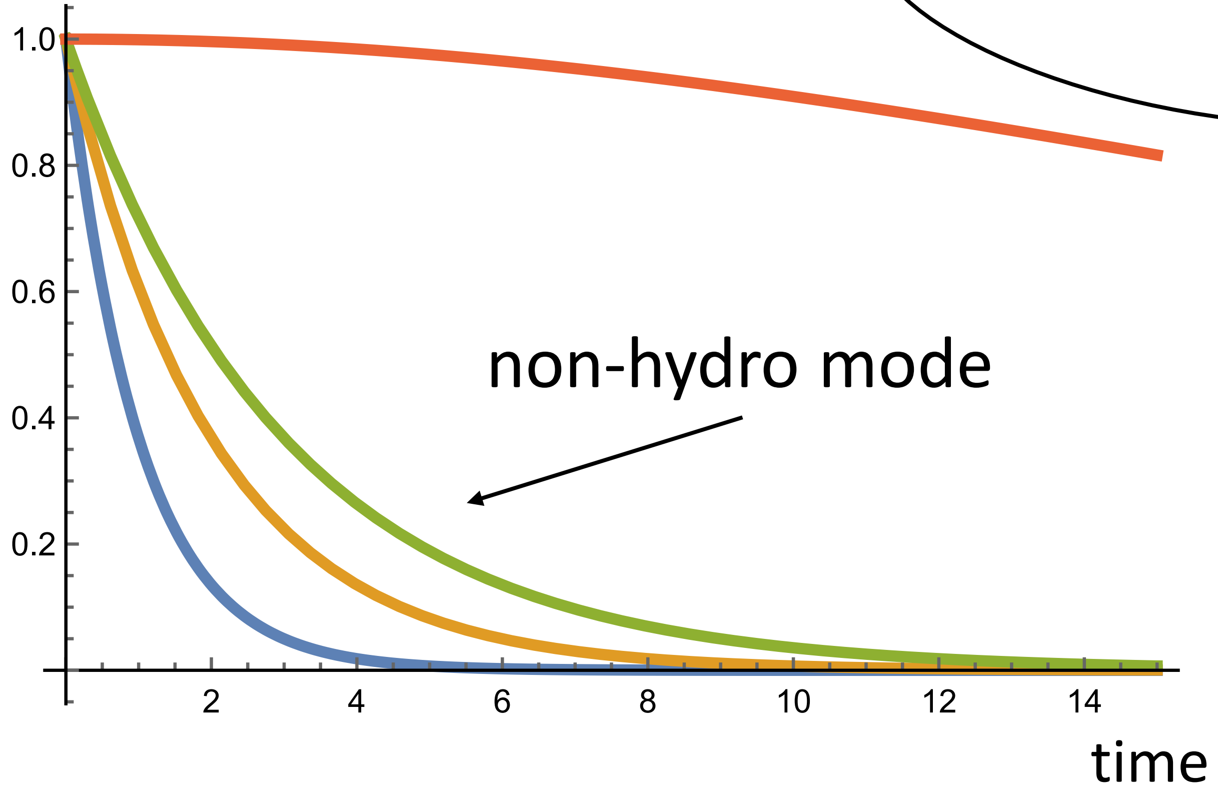
local thermal equilibrium



Effective

Non-conserved quantities relax locally [irrelevant at late time]

Conserved quantities relax only via transport



hydrodynamic modes

$$\omega(k=0)=0$$

[only relevant degrees of freedom]

Thermal equilibrium

In the classical formalism, hydrodynamics describes fluctuations around thermal equilibrium

Example: promote temperature T to local temperature T(x)

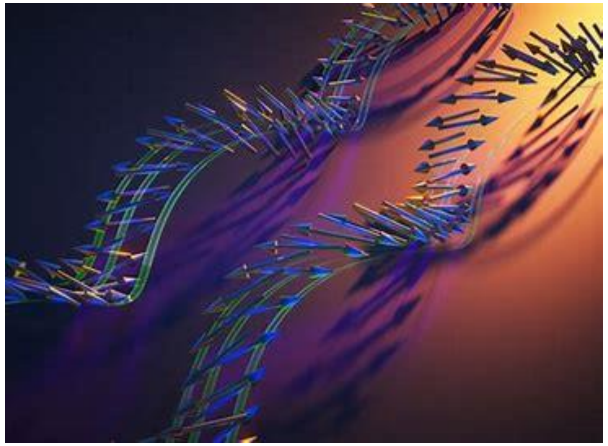
In a way, it is a order 1 extension of thermodynamics

Several constraints:

- Detailed balance
- Fluctuation-dissipation theorem
- Thermodynamic laws
- Onsager relations
- Microscopic time-reversal symmetry
- …

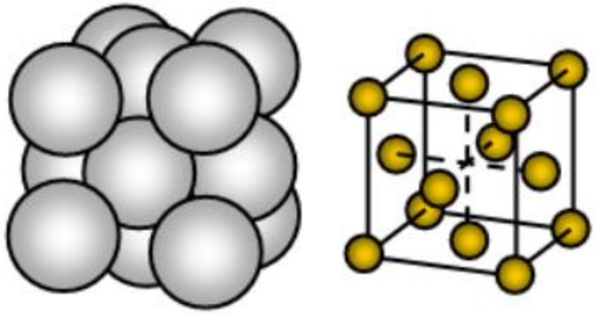
Very general (beyond fluids)

[find the intruder]



Hydrodynamic Theory of Spin Waves

B. I. HALPERIN and P. C. HOHENBERG
Phys. Rev. **188**, 898 – Published 10 December 1969

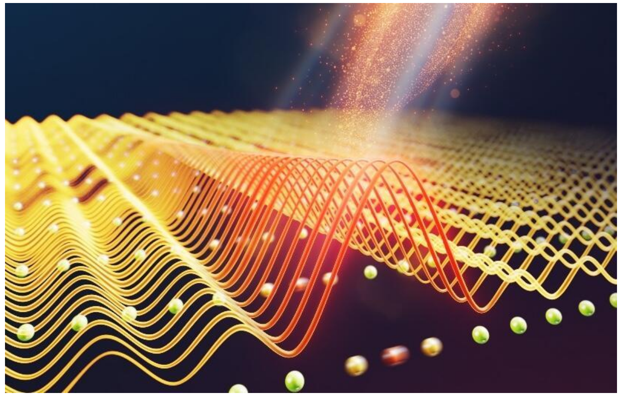


Unified Hydrodynamic Theory for Crystals, Liquid Crystals, and Normal Fluids

P. C. Martin, O. Parodi, and P. S. Pershan
Phys. Rev. A **6**, 2401 – Published 1 December 1972

Hydrodynamics of soft active matter

M. C. Marchetti, J. F. Joanny, S. Ramaswamy, T. B. Liverpool, J. Prost, Madan Rao, and R. Aditi Simha
Rev. Mod. Phys. **85**, 1143 – Published 19 July 2013



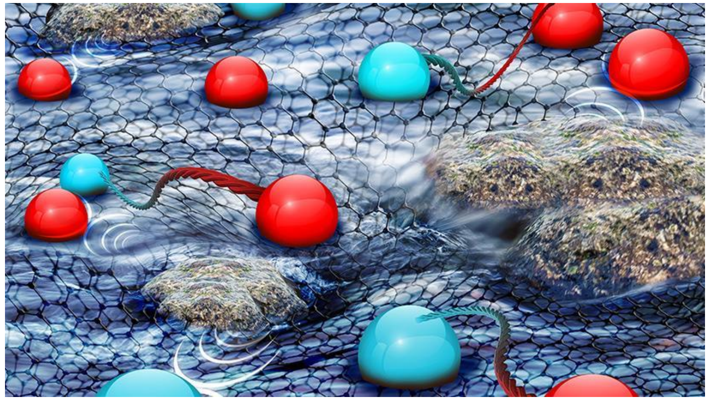
Colloquium: Hydrodynamics and holography of charge density wave phases

Matteo Baggioli and Blaise Goutéraux
Rev. Mod. Phys. **95**, 011001 – Published 4 January 2023

Crowds flow like rivers

Mary Elizabeth Sutherland

Nature Human Behaviour **3**, 207 (2019) | C



Charge transport and hydrodynamics in materials

Georgios Varnavides, Amir Yacoby, Claudia Felser & Prineha Narang

Nature Reviews Materials **8**, 726–741 (2023) | Cite this article

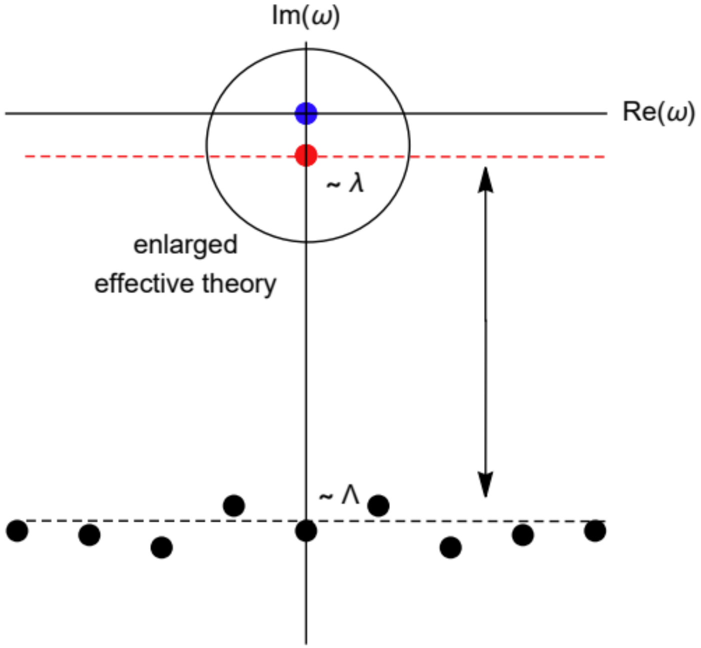
(Some) modern challenges

(1) Almost conserved quantities/ soft modes
(broken symmetries, critical points)
quasi-hydrodynamics

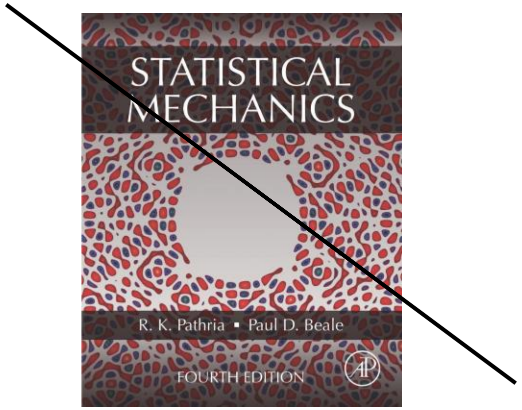
arXiv:2403.14254 [pdf, other] hep-th

Developments in quasihydrodynamics

Authors: Luca Martinoia



(2) Non-equilibrium steady states
(driven systems, active matter, open systems)



A simple scenario with a not so simple question

Imagine throwing color into a glass of water

At large scales and late times, what will happen after can be predicted by hydrodynamics

Now, do the same in a steadily moving river

Can hydrodynamics predict fluctuations around non-equilibrium steady states?

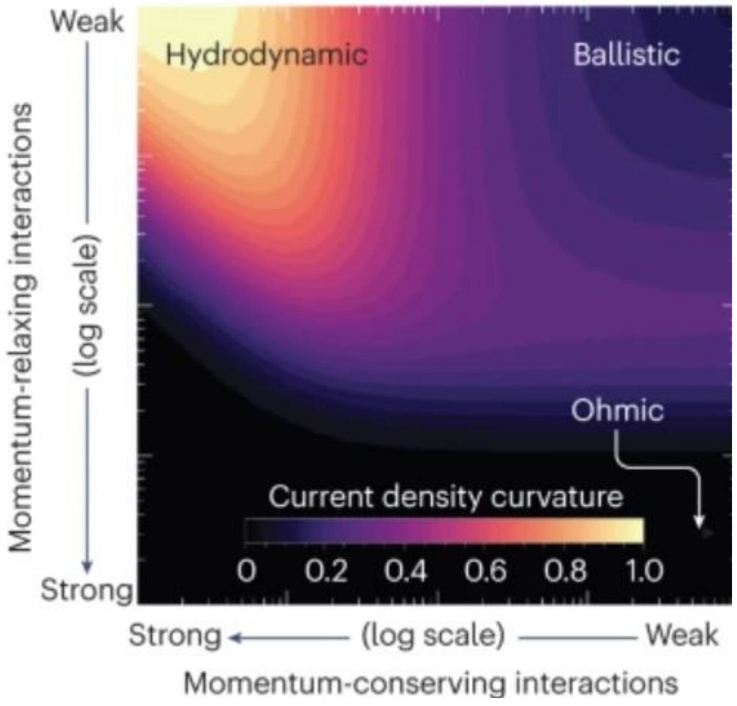
Why it is relevant (one example)

In ultra-clean systems, electrons can flow collectively as a fluid (electron hydrodynamics)

Charge transport and hydrodynamics in materials

Georgios Varnavides, Amir Yacoby, Claudia Felser & Prineha Narang

Nature Reviews Materials **8**, 726–741 (2023) | Cite this article

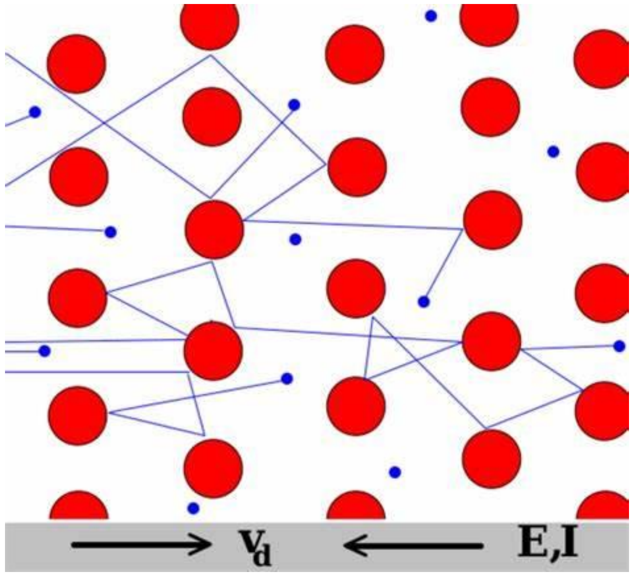


Drude model

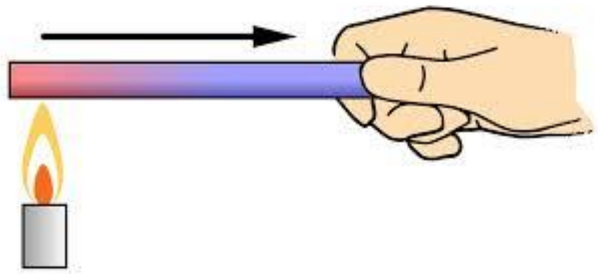
$$\frac{d}{dt}\langle \mathbf{p}(t)\rangle = q\left(\mathbf{E} + \frac{\langle \mathbf{p}(t)\rangle}{m} \times \mathbf{B}\right) - \frac{\langle \mathbf{p}(t)\rangle}{\tau},$$

Electric conduction in metals is a non-equilibrium steady state with a drift velocity

in the order of 10^{-3} meters per second



Historical analogy: thermal conduction



Heat waves

D. D. Joseph and Luigi Preziosi
Rev. Mod. Phys. **61**, 41 – Published 1 January 1989

$$\mathbf{q} = -k\nabla T,$$

Fourier law

Add ``energy conservation'' and you get a diffusion equation: heat diffuses

Infinite speeds of propagation are generated by diffusion

$$(1 + \tau\partial_t)\ \vec{Q} = -\kappa\vec{\nabla}T.$$

Maxwell-Cattaneo equation

``thermal inertia'', usually very small (ps) [but not always! e.g. sand]

Historical analogy: thermal conduction

Heat Conduction Paradox Involving Second-Sound Propagation in Moving Media

C. I. Christov and P. M. Jordan
Phys. Rev. Lett. **94**, 154301 – Published 22 April 2005

In this Letter, we revisit the Maxwell-Cattaneo law of finite-speed heat conduction. We point out that the usual form of this law, which involves a partial time derivative, leads to a paradoxical result if the body is in motion. We then show that by using the material derivative of the thermal flux, in lieu of the local one, the paradox is completely resolved. Specifically, that using the material derivative yields a

$$\left[1+\tau\left(\partial_t+\vec{v}_{\mathrm{drift}}\cdot\vec{\nabla}\right)\right]\vec{Q}=-\kappa\vec{\nabla}T.$$

All in all, this leads to the final result

$$\tau\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}+\vec{v}_{\mathrm{drift}}\cdot\frac{\partial\vec{\nabla}T}{\partial t}-D\nabla^2 T=\mathcal{R},$$

A few important (and recurrent) points

$$\tau\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} + \vec{v}_{\text{drift}} \cdot \frac{\partial \vec{\nabla} T}{\partial t} - D\nabla^2 T = \mathcal{R},$$

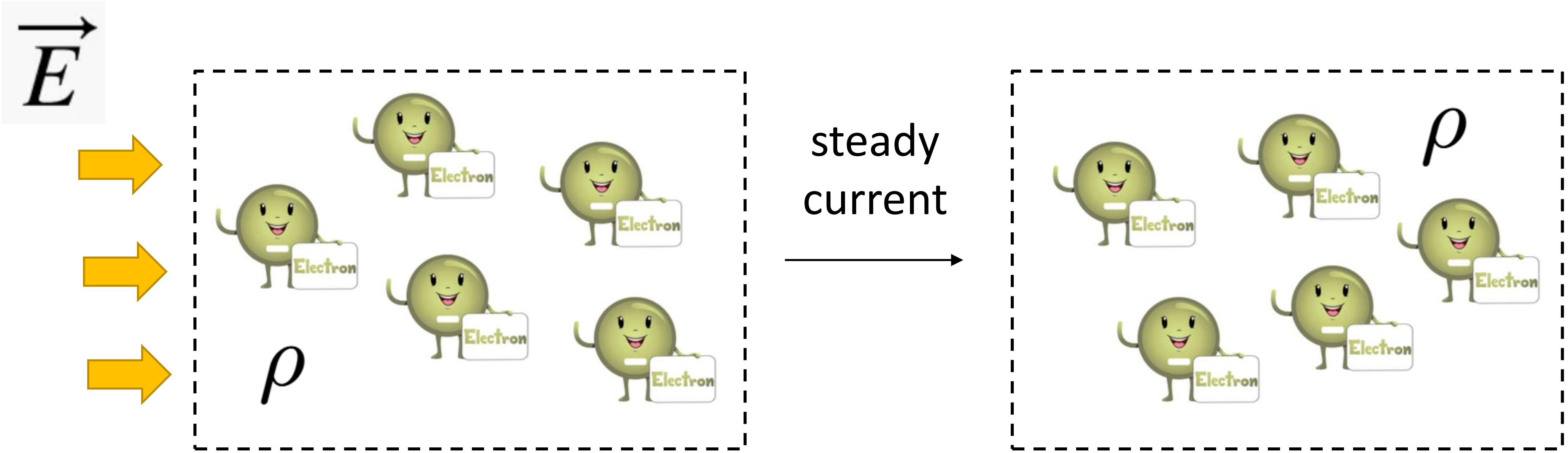
(1) Relaxation mechanisms are unavoidable (quasi-hydro)
(2) External driving/ drift velocity also

Consequences:

(1) Background state is not thermal equilibrium
(so thermodynamic requirements do not apply)
(2) Boost invariance is necessarily broken
(3) Non-hydro relaxing modes are necessarily present

Our system

A toy model for electric conduction in real materials



- Stationary system
- Non-uniform charge density
- Constant driving electric field
- Uniform spatial charge current

$$\vec{J} = \sigma_{\mathrm{DC}}(\rho, \vec{E}^2)\vec{E}$$

Modeling

Obvious piece
[charge conservation, U(1) symmetry]

$$\partial_t \delta\rho + \vec{\nabla} \cdot \delta\vec{J} = 0$$

Further assumptions:

1. We consider only the dynamics of the charge and the spatial current [``probe limit'']
2. We assume the electric field as an order 0 background quantity [not a small perturbation]
3. We go up to order 2 in derivatives and order E^2 in the electric field

Modeling

[1] = - [2]
condition for
homogeneous steady state

$$\partial_t \delta J^i + \underbrace{\partial_j \mathcal{T}^{ij}}_{[3]} - \underbrace{\alpha E^i \delta\rho}_{[2]} + \underbrace{\frac{1}{\tau}\delta J^i}_{[1]} = 0 \; ,$$

[1]: relaxation of the spatial current
[2]: driving of the electric field (drift velocity)
[3]: generalized stress tensor [at order E, what is shown below]

$$\mathcal{T}^{ij} = v^2 \delta^{ij}\delta\rho + (\eta_1 + \eta_2)\,\frac{E^i E^j E_k}{\vec{E}^2}\delta J^k + (\eta_2 + \eta_3)\,E^{(i}\Pi_k{}^{j)}\delta J^k$$

$$+ (\eta_2 - \eta_3)\,E^{[i}\Pi_k{}^{j]}\delta J^k + \eta_4 \Pi^{ij}\delta J^k E_k,$$

and $\Pi^{ij} \equiv \delta^{ij} - \frac{E^i E^j}{\vec{E}^2}$ is a projector with respect to the electric vector field.

(Re)modeling

$$\tau \frac{\partial^2 \delta\rho}{\partial t} + \frac{\partial \delta\rho}{\partial t} + \vec{v}_{\mathrm{drift}} \cdot \nabla \delta\rho - D \nabla^2 \delta\rho = \mathcal{R}$$

relaxation

drift

diffusion

source terms

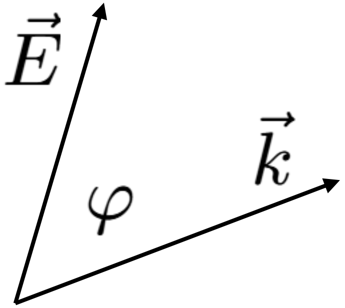
$$\vec{v}_{\mathrm{drift}} = -\alpha\tau\vec{E}\,, \qquad D = \tau v^2\,,$$

$$\mathcal{R} = \tau\eta_4 \nabla_\perp^2 (\vec{E} \cdot \delta\vec{J}) + \tau\,(\eta_1 + \eta_2)\,(\vec{E} \cdot \nabla)^2 \vec{E} \cdot \delta\vec{J} + \tau\,(\eta_2 + \eta_3)\,(\vec{E} \cdot \nabla)(\nabla_\perp \cdot \delta\vec{J})$$

Complex interplay between diffusion, relaxation and advection

Familiar structure (compare to thermal transport problem)

Predictions [up to order E]



(1) Static limit $\delta\vec{E} = i\vec{k}\delta A_t$

$$\langle\rho\rho\rangle_{\mathrm{R}}(0,\vec{k}) = \frac{|\vec{k}|\chi}{v^2|\vec{k}| + i\alpha|\vec{E}|\cos\varphi}$$

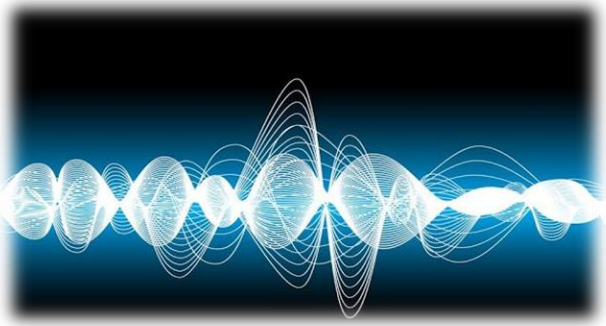
$\varphi = \pi/2$: all ``normal'' . Charge susceptibility $\chi_{\rho\rho} = \chi/v^2$

For all other angles, it vanishes at k=0 !!!

$\lambda = \frac{D}{|\vec{v}_{\mathrm{drift}}|}$: emergent length-scale (competition between advection and diffusion)

Predictions [up to order E]

(2) Collective modes



For $\vec{k}$ parallel to $\vec{E}$, they are characterized by the following dispersion relations

$$\omega_{\rm gapless} = \alpha\,\tau\,\vec{k}\cdot\vec{E} + \mathcal{O}(\vec{k}^2,\vec{E}^2)\ , \qquad [10a]$$

$$\omega_{\rm gapped} = -\frac{i}{\tau} + (\eta_1+\eta_2-\alpha\,\tau)\,\vec{k}\cdot\vec{E} + \mathcal{O}(\vec{k}^2,\vec{E}^2)\ , \qquad [10b]$$

in the longitudinal sector, while there is a multiplicity two gapped mode with

$$\omega_\perp = -\frac{i}{\tau} + \eta_3\,\vec{k}\cdot\vec{E} + \mathcal{O}(\vec{k}^2,\vec{E}^2)\ , \qquad [10c]$$

in the transverse sector.

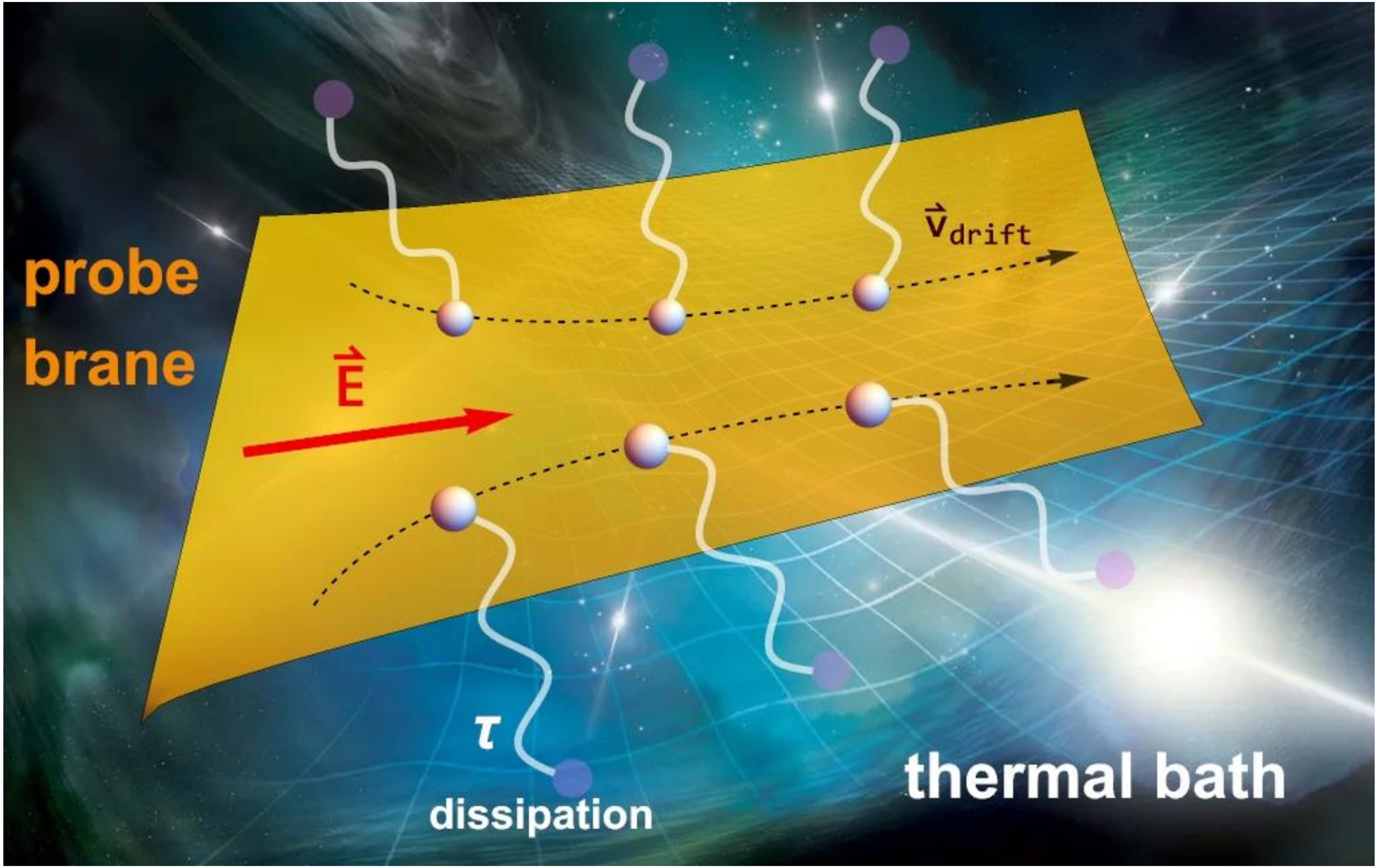
More predictions valid to $\mathcal{O}(\vec{E}^2)$

$$\partial_t \delta J^i + \partial_j \Big(v_\parallel^2 \frac{E^i E^j}{\vec{E}^2}\delta\rho + v_\perp^2 \Pi^{ij}\delta\rho + (\eta_1+\eta_2)\frac{E^i E^j E_k}{\vec{E}^2}\delta J^k + (\eta_2+\eta_3) E^{(i}\Pi_k{}^{j)}\delta J^k$$
$$+ (\eta_2-\eta_3) E^{[i}\Pi_k{}^{j]}\delta J^k + \eta_4 \Pi^{ij}\delta J^k E_k \Big) - \alpha E^i \delta\rho + \frac{1}{\tau_\parallel}\frac{E^i E^j}{\vec{E}^2}\delta J_j + \frac{1}{\tau_\perp}\Pi^{ij}\delta J_j$$
$$- \chi_\parallel \frac{E^i E^j}{\vec{E}^2}\delta E_j - \chi_\perp \Pi^{ij}\delta E_j + \chi_B \Pi^i{}_j \epsilon^{jkl} E_k \delta B_l + \zeta_\parallel \frac{E^i E^j}{\vec{E}^2}\partial_t \delta E_j + \zeta_\perp \Pi^{ij}\partial_t \delta E_j$$
$$+ \theta_\parallel \frac{E^i E^j}{\vec{E}^2} E^k \partial_k \delta E_j + \theta_\perp \Pi^{ij} E^k \partial_k \delta E_j + \theta_\otimes E^i \Pi^{jk}\partial_j \delta E_k + \theta_\bullet \Pi^{ij} E^k \partial_j \delta E_k$$
$$+ \frac{\theta_B}{\vec{E}^2}\Pi^i{}_j \epsilon^{jkl} E_k E^m \partial_m \delta B_l + \theta_b E^i \epsilon^{jkl} E_k \Pi_l{}^m \partial_m \delta B_j + \theta_\beta \epsilon^{ijk} E_j \Pi_k{}^l E^k \partial_l \delta B_k = 0 + \mathcal{O}(\partial^2, \delta^2) \, ,$$

We will see after why we need to go to second order…

So far, hydrodynamic theory…

We need a microscopic model to confirm its validity !!


probe
brane
$\vec{E}$
$\vec{v}_{drift}$
τ
dissipation
thermal bath

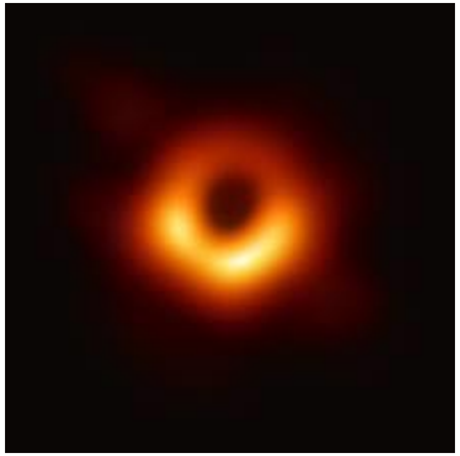
Thermal bath: Black brane in asymptotically AdS spacetime

$$ds^2 = \frac{\ell^2}{u^2}\left(-f(u)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 + \frac{\mathrm{d}u^2}{f(u)}\right) + \ell^2 \mathrm{d}\Omega_5^2,$$

$$f(u) = 1 - u^4/u_{\mathrm{H}}^4$$

$$T = 1/(\pi u_{\mathrm{H}}).$$

Matter degrees of freedom: D7 probe branes (DBI action)



$$S_{D7} = -T_{D7}\int \mathrm{d}^8\xi \sqrt{-\det\left(g_{ab} + 2\pi\alpha' F_{ab}\right)},$$

Background (order 0) electric field and finite charge density

$$A_t = A_t(u) \text{ and } A_x = -Et + h(u)$$

A few details

Nonlinear electric conductivity

$$\rho = \frac{\mathcal{N} C^2 A_t'}{u\sqrt{1 - C^2 u^4 (A_t'^2 + f^{-1} E^2 - f h'^2)}},$$

$$J = \frac{\mathcal{N} C^2 f h'}{u\sqrt{1 - C^2 u^4 (A_t'^2 + f^{-1} E^2 - f h'^2)}},$$

$$\frac{\sigma_{\rm DC}}{\pi T} = \frac{\tilde{J}}{\tilde{E}} = \left(\frac{C^2 \tilde{\rho}^2}{1 + C^2 \tilde{E}^2} + \mathcal{N} C^4 \sqrt{1 + C^2 \tilde{E}^2} \right)^{1/2}$$

$$(\tilde{J}, \tilde{E}, \tilde{\rho}) \equiv (J/(\pi T)^3, E/(\pi T)^2, \rho/(\pi T)^3)$$

Linear fluctuations

$$A_t \to A_t(u) + \delta A_t(t, x, y, u),$$
$$A_x \to -Et + h(u) + \delta A_x(t, x, y, u),$$
$$A_y \to 0 + \delta A_y(t, x, y, u),$$

All excitations
[dispersion relations w(k)]

All retarded Green functions

What has been done before

arXiv > hep-th > arXiv:1709.01520

High Energy Physics - Theory

[Submitted on 5 Sep 2017]

Origin of the Drude peak and of zero sound in probe brane holography

Chi-Fang Chen, Andrew Lucas

$$\partial_t \rho + \partial_i J^i = 0,$$

$$\partial_t J^i + v^2 \partial_i \rho = \chi E^i - \frac{J^i}{\tau},$$

$$\frac{1}{\tau} = \frac{(4\pi T)^2}{16 c \rho^{1/3}}, \qquad \chi_{\rho\rho} = \frac{\rho^{2/3}}{c}, \qquad v^2 = \frac{1}{3},$$

Electric field
is order O(1)
[not a non-equilibrium steady state
since there is no background current]

Reminder at E~O(1)

$$\partial_t \delta J^i + \partial_j \mathcal{T}^{ij} - \boxed{\alpha} E^i \delta\rho + \frac{1}{\tau}\delta J^i = 0 \,, \qquad [3b]$$

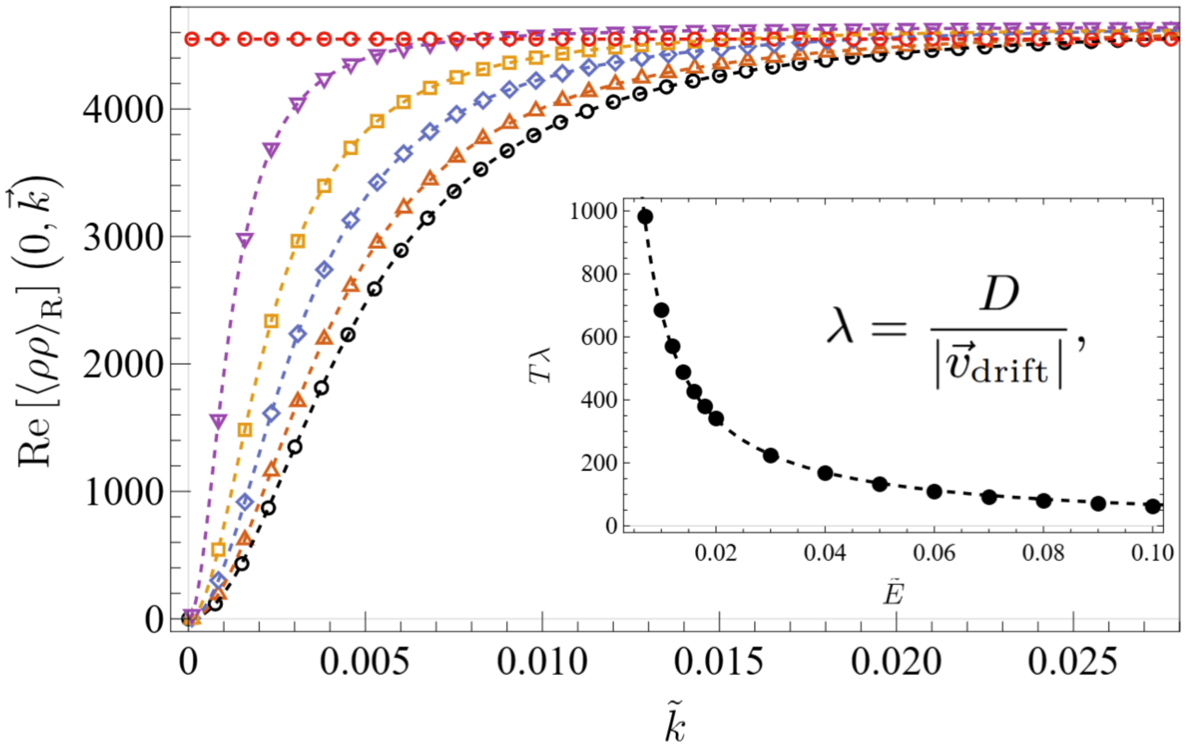
where

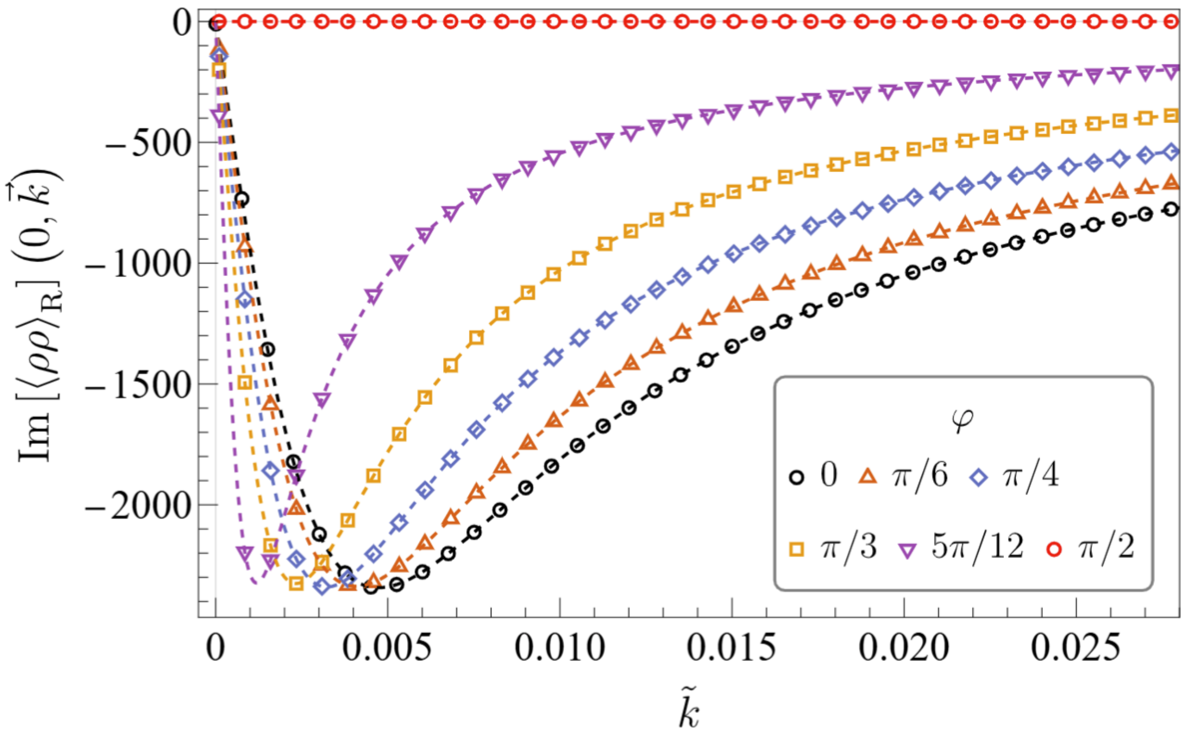
$$\mathcal{T}^{ij} = v^2 \delta^{ij} \delta\rho + \boxed{(\eta_1 + \eta_2)} \frac{E^i E^j E_k}{\vec{E}^2} \delta J^k + \boxed{(\eta_2 + \eta_3)} E^{(i} \Pi_k{}^{j)} \delta J^k$$

$$+ \boxed{(\eta_2 - \eta_3)} E^{[i} \Pi_k{}^{j]} \delta J^k + \boxed{\eta_4} \Pi^{ij} \delta J^k E_k \,, \qquad [4]$$

→ 5 unknowns

Results





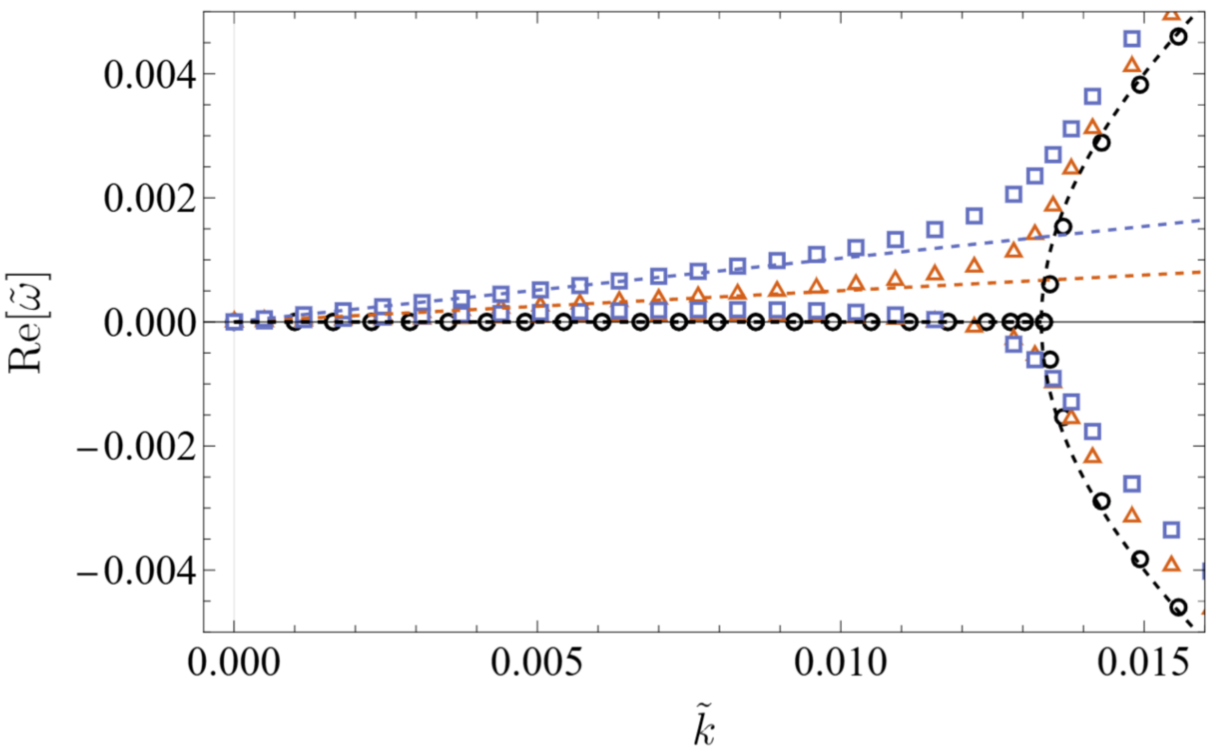
$$\langle\rho\rho\rangle_{\mathrm{R}}(0,\vec{k}) = \frac{|\vec{k}|\chi}{v^2|\vec{k}| + i\alpha|\vec{E}|\cos\varphi}$$

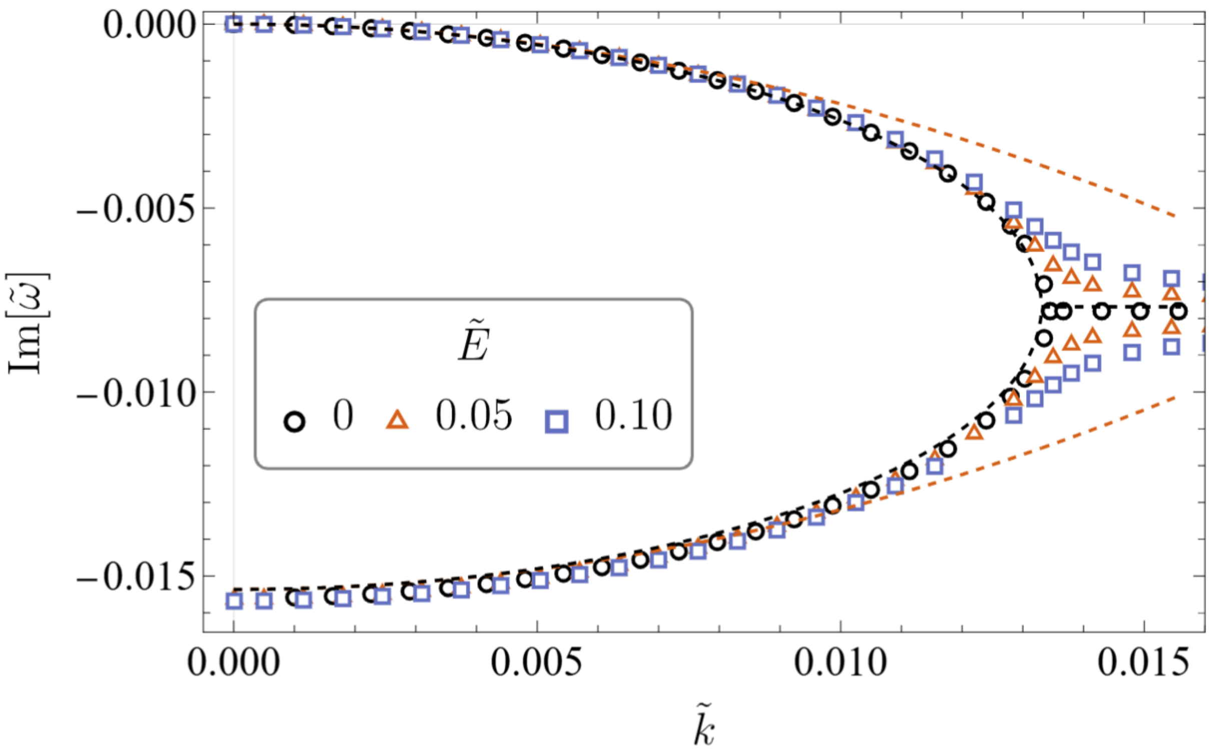
Fit one angle: get alpha
All other curves predicted

Results

$$\omega_{\text{gapless}} = \alpha\,\tau\,\vec{k}\cdot\vec{E} + \mathcal{O}(\vec{k}^2, \vec{E}^2)\,,$$

$$\omega_{\text{gapped}} = -\frac{i}{\tau} + (\eta_1 + \eta_2 - \alpha\,\tau)\,\vec{k}\cdot\vec{E} + \mathcal{O}(\vec{k}^2, \vec{E}^2)$$



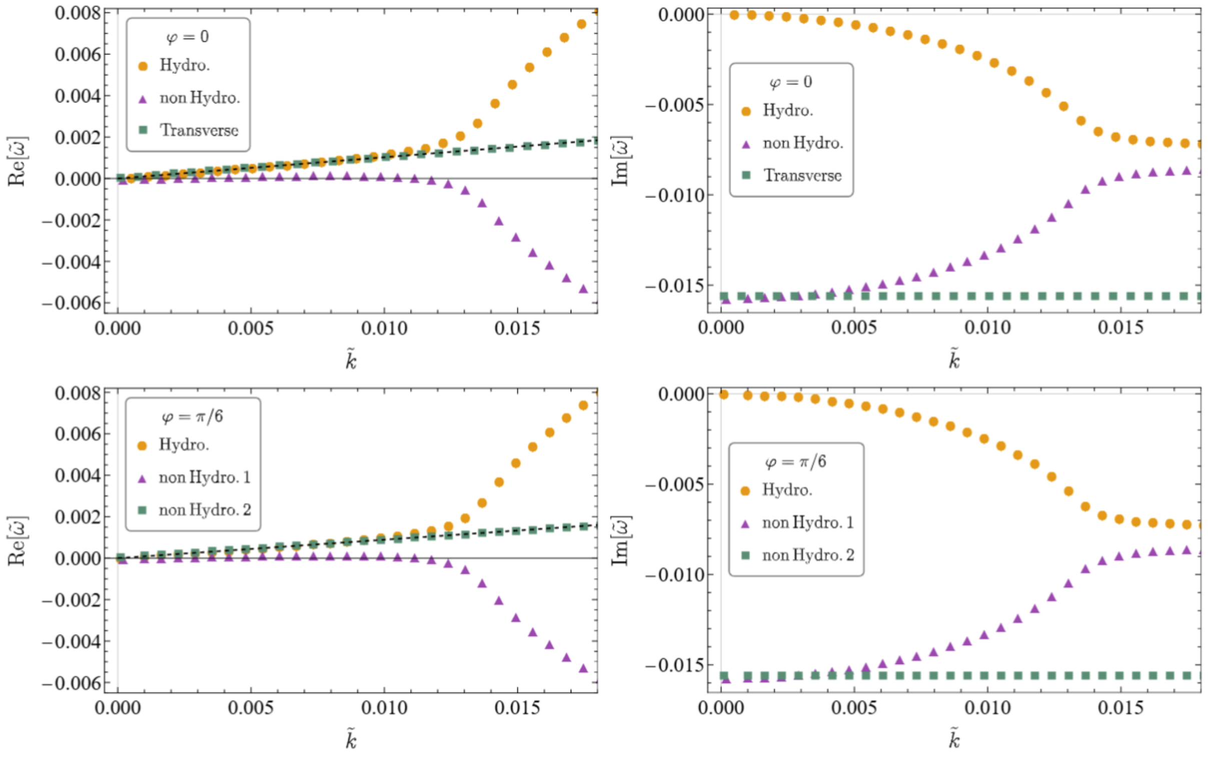


We determine

$$\eta_{1+2} \equiv \eta_1 + \eta_2$$

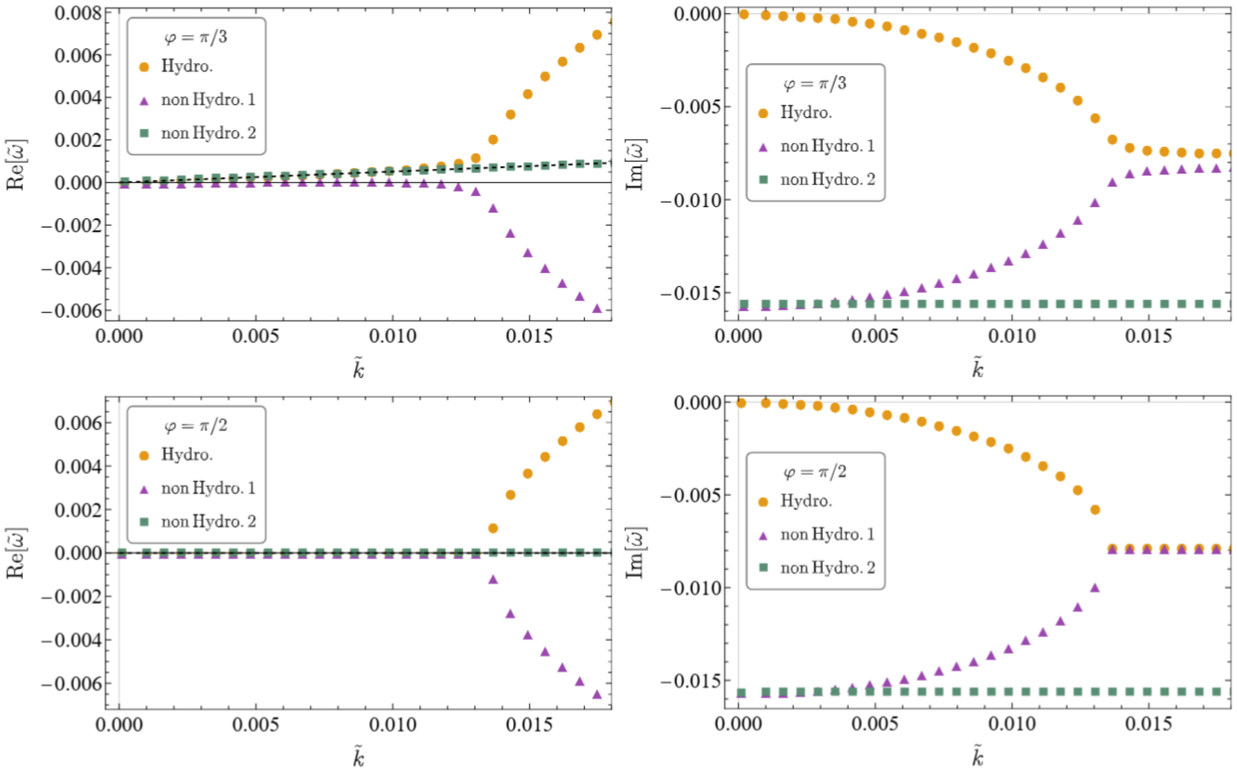
Interestingly, at this order

$$\eta_{1+2} = 4/3\alpha\tau$$


$\varphi = 0$
Hydro.
non Hydro.
Transverse
$\mathrm{Re}[\tilde{\omega}]$
$\mathrm{Im}[\tilde{\omega}]$
$\tilde{k}$
$\varphi = \pi/6$
Hydro.
non Hydro. 1
non Hydro. 2
0.000
0.005
0.010
0.015
0.008
0.006
0.004
0.002
−0.002
−0.004
−0.006
−0.005
−0.010
−0.015


MORE


RELAX, IT'S GOOD
IT'S ALLLLL GOOD


$\varphi = \pi/3$
Hydro.
non Hydro. 1
non Hydro. 2
$\varphi = \pi/2$
$\mathrm{Re}[\tilde{\omega}]$
$\mathrm{Im}[\tilde{\omega}]$
$\tilde{k}$
0.000
0.005
0.010
0.015
0.008
0.006
0.004
0.002
−0.002
−0.004
−0.006
−0.005
−0.010
−0.015

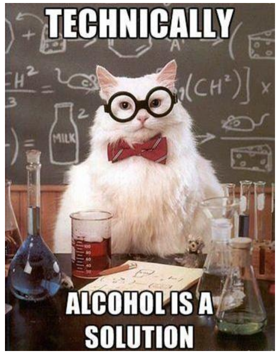
Problem

$$\omega_\perp = -\frac{i}{\tau} + \eta_3\,\vec{k}\cdot\vec{E} + \mathcal{O}(\vec{k}^2,\vec{E}^2)$$

We can also fit this other mode and get η_3

Interestingly, we obtain: $\eta_3 = \alpha\tau$ *[relics of thermodynamic constraints ?!]*

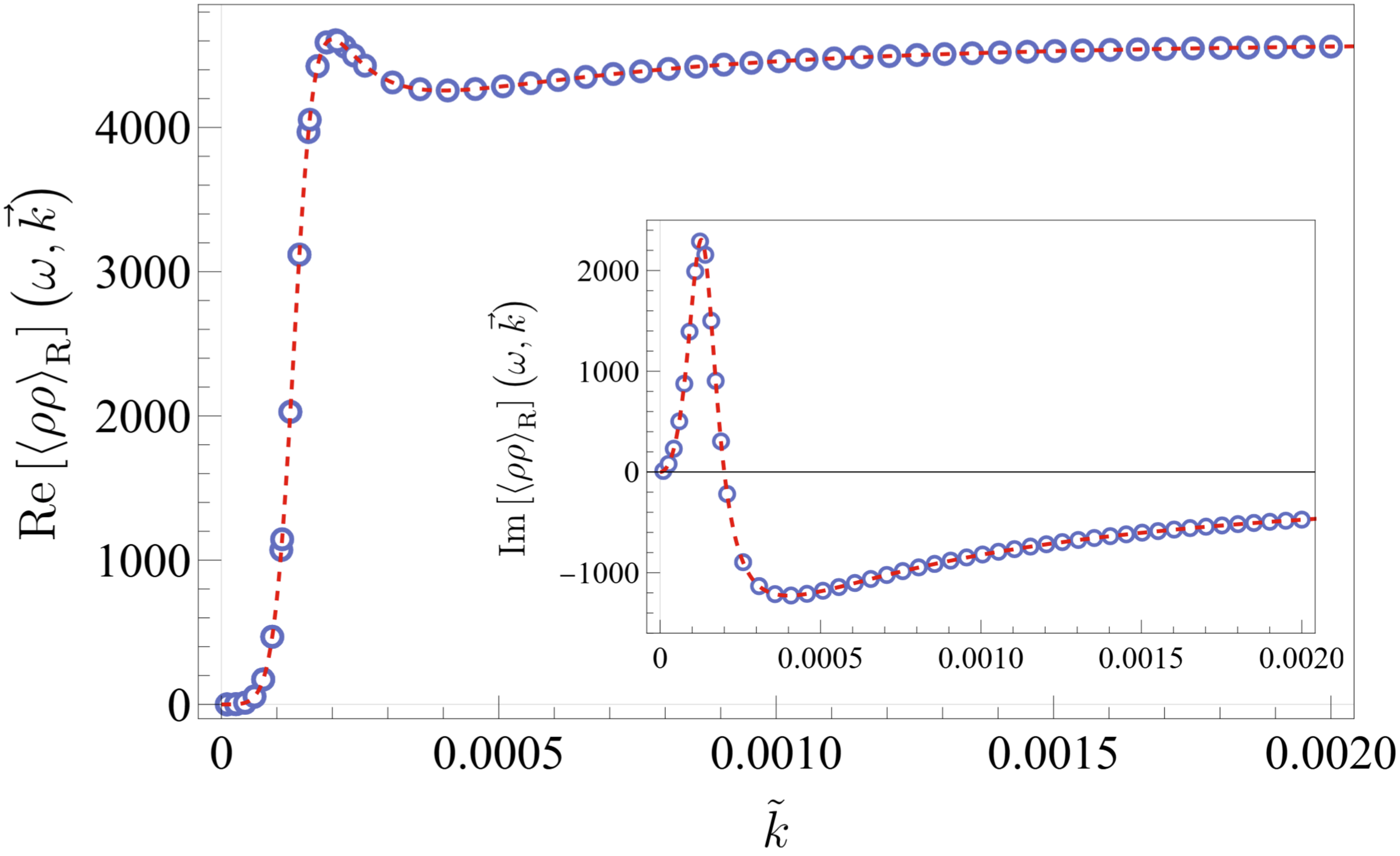
Firstly, we are only able to determine the sum of $\eta_1+\eta_2$; second, η_4 does not appear in the dispersion of the collective excitations in the limit $\tilde{E} \ll 1$. This would not be the case if the thermodynamics was known.



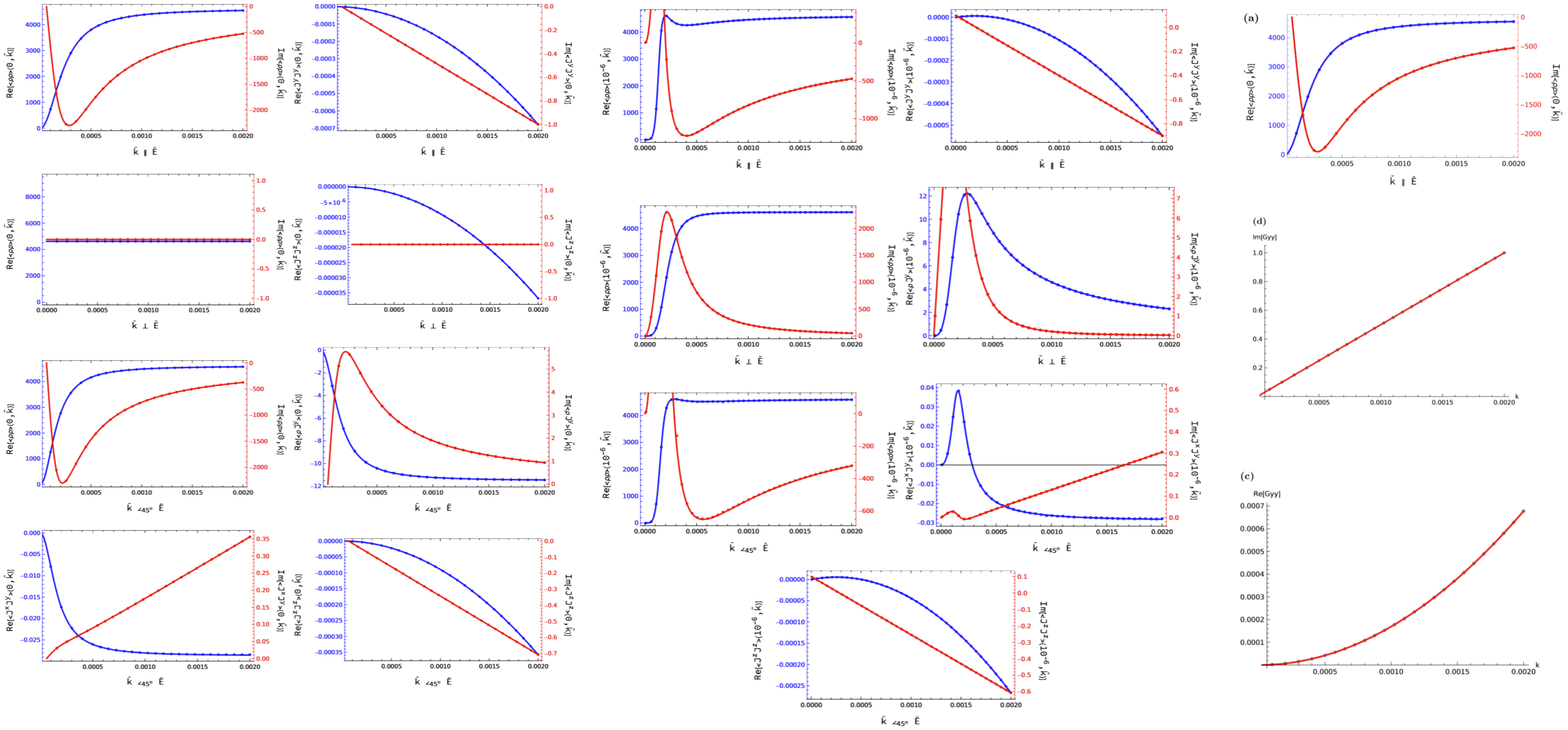
Or … go to O(E^2) and solve all correlators

One example

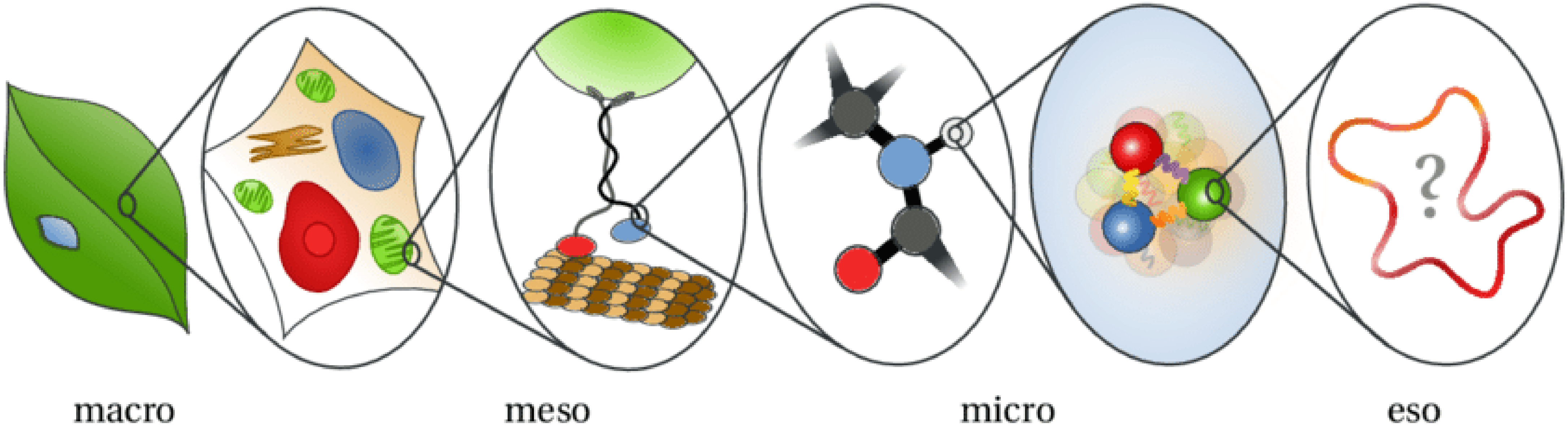
$$v_{\parallel}^2 \,,\ v_{\perp}^2 \,,\ \alpha \,,\ \tau_{\parallel} \,,\ \tau_{\perp} \,,\ \chi_{\parallel} \,,\ \chi_{\perp} \,,\ \theta_{\parallel} \,,\ \theta_{\beta} \,,\ \theta_b \,,\ \eta_1 \,,\ \eta_2 \,,\ \eta_3 \,,\ \eta_4 \,.$$



If you do not trust me



Summary



Hydrodynamic theory

Holographic model

General conclusions

Hydrodynamics is applicable and predictive for non equilibrium steady states

Late time and long wave-length dynamics are perfectly captured even if the background is not a thermal equilibrium state

Still, significant and measurable differences due to relaxation and steady state nature

Open questions

- More formal derivation of hydro
- Nonlinear evolution and other non-hydro modes
- Adding energy and momentum dynamics
- Thermodynamics of NESS [Can we use holography?]
- Universality
- …

Thanks Shingo, a good old friend!

Thanks for listening

TheoryLab

上海交通大學
Shanghai Jiao Tong University