Symmetry Enhancement in a Two-Logarithm Matrix Model and Canonical Tensor Model

ArXiv:2008.xxxxx

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§ Introduction

Quantum gravity is a fundamental problem in theoretical physics.

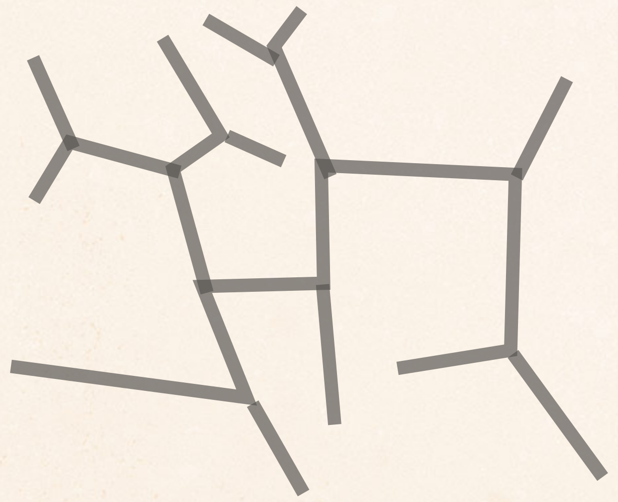
Study spacetime through efforts to construct models which can generate spacetime-like objects.

Tensor Model —Extension of Matrix Model proposed for quantum gravity in D>2.
Ambjorn et al, NS, Gross et al ’90

Usually considered in the context of Euclidean quantum gravity.

Feynman diagrams ⇆ Simplicial complexes (Colored tensor model - Gurau ’09)

Suffers from the dominance of singular spaces like branched polymers



Macroscopic spaces ?

Seems difficult to regard Tensor Model as quantum gravity in D>2

How about Lorentzian version ?

Simplicial quantum gravity seems much more successful in Lorentzian context

Causal Dynamical Triangulation (CDT) — Emergence of macroscopic spacetime

Ambjorn, Jurkiewicz, Loll, PRL 93 ('04) 131301

Canonical(Causal) Tensor Model (CTM) — A Lorentzian version of Tensor Model

NS '12

Follow the structure of ADM formalism of GR.

Formulated as a first-class constrained system in Hamiltonian formalism.

$$\mathcal{H}_a(Q,P) = \mathcal{H}_{ab}(Q,P) = 0 \qquad \{Q_{abc}, P_{def}\} = \delta_{abc,def} \qquad a, b, \ldots = 1,2,\ldots,N$$

$\mathcal{H}_a$, $\mathcal{H}_{ab}$ form a closed Poisson algebra with variable dependent structure coefficients (First-class constraints).

Classical CTM has various connections to GR

- N=1 agrees with the mini-superspace approx. of GR

NS, Sato, Phys.Lett.B 732 (2014) 32-35

- In a formal continuum limit $a \to \mathbb{R}^D$

  - Constraint algebra of CTM agrees wth that of ADM

  NS, Sato, JHEP 10 (2015) 109

  - The classical dynamics of CTM agrees with GR in the Hamilton-Jacobi formalism with a certain Hamilton's principal function

Chen, NS, Sato, Phys.Rev.D 95 (2017) 6, 066008

Quantum CTM may provide a model for quantum gravity

What do we know about quantum CTM so far ?

- $\hat{\mathcal{H}}_a(\hat{Q},\hat{P})|\Psi\rangle = \hat{\mathcal{H}}_{ab}(\hat{Q},\hat{P})|\Psi\rangle = 0$ has an exact solution.

Gaurav, NS, Sato, JHEP 01 (2015) 010

$$\Psi(P) = \langle P|\Psi\rangle = \varphi(P)^{\lambda_H/2}$$

$$\varphi(P) := \int_{\mathscr{C}} d\tilde{\phi} \prod_{a=1}^{N} d\phi_a \exp\left[i P_{abc}\phi_a\phi_b\phi_c - i\,k\,\phi^2\tilde{\phi} + i\,\tilde{\phi}^3\right]$$

$$\hat{\mathcal{H}}_a = \hat{P}_{abc}\hat{P}_{bde}\hat{Q}_{cde} - \lambda\,\hat{Q}_{abb} + i\,\lambda_H\,\hat{P}_{abb}$$

$\lambda \propto k^3$: Cosmological constant $\quad \lambda = \pm 1$ or 0

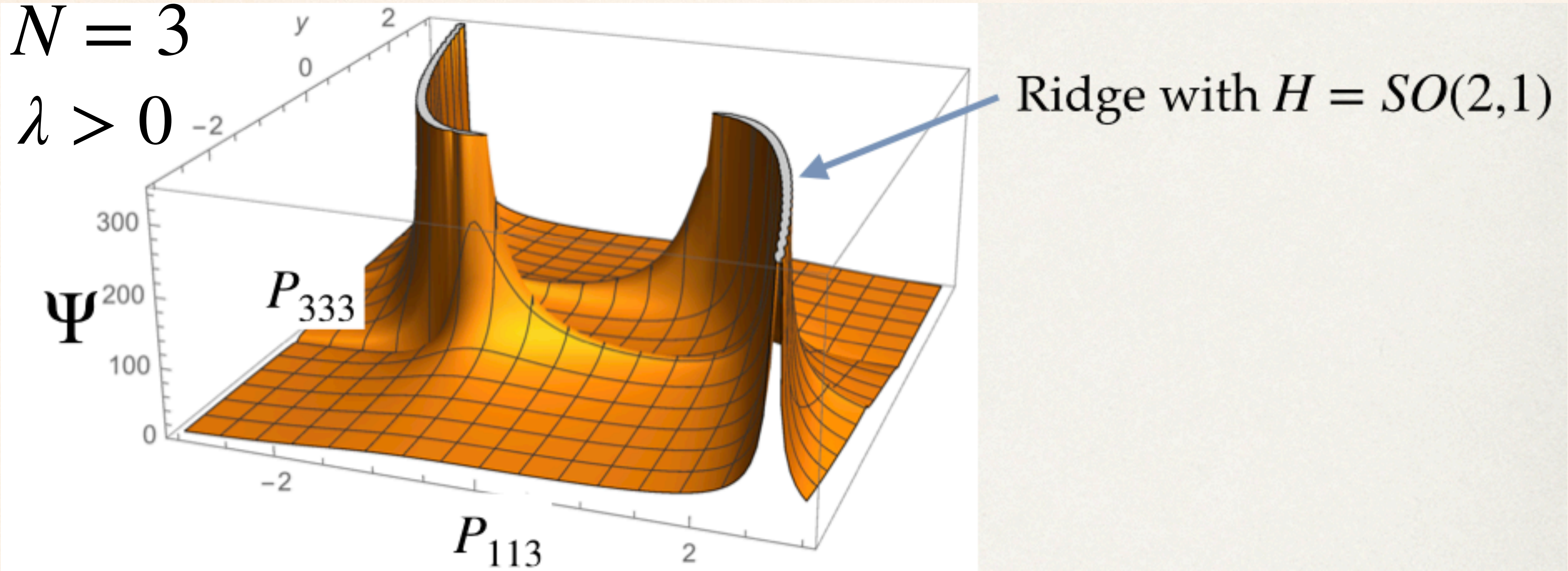
$\lambda_H = (N+2)(N+3)/2$ $\quad$ Determined by hermiticity of $\hat{\mathcal{H}}_a$

Emergence of Lie-group symmetries

$\Psi(P)$ has peaks at Lie-group invariant configurations $(g_a^{a'} g_b^{b'} g_b^{b'} P_{a'b'c'} = P_{abc},\ g \in G)$.

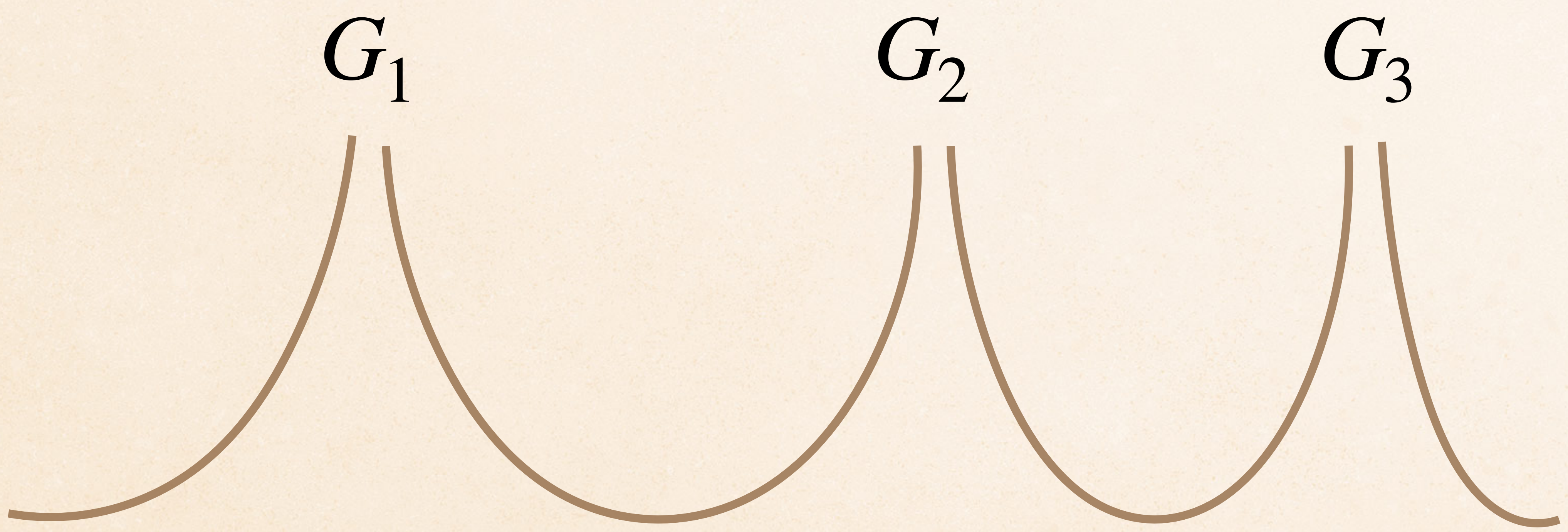
Strong peaks observed for positive cosmological constant $\lambda > 0$. Not much for $\lambda < 0$.

Obster, NS, PTEP 2018 (2018) 4, 043A01



Can ridges be interpreted as spacetime trajectory ?

But N is too small so far.

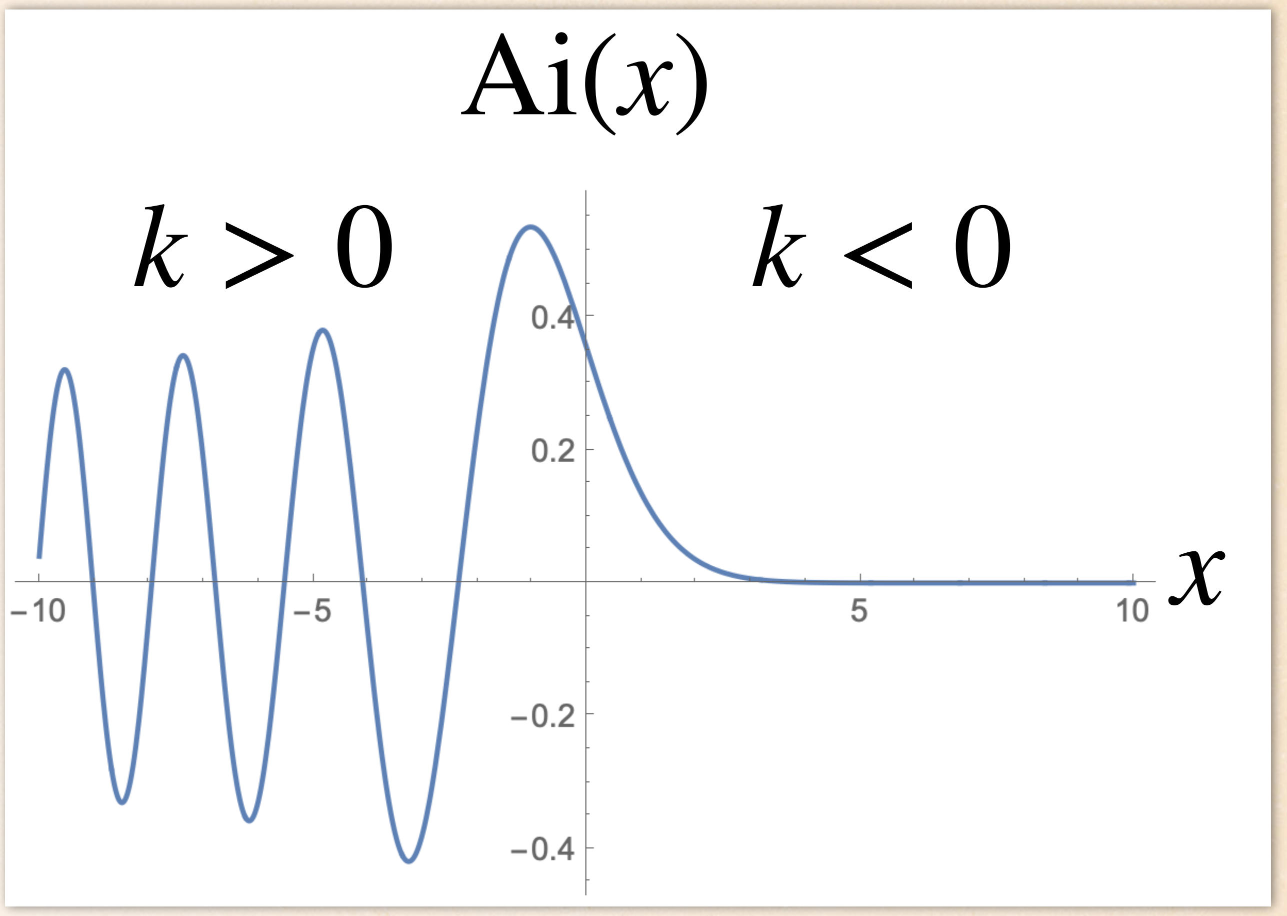


How are the peaks distributed ?

What are the dominant Lie-group syms. ?

We need more understanding of the wave function.

$$\varphi(P) := \int_{\mathbb{R}^{N+1}} d\tilde{\phi} \prod_{a=1}^{N} d\phi_a \exp\left[i P_{abc}\phi_a\phi_b\phi_c \underbrace{- i k \phi^2 \tilde{\phi} + i \tilde{\phi}^3}_{\mathrm{Ai}(-k\phi^2)} \right]$$



Negative cosmological constant case ($k < 0$) may be approximated by $\mathrm{Ai}(x) \to \exp(kx^2)$.

With this approximation, $\langle \Psi | e^{-\alpha \hat{P}^2} | \Psi \rangle$ has been studied by Monte Carlo simulations.

$\lambda_H = (N+2)(N+3)/2$ seems a continuous phase transition point.

Takeuchi, NS, Eur.Phys.J.C 80 (2020) 2, 118; Obster, NS, PTEP 2020, 073B06; Lionni, NS, PTEP 2019 (2019) 7, 073A01.

However, positive cosmological constant case ($k > 0$) is oscillatory and suffers from sign problem, making Monte Carlo simulations very difficult.

In this talk, we study a similar problem with much simplicity $P_{abc} \to M_{ab}$ to obtain insights for the positive cosmological constant case.

$$\Psi(M) = \langle M | \Psi \rangle = \varphi(M)^R$$

$$\varphi(M) := \int_{\mathbb{R}^N} \prod_{a=1}^{N} d\phi_a \exp\left[i\, M_{ab}\phi_a\phi_b - (k_1 + i\, k_2)\, \phi^2 \right]$$

$k_2 \neq 0$: Positive cosmological constant case (Oscillatory)

k_1 is not so important in this talk: $k_1 \ll 1$

$$Z = \langle \Psi | e^{-\alpha \hat{M}^2} | \Psi \rangle = \int dM\, e^{-\alpha M^2} | \varphi(M) |^{2R}$$

Simple thought :

$$\varphi(M) \sim \mathrm{Det}\left(M - k_2 I\right)^{-\frac{1}{2}} \quad (k_1 \ll 1)$$

$$M \sim \begin{pmatrix} k_2 & & 0 \\ & \ddots & \\ 0 & & k_2 \end{pmatrix}$$

$SO(N)$ symmetric configuration is always dominant

Wrong !

More details exist :

- Phase structure
- $SO(R)$ symmetric configuration is dominant for $R < N$.
- Symmetry is stable only in a phase (unstable in the other) for $R < N$.
  $\Rightarrow$ Dimension of emergent space is stable/unstable only in a/other phase.

§ A Two-Logarithm Matrix Model

By integrating out ϕ_a

$$\langle \Psi | e^{-\alpha \hat{M}^2} | \Psi \rangle = \int dM \; e^{-\alpha M^2} | \varphi(M) |^{2R} = \int dM \, e^{-S(M)}$$

$$S(M) = \mathrm{Tr}\left[\frac{R}{2}\log(M - k_2 + i\,k_1) + \frac{R}{2}\log(M - k_2 - i\,k_1) + \alpha M^2\right]$$

M : $N \times N$ real symmetric matrix

$k_2 \neq 0$: Positive cosmological constant case

The matrix model can be analyzed by

- Aligned Coulomb gas picture
- SD-eq.

§ Aligned Coulomb gas picture

$$dM \to \prod_{a=1}^{N} d\lambda_a \prod_{\substack{a,b=1 \\ a<b}}^{N} |\lambda_a - \lambda_b|$$

for real symmetric matrix M

λ_a : eigenvalues

$$Z = \int \prod_a d\lambda_a \, e^{-S_{Coul}(\lambda)}$$

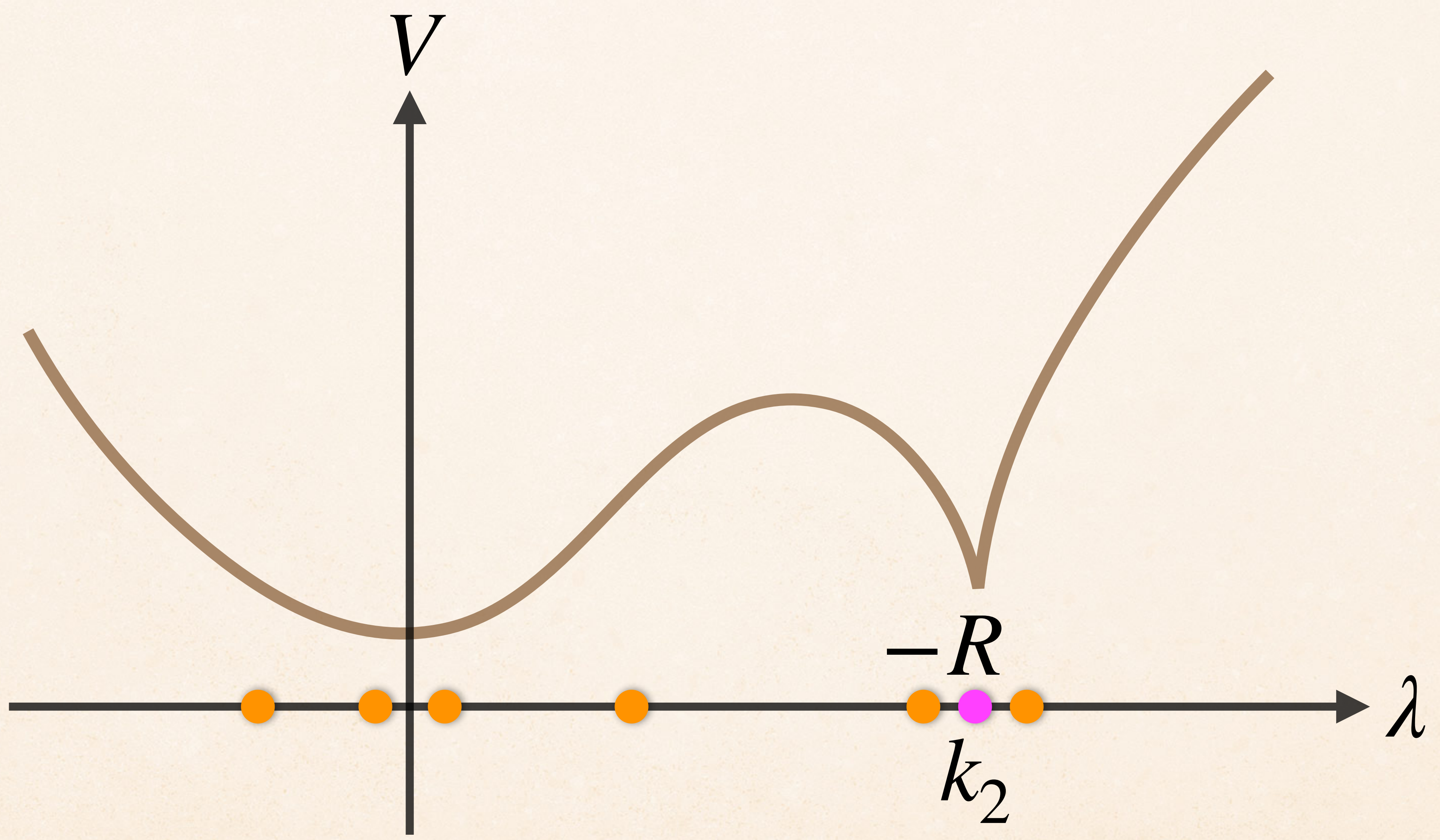
$$S_{Coul}(\lambda) = -\sum_{a<b} \log|\lambda_a - \lambda_b| + \sum_a \left(R \log|\lambda_a - k_2 - ik_1| + \alpha\lambda_a^2\right)$$

The system can be understood as particles of unit charges at 1-dim locations λ_a

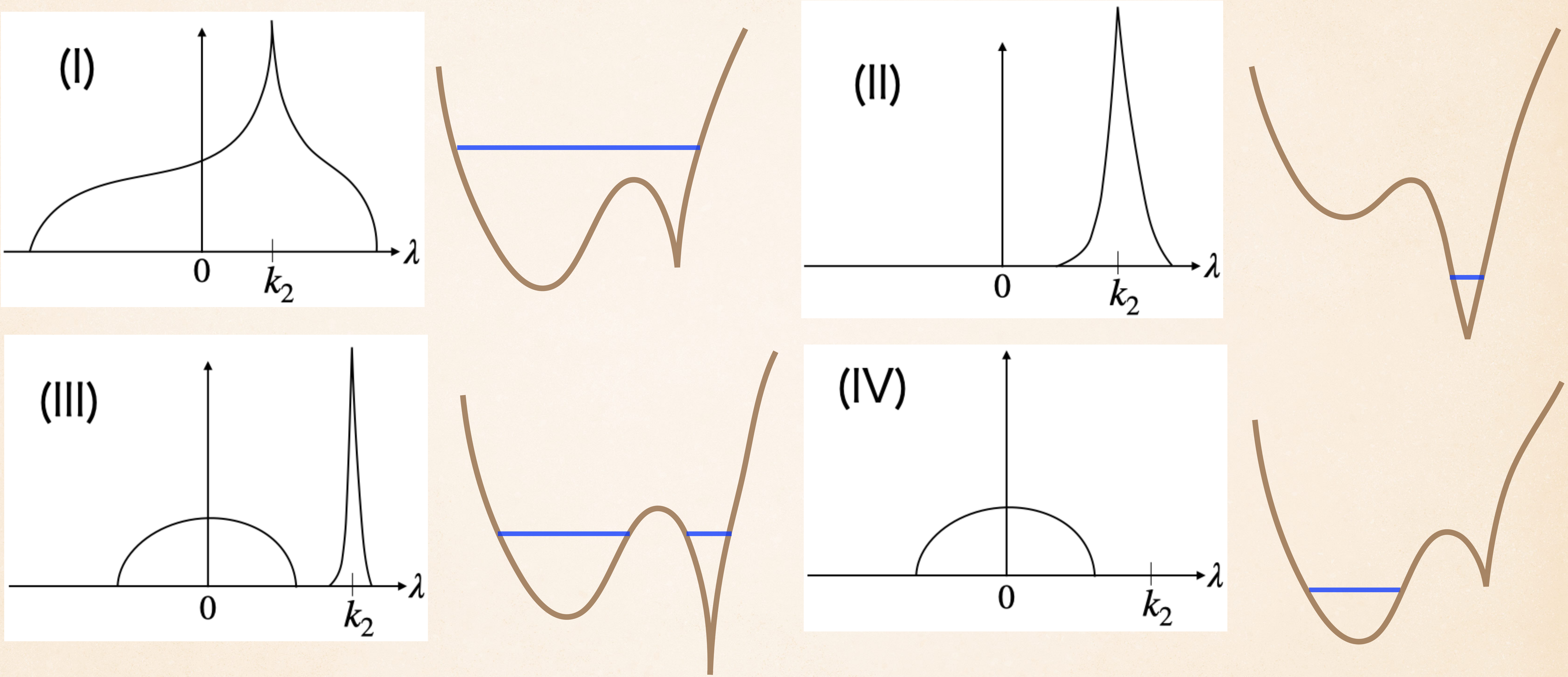
$$S_{Coul}(\lambda) = -\sum_{a<b} \log|\lambda_a - \lambda_b| + \sum_a \left(R \log|\lambda_a - k_2 - i\,k_1| + \alpha\,\lambda_a^2\right)$$

Coulomb repulsive forces among particles of unit charges

$-R$ charge located at fixed location k_2 $(k_1 \ll 1)$



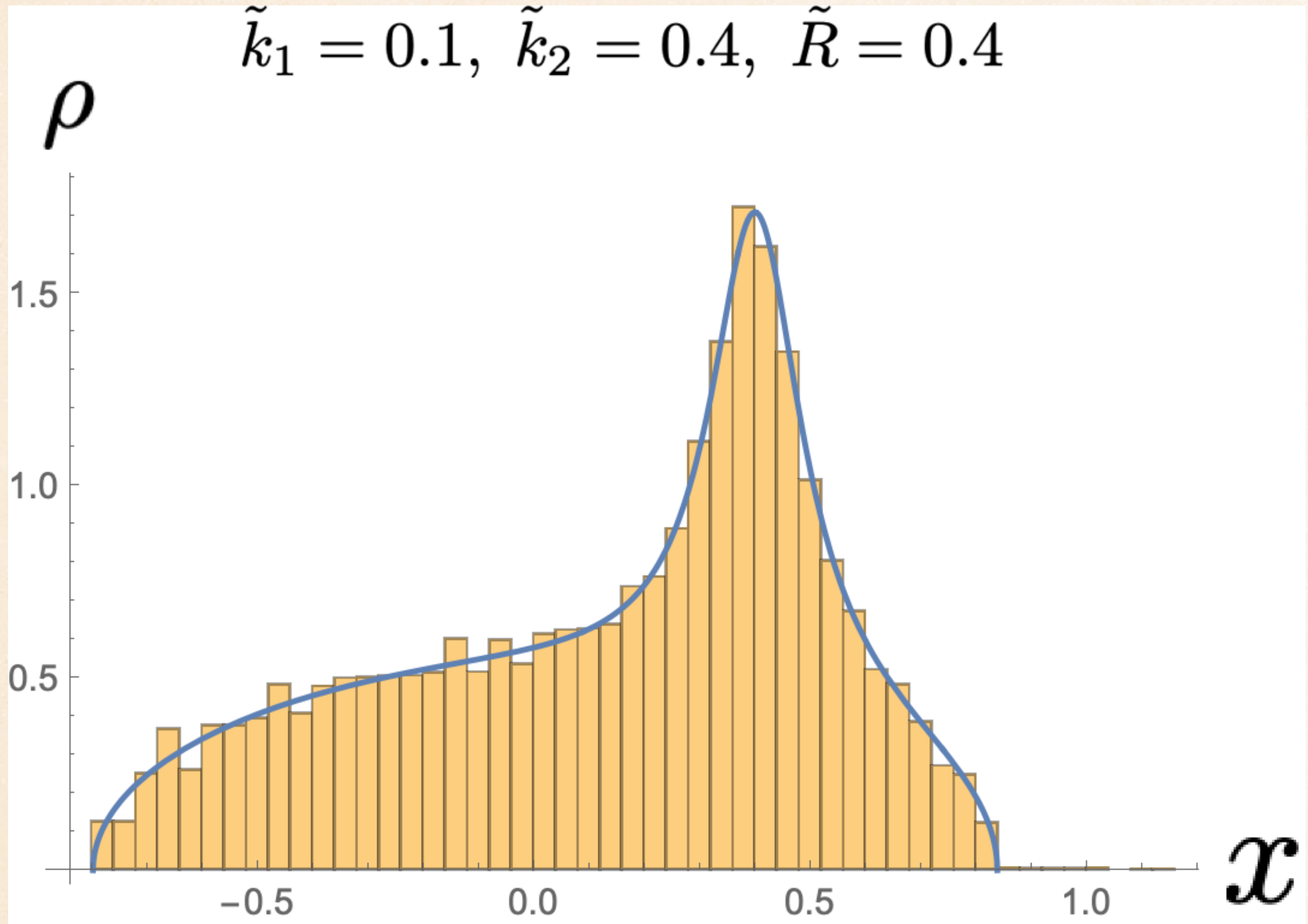
There are four cases classified by the profile of particle density $\rho(\lambda)$.



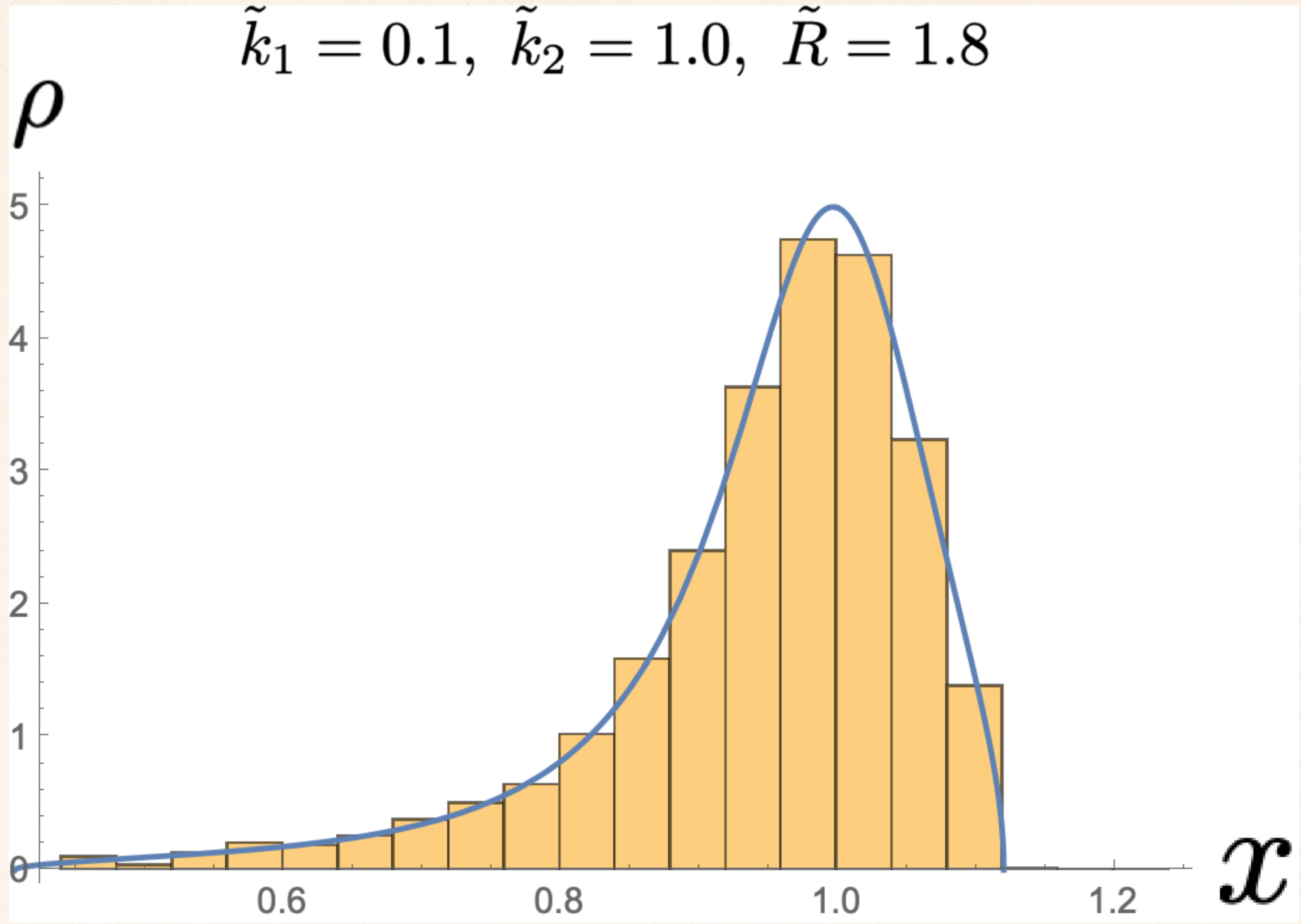
However, the separations among (I),(II),(IV) are arbitrary, while (III) is distinct.

Examples of the eigenvalue distribution

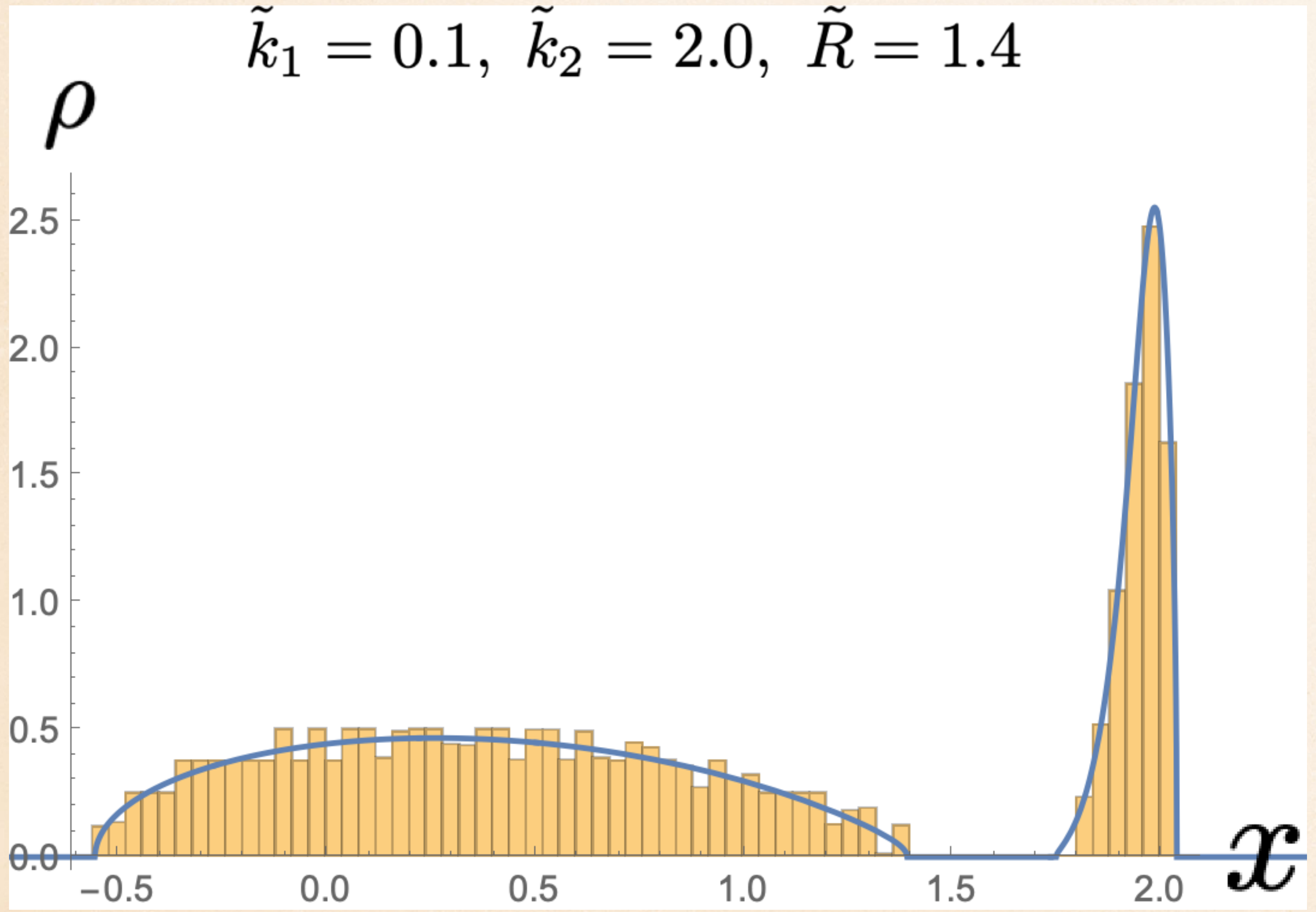
(I)



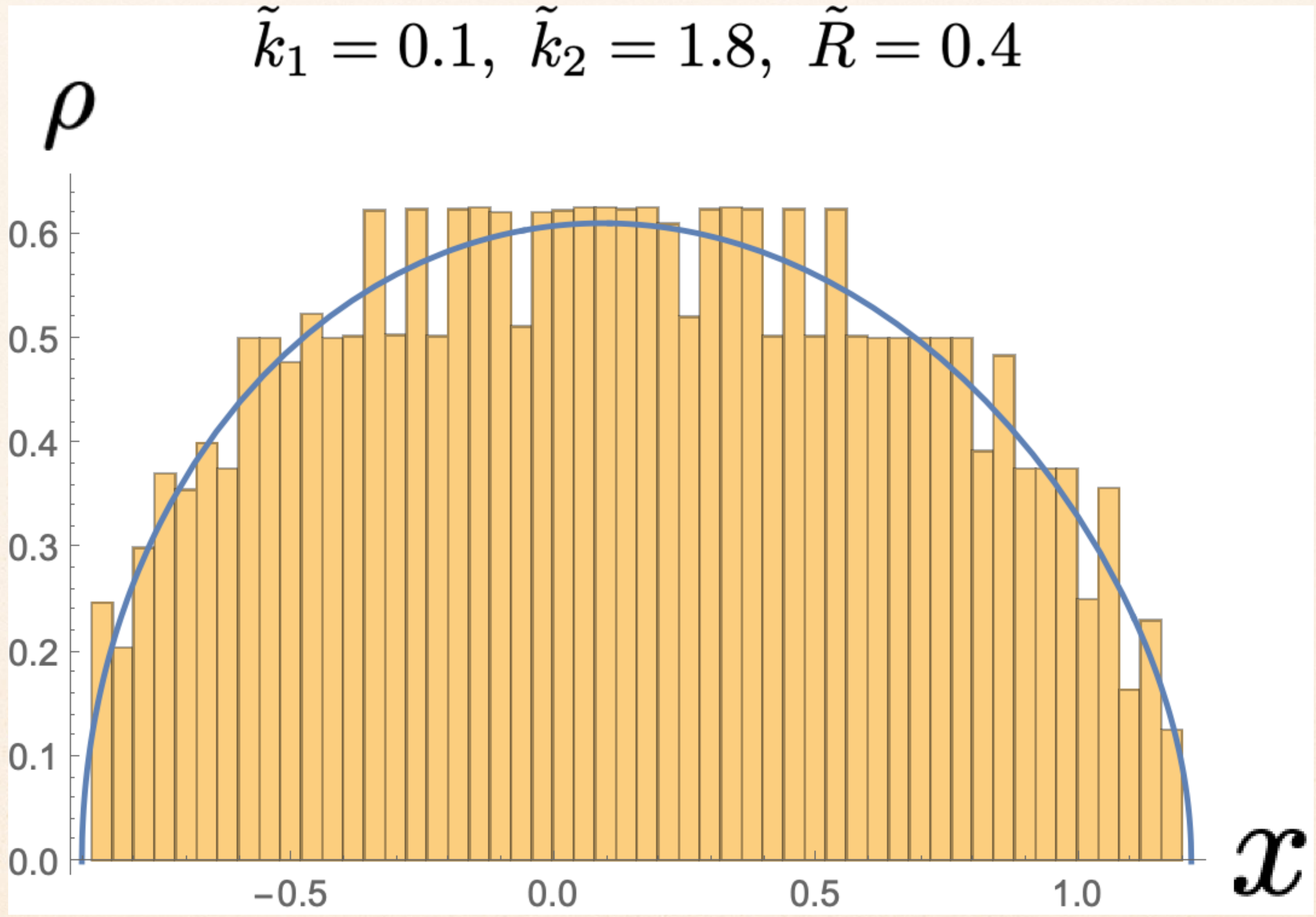
(II)



(III)



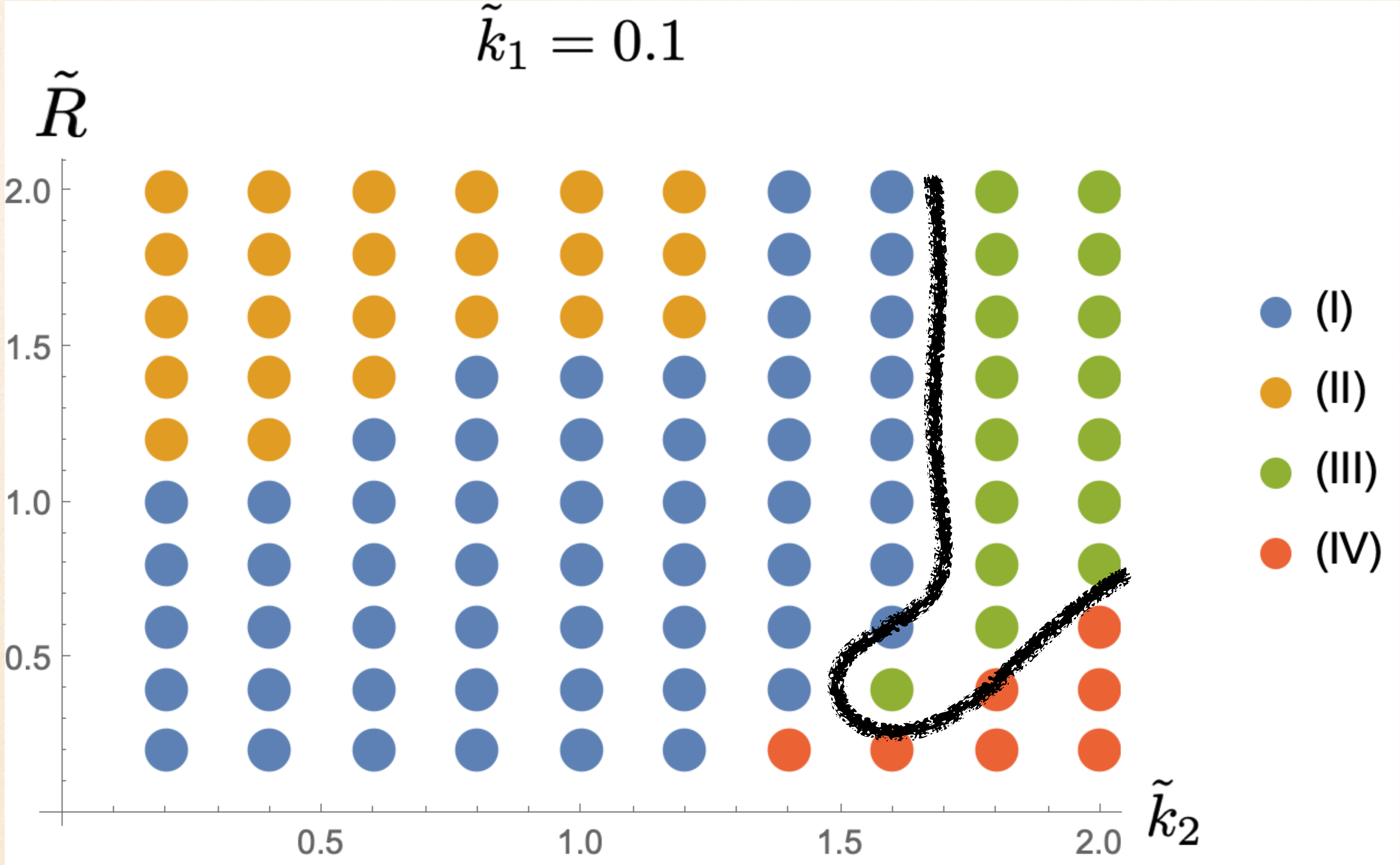
(IV)



Histogram:
Monte Carlo of the Coulomb gas system with $N = 200$

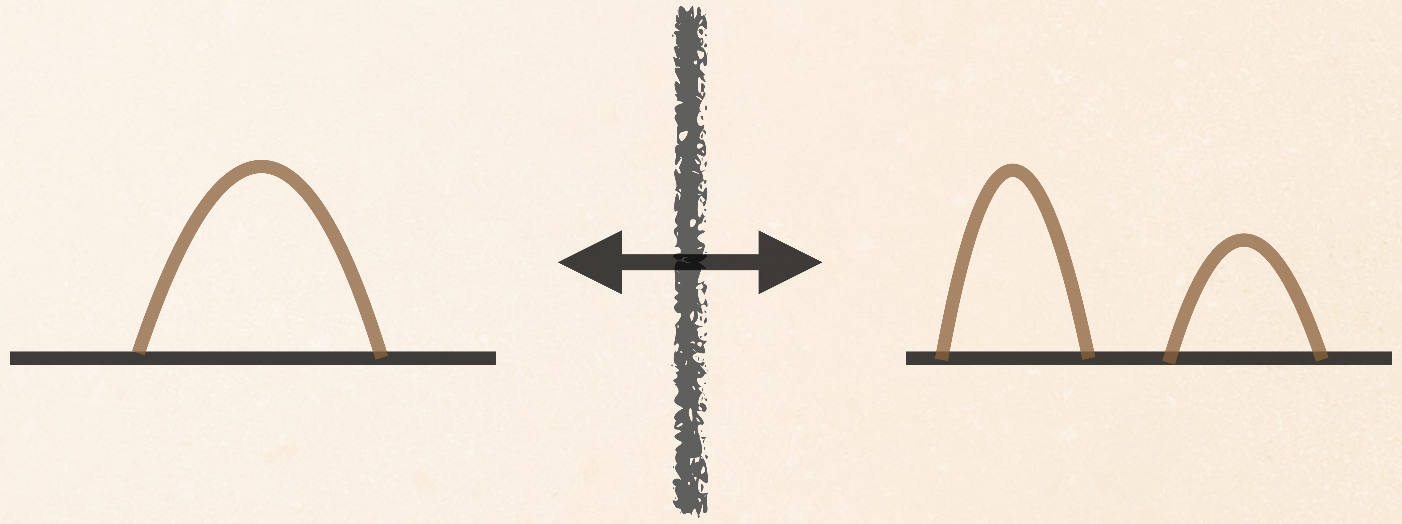
Blue lines:
Solutions of SD eq. in $N \to \infty$

Phase diagram (for $\tilde{k}_1 = 0.1$)



$$R = N\tilde{R}$$

$$k_{1,2} = \sqrt{N}\,\tilde{k}_{1,2}$$



$N \to \infty$ limit

$$R = N\tilde{R} \qquad k_{1,2} = \sqrt{N}\,\tilde{k}_{1,2} \qquad \lambda_a = \sqrt{N}\,\lambda_a$$

$$\sum_{a=1}^{N} \to N\int dx\,\rho(x) \qquad \left(\int dx\,\rho(x) = 1\right)$$

$$S_{cont}(\rho) = N^2\left[-\frac{1}{2}\int_{\mathbb{R}^2} dxdy\,\rho(x)\rho(y)\log|x-y| + \int_{\mathbb{R}} dx\,\rho(x)\left(\tilde{R}\log|x-\tilde{k}_2+i\tilde{k}_1| + \alpha x^2\right)\right]$$

$$\mathrm{EOM}: \quad \mathrm{P}\int_{\mathbb{R}} dy\frac{1}{x-y}\rho(y) = \frac{\tilde{R}(x-\tilde{k}_2)}{(x-\tilde{k}_2)^2+\tilde{k}_1^2} + 2\alpha x$$

(cf. Brezin et al., CMP 59 ('78) 35)

§ SD-eq.

$$W(z) := \frac{1}{N}\left\langle \mathrm{Tr}\left[\frac{1}{z-M}\right]\right\rangle \qquad \rho(x) = \frac{i}{2\pi}\big(W(x+i\epsilon) - W(x-i\epsilon)\big)$$

SD-eq. $$\int dM \, \frac{\partial}{\partial M_{ab}}\left\{\left(\frac{1}{z-M}\right)_{ab} e^{-S(M)}\right\} = 0$$

$N \to \infty$

$$W(z)^2 - 2\tilde{S}'(z)W(z) + \tilde{R}\left(\frac{W(\tilde{k}_2 + i\tilde{k}_1)}{z - \tilde{k}_2 - i\tilde{k}_1} - \frac{W(\tilde{k}_2 - i\tilde{k}_1)}{z - \tilde{k}_2 + i\tilde{k}_1}\right) + 4\alpha = 0$$

where $$\tilde{S}'(M) = \frac{\tilde{R}}{2(M - \tilde{k}_2 - i\tilde{k}_1)} + \frac{\tilde{R}}{2(M - \tilde{k}_2 + i\tilde{k}_1)} + 2\alpha M$$

(cf. Paniak, Weiss, JMP 36 ('95) 2512)

Solution

$$W(z) = \tilde{S}'(z) - \sqrt{\tilde{S}'(z)^2 - \left[\tilde{R}\left(\frac{W(\tilde{k}_2 + i\tilde{k}_1)}{z - \tilde{k}_2 - i\tilde{k}_1} + \frac{W(\tilde{k}_2 - i\tilde{k}_1)}{z - \tilde{k}_2 + i\tilde{k}_1}\right) + 4\alpha\right]}$$

This satisfies (under appropriate choice of branches)

- $W(z) \sim \dfrac{1}{z}$ for $z \to \infty$
- The poles of $\tilde{S}'(z)$ canceled

Further, $W(\tilde{k}_2 + i\tilde{k}_1)$ must be chosen so that (not so easy)

- Branch cuts exist only on the real axis
- The number of cuts is one or two, corresponding to (I,II,IV) or (III)
- $\rho(x)$ must be positive $\left(\rho(x) = i\left(W(x + i\epsilon) - W(x - i\epsilon)\right)/2\pi\right)$

$$W(z) = \tilde{S}'(z) - \frac{\sqrt{f(z)}}{(z-\tilde{k}_2)^2 + \tilde{k}_1^2}$$

$f(z)$: 6th-order real polynomials of z

Instead it is easier to assume the form below and determine c's by the conditions.

One-cut solution (I,II,IV) :

$$W(z) = \tilde{S}'(z) - \left(\frac{c}{z - \tilde{k}_2 - i\tilde{k}_1} + \frac{c^*}{z - \tilde{k}_2 + i\tilde{k}_1} + 2\alpha \right) \sqrt{z - c_+}\sqrt{z - c_-}$$

Two-cut solution (III) :

$$W(z) = \tilde{S}'(z) - \frac{2\alpha(z-c)\sqrt{z-c_1}\sqrt{z-c_2}\sqrt{z-c_3}\sqrt{z-c_4}}{(z-\tilde{k}_2)^2 + \tilde{k}_1^2}$$

$c_1 < c_2 < c < c_3 < c_4$

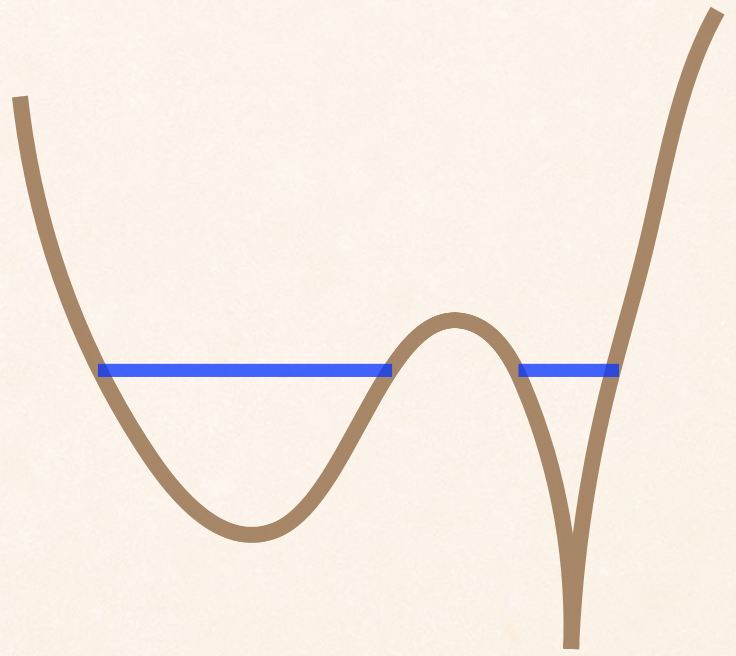
The conditions can be written down as a number of algebraic equations.

$$\tilde{R}/2 - c\sqrt{\tilde{k}_2 + i\tilde{k}_1 - c_+}\sqrt{\tilde{k}_2 + i\tilde{k}_1 - c_-} = 0$$

etc,$\cdots$

Further, for multi-cut solution, chemical potential should be equal among bunches.
(Jurkiewicz, PLB 245 ('90) 178)

$$\int_{c_2}^{c_3} dx \left(W(x) - \tilde{S}'(x)\right) = 0$$

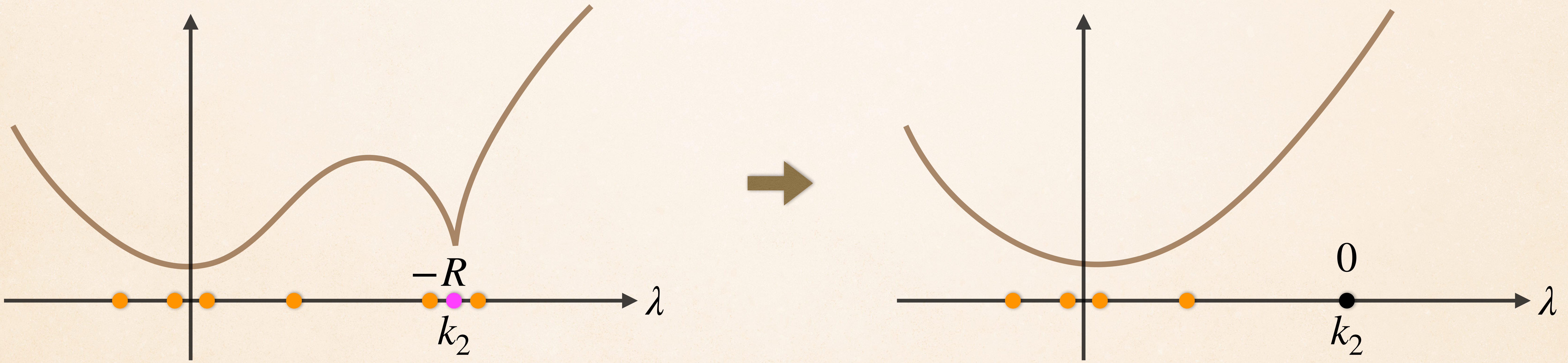


These conditions do not seem possible to be solved explicitly.
But they can be solved numerically for each case of given parameters.

§ $\tilde{k}_1 \to 0$ limit ($k/\sqrt{N} \to 0$ limit)

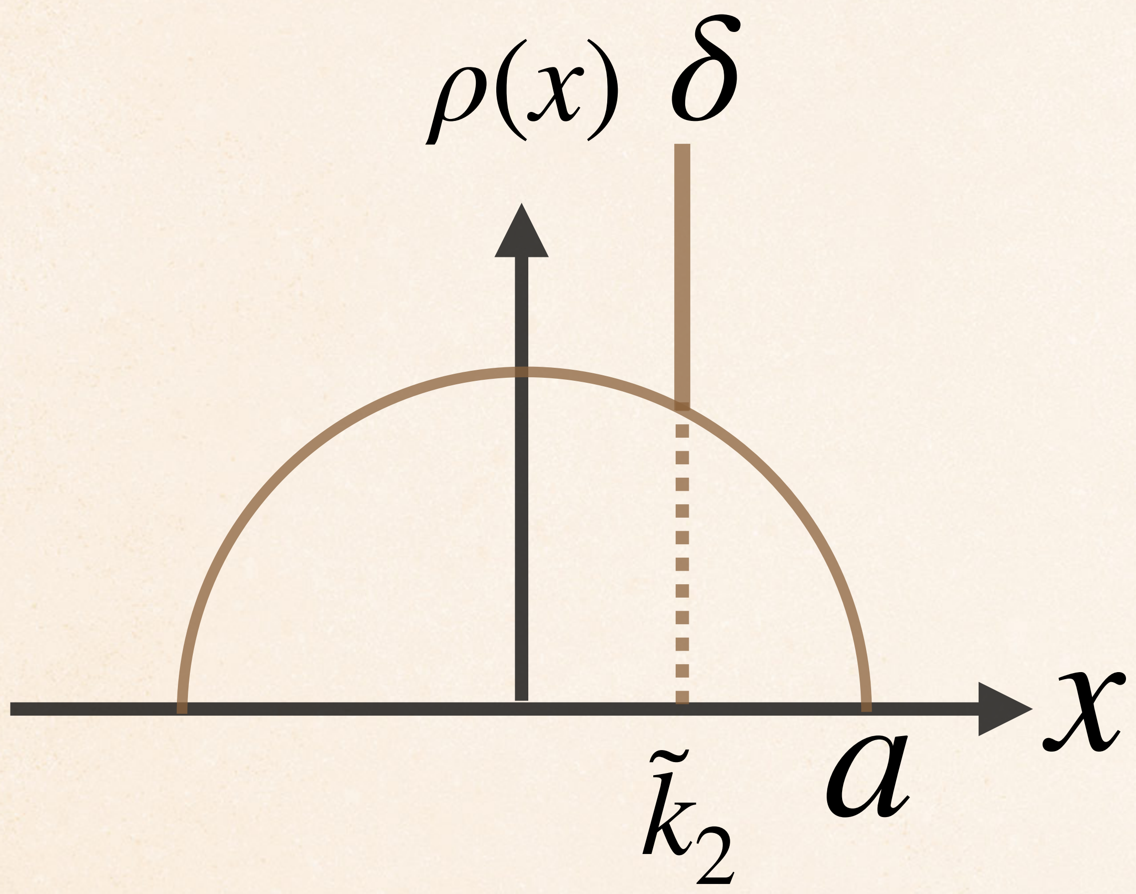
The algebraic equations become simple in the $\tilde{k}_1 \to 0$ limit, and can explicitly be solved. The reason is clear in the aligned Coulomb gas picture.

The potential $\tilde{R} \log|\lambda - \tilde{k}_2 + i\tilde{k}_1|$ becomes infinitely deep in $\tilde{k}_1 \to 0$, and the $-R$ charge is totally screened by R particles of unit charges (if $N \geq R$).



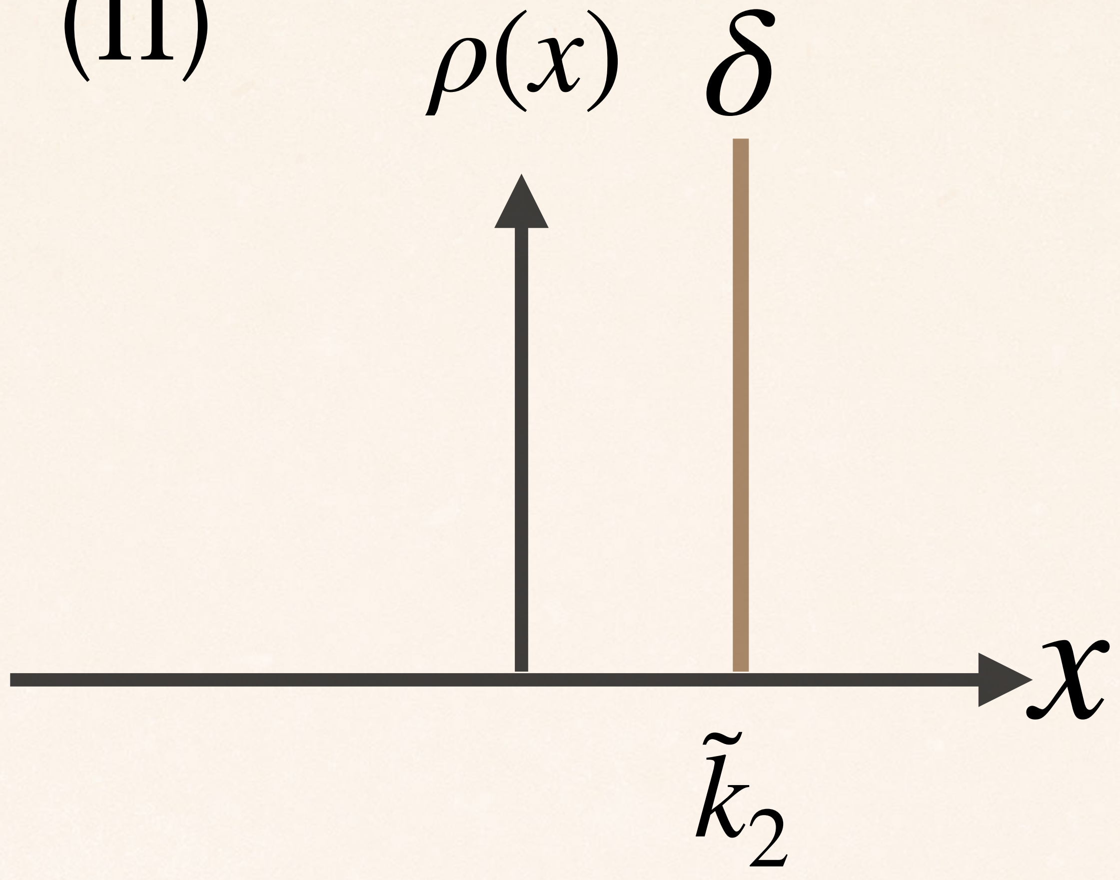
The profile of the eigenvalue densities in the $\tilde{k}_1 \to 0$ limit

(I) Semi-circle + $\delta(x - \tilde{k}_2)$

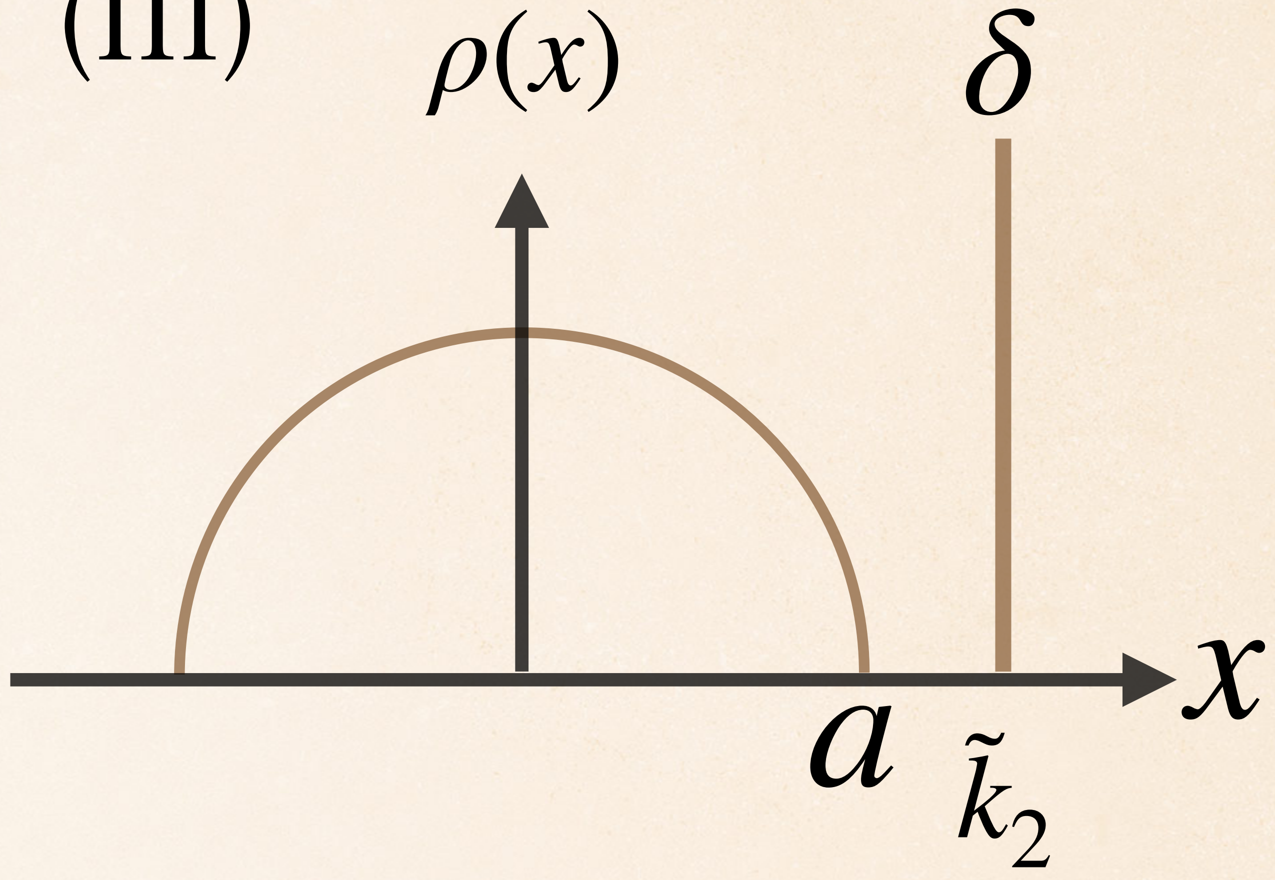


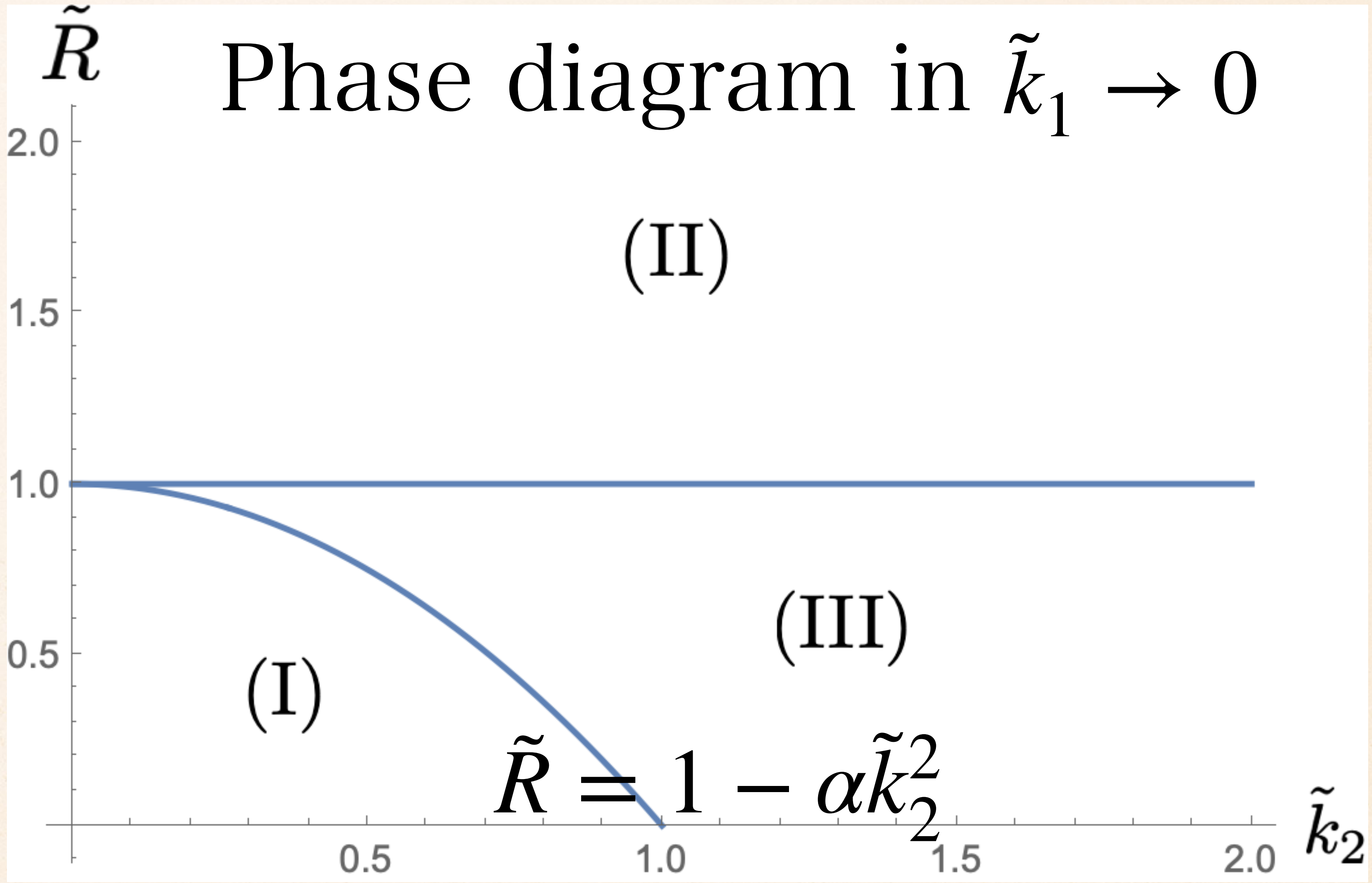
$$a = \sqrt{(1 - \tilde{R})/\alpha}$$

(II)



(III)





(IV) disappears

§ Symmetry Enhancement

When $\tilde{k}_1 = k_1/\sqrt{N} \to 0$, R of the eigenvalues concentrate around $\lambda \sim \tilde{k}_2$ to totally screen the $-R$ charge $(R < N)$.

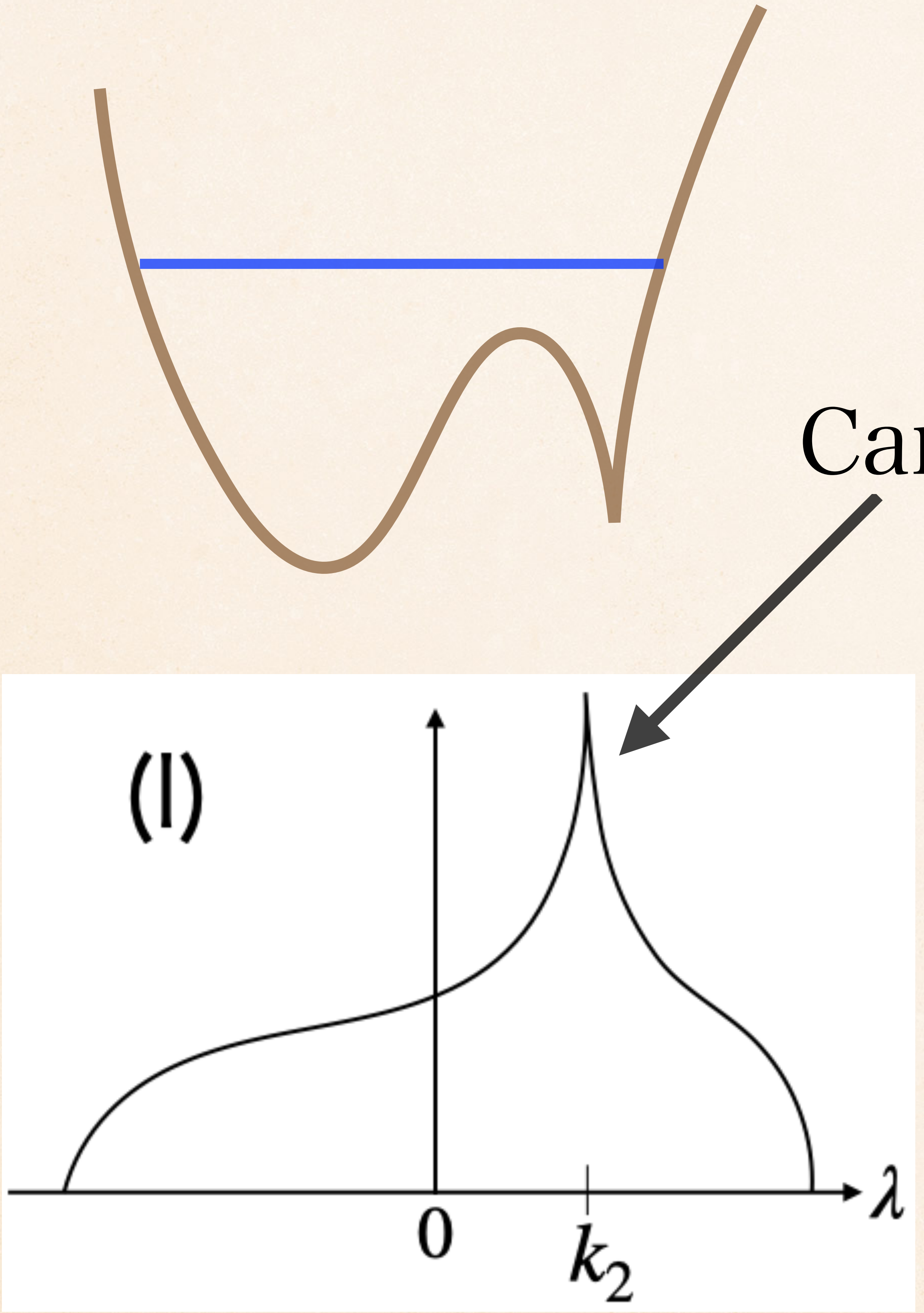
$$M = \begin{pmatrix} \overbrace{\begin{matrix} \tilde{k}_2 & & \\ & \tilde{k}_2 & \\ & & \ddots \end{matrix}}^{R} & 0 \\ 0 & \begin{matrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{matrix} \end{pmatrix}$$

$SO(R)$ symmetry is enhanced in the limit $\tilde{k}_1 \to 0$.

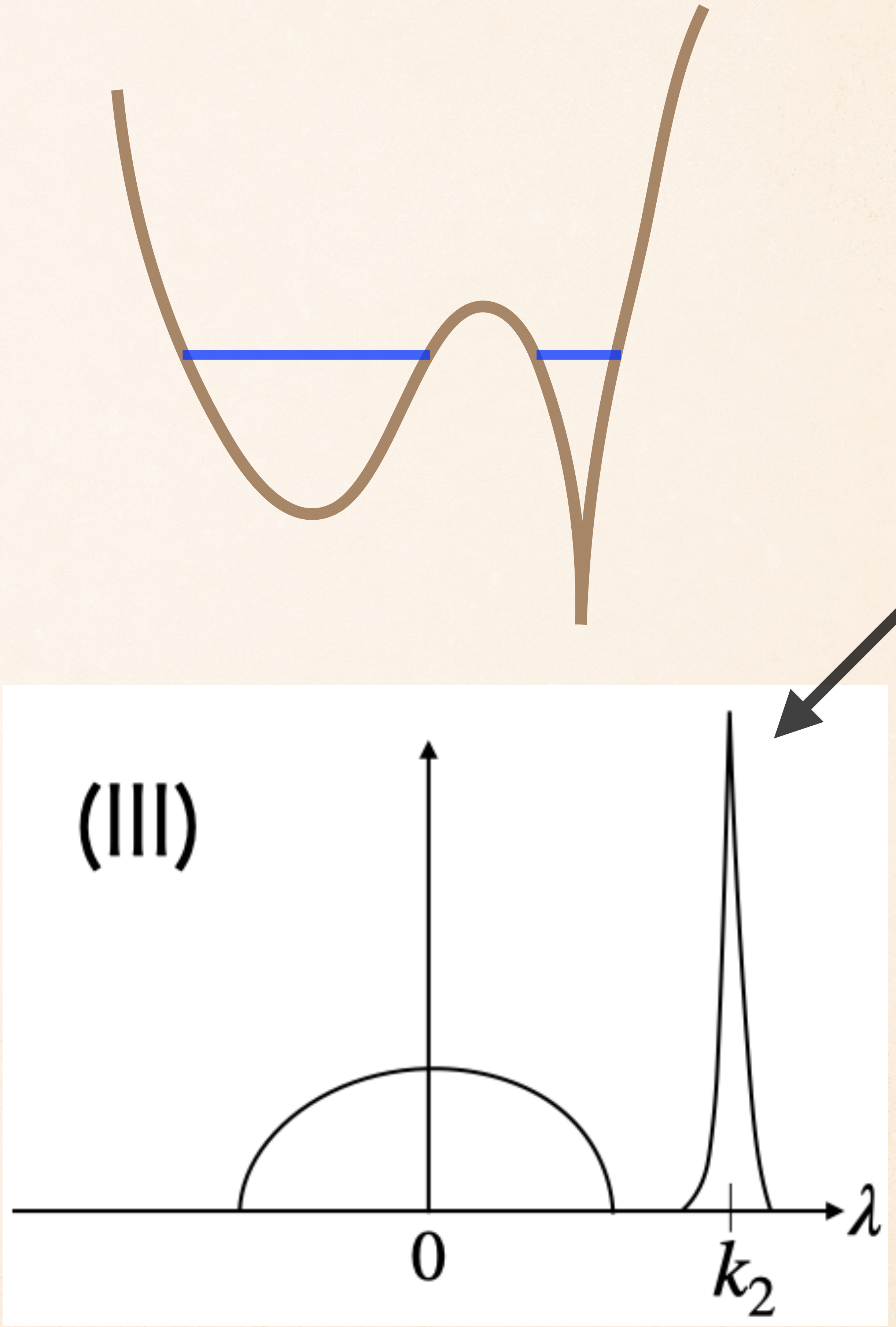
There are two phases (I) and (III). Consider $\tilde{k}_1 \ll 1$.

In (I), enhanced symmetry is ambiguous, because of nearby eigenvalues.

In (III), enhanced symmetry is rather definite, since the other eigenvalues are separated by a potential barrier.

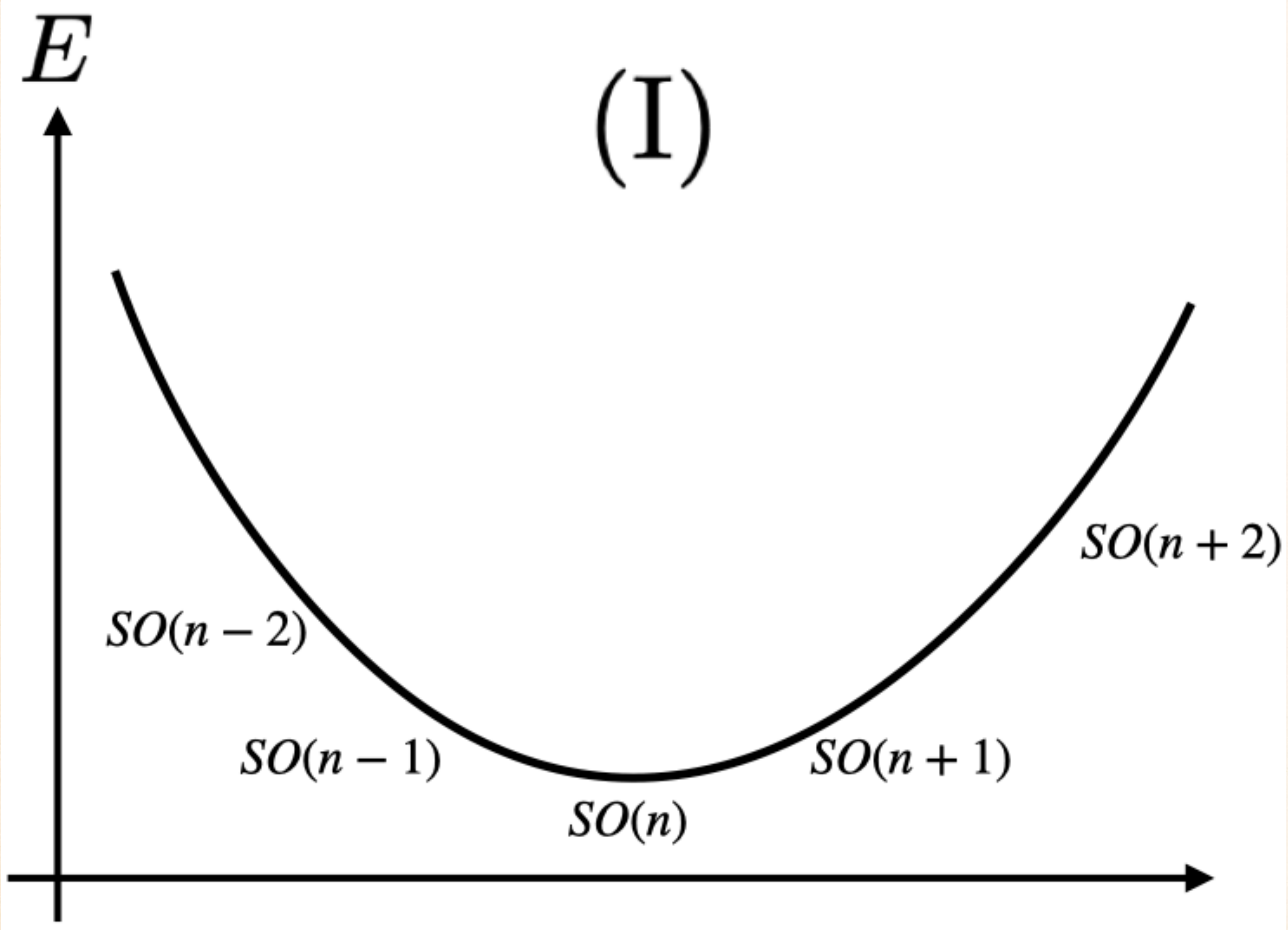


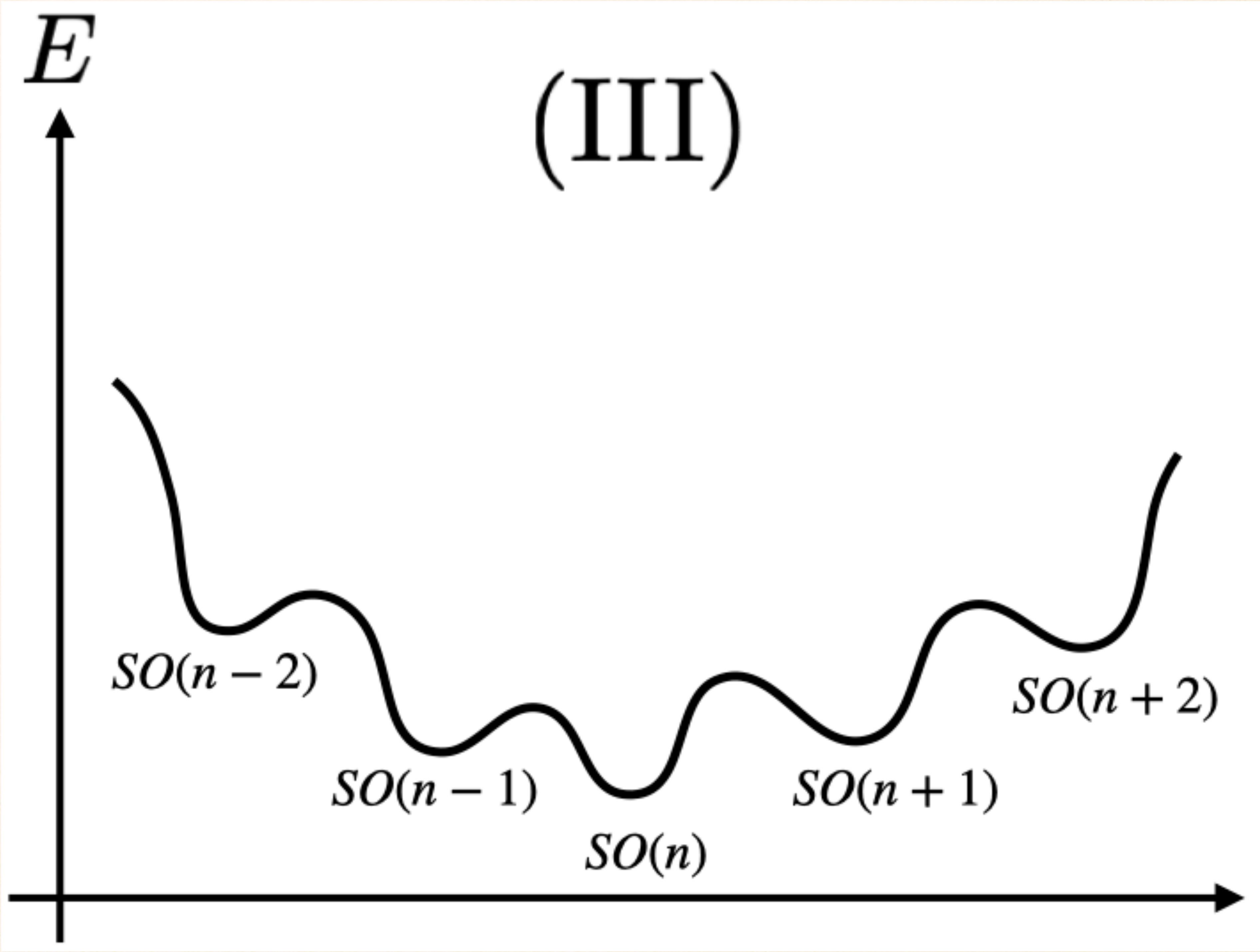
Cannot count well



Can count well

In other words, in phase (I), an enhanced symmetry is ambiguous and sensitive to perturbations, while, in phase (III), it is more definite and protected by potential barrier from perturbations.





§ Dimension of emergent space

$$Z = \int dM\ e^{-\alpha M^2} |\varphi(M)|^{2R}$$

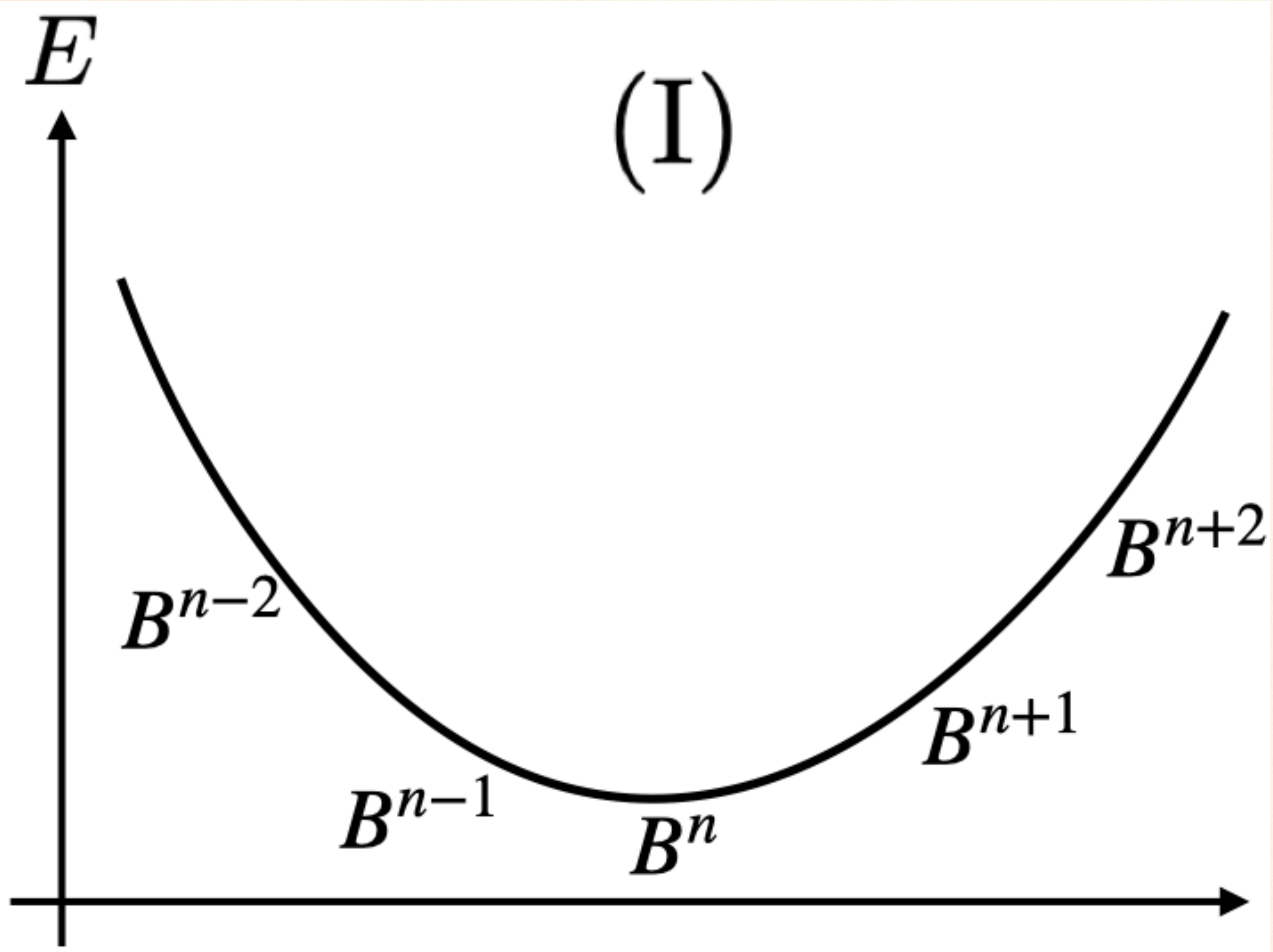
$$= \int dM\ e^{-\alpha M^2} \varphi(M)^{*R} \varphi(M)^{R-1} \int_{\mathbb{R}^N} d\phi \exp\left[i M_{ab}\phi_a\phi_b - (k_1 + i\, k_2)\, \phi^2\right]$$

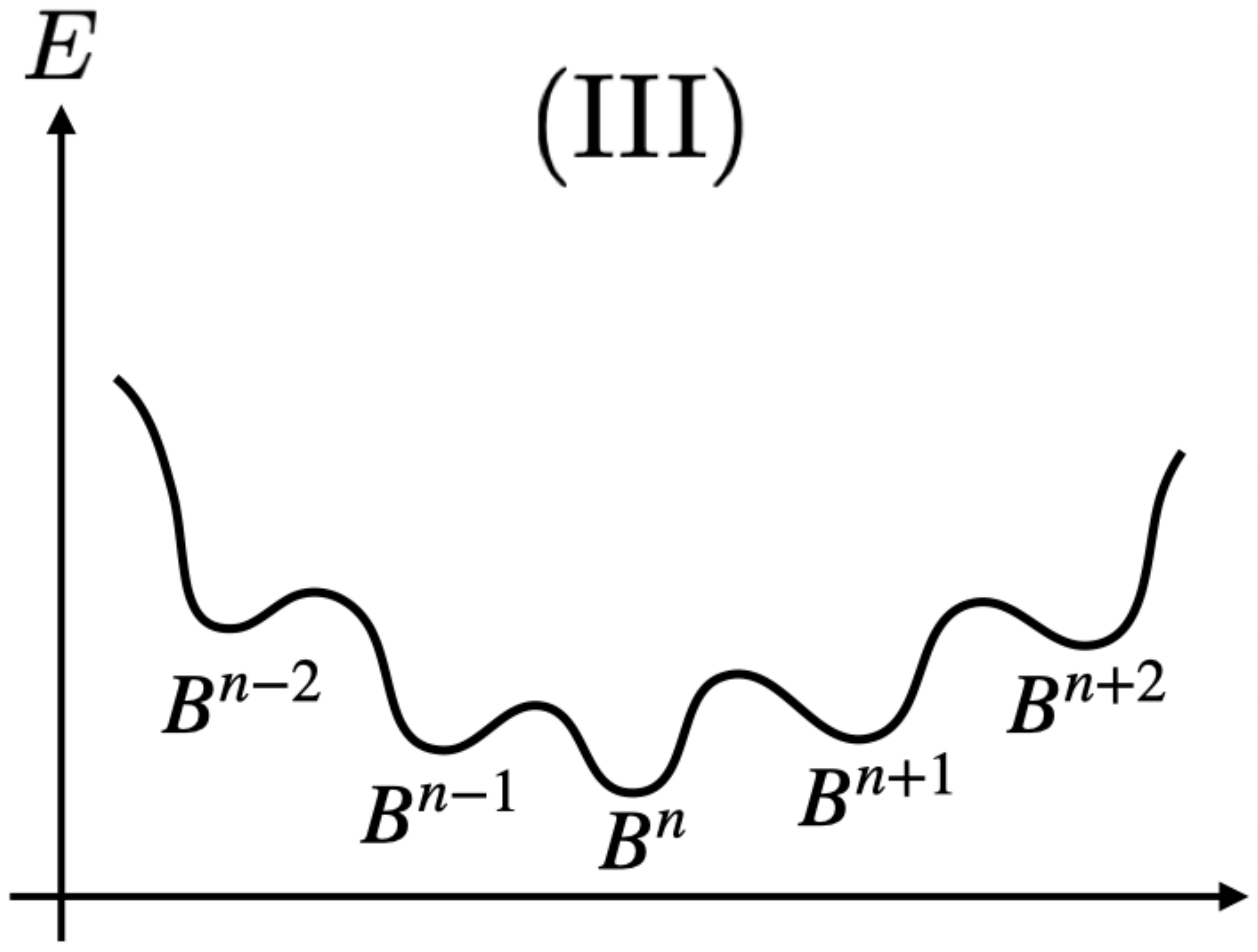
$$\sim \int_{\mathbb{R}^D} d\phi \exp\left[i\, \phi^T \left(M - (k_2 - ik_1)I\right) \phi\right]$$

$$M = \begin{pmatrix} \overbrace{\begin{matrix} k_2 & & \\ & k_2 & \\ & & \ddots \end{matrix}}^{R} & 0 \\ 0 & \begin{matrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{matrix} \end{pmatrix}$$

In $\tilde{k}_1 \to 0$, ϕ forms an R-dimensional ball B^R of radial size ~ $1/\sqrt{k_1}$ and thickness $\sim 1/\sqrt{|k_2 - *|}$

In (I), dimension of the ball is ambiguous and sensitive to perturbations.
But definite and stable in (III).



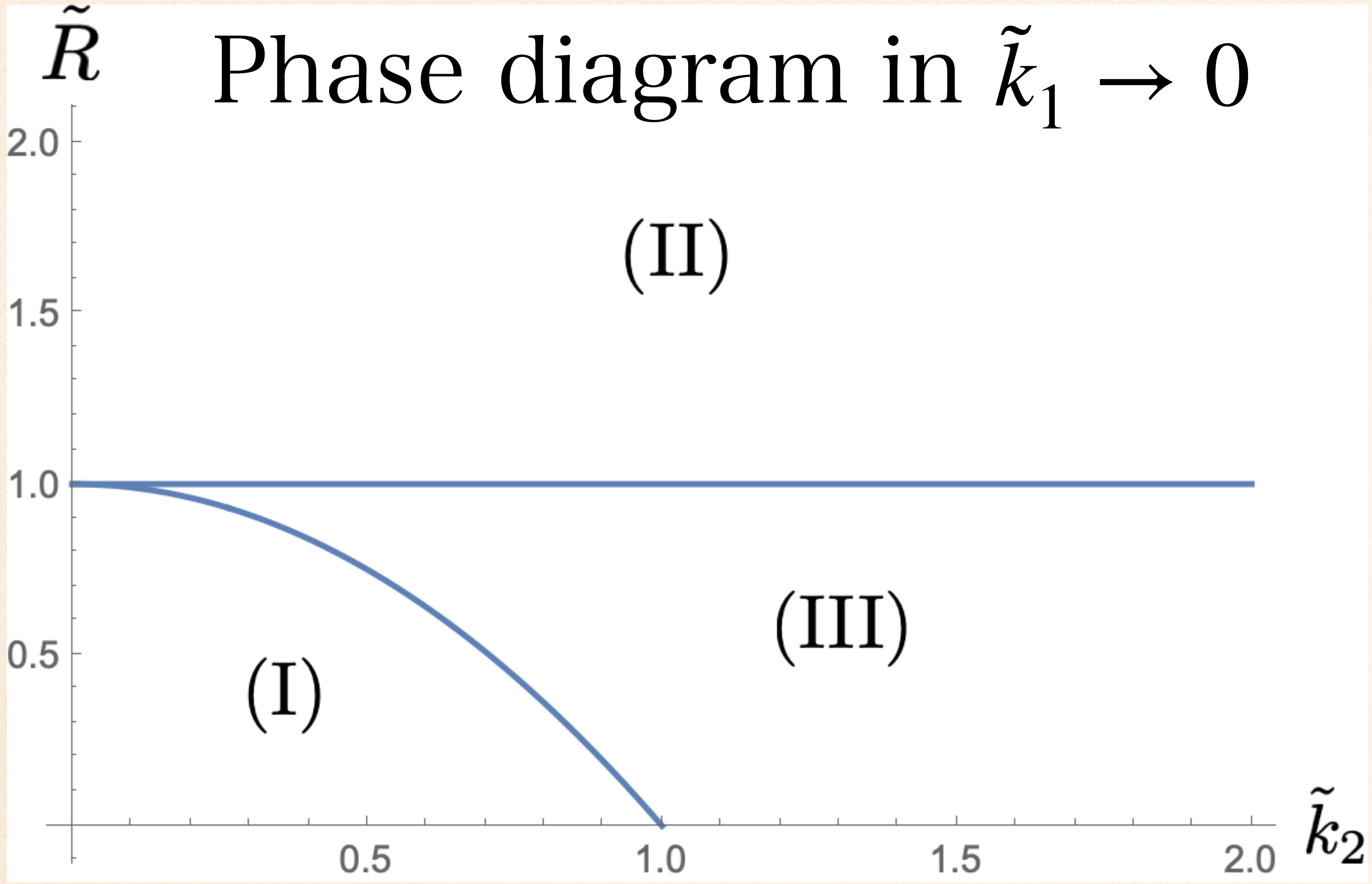


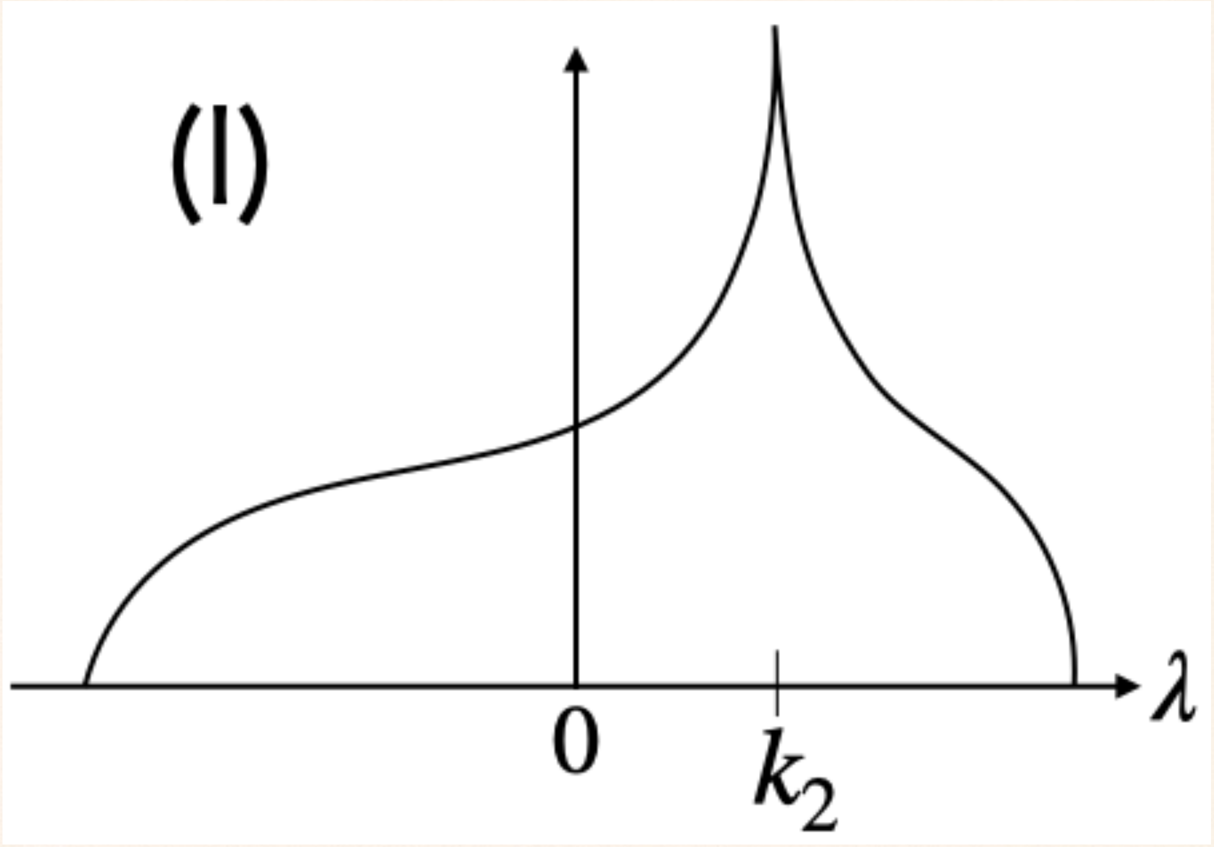
Ambiguous
Sensitive to perturbations

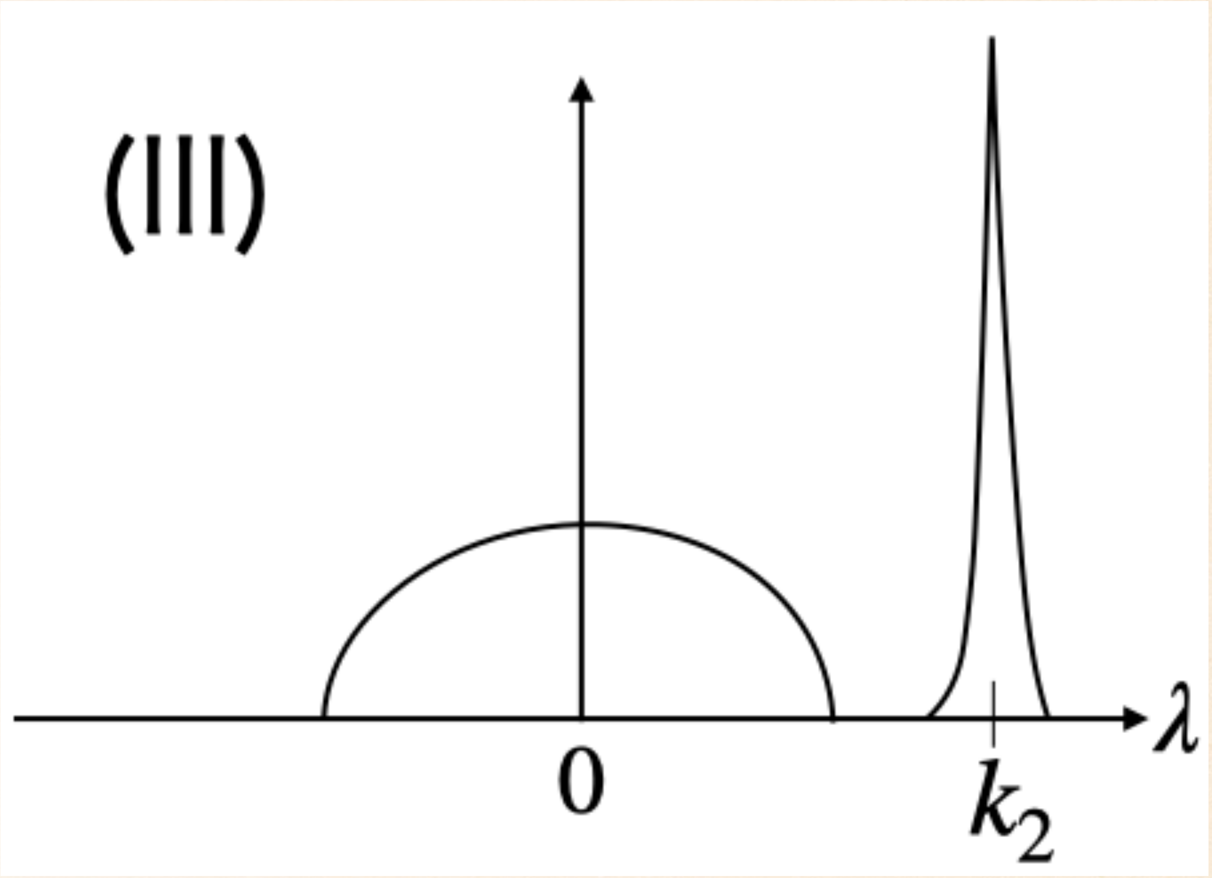
Definite
Stable

§ Summary and Discussions

Considered a toy version of $\int dP\, e^{-\alpha P^2} |\Psi(P)|^2$ by simplification $P_{abc} \to M_{ab}$.







Phase (III) is important for emergence of space

Symmetry/dimension are stable because condensation of eigenvalues is separated.

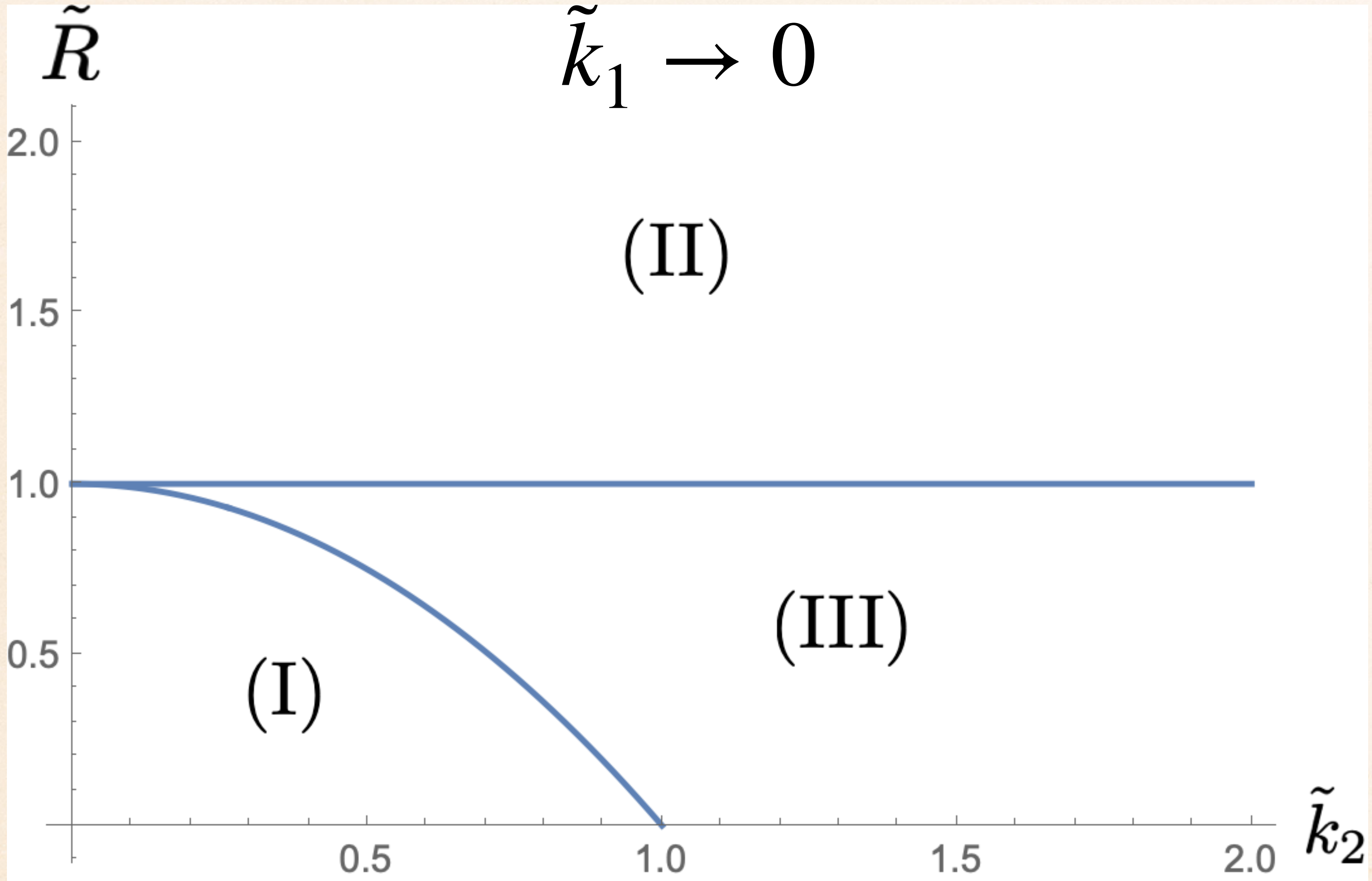
$k_2 \neq 0$ is vital, corresponding to the positive cosmological constant.

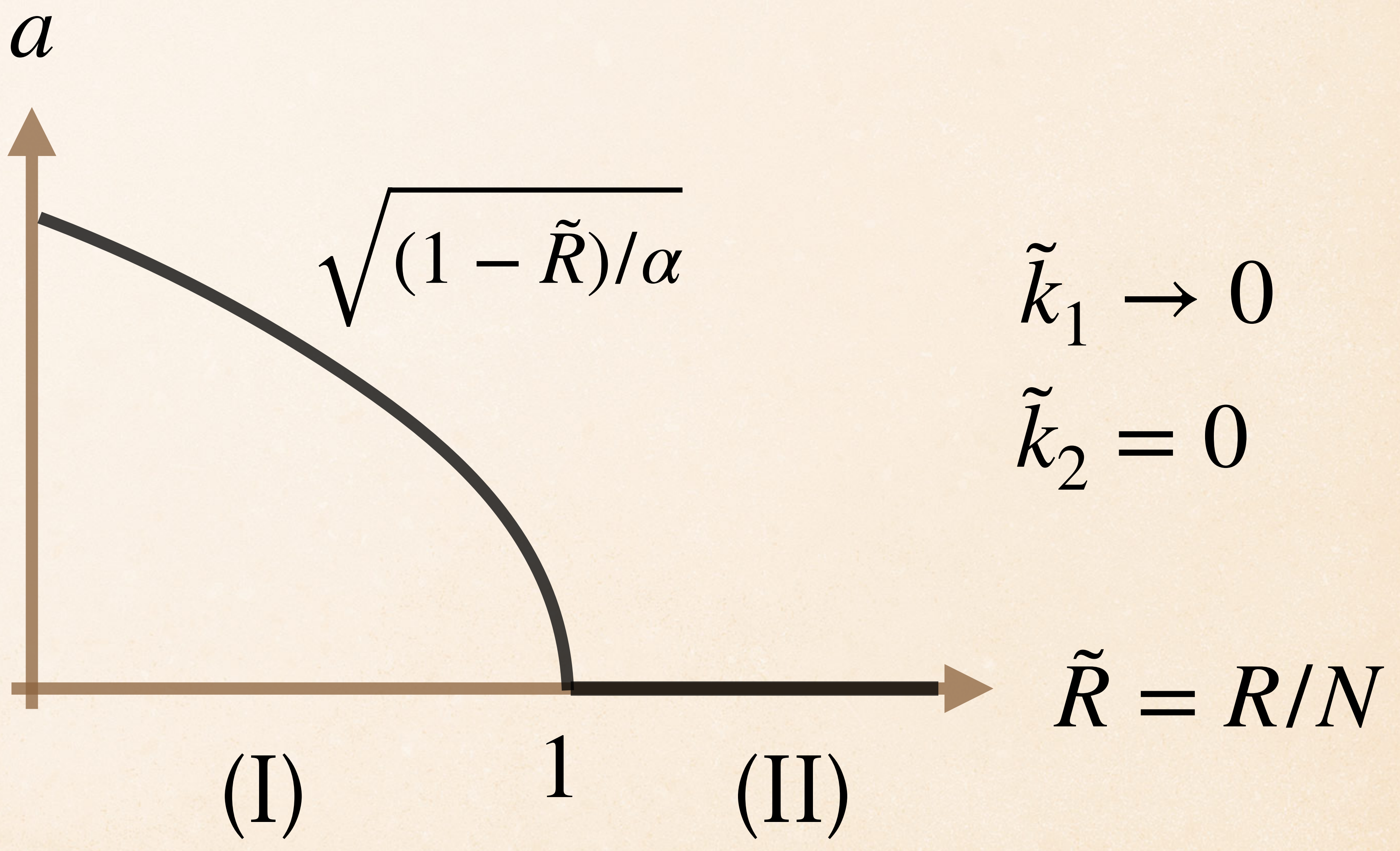
- An indirect crosscheck for the previous Monte Carlo result for the tensor model.

Tensor model : $\lambda_H = (N+2)(N+3)/2$ seems the critical point of a continuous phase transition

Takeuchi, NS, Eur.Phys.J.C 80 (2020) 2, 118; Obster, NS, PTEP 2020, 073B06; Lionni, NS, PTEP 2019 (2019) 7, 073A01.

Matrix model : A continuous phase transition at $R = N$ in the limit $\tilde{k}_1 \to 0$





§ Future perspective

- For a tensor, eigenvalue/vector can be defined by

$$P_{abc}\phi_b\phi_c = \lambda\,\phi_a$$

Qi, 1201.3424 [math.SP]

  This agrees with the saddle point equation for the wave function.

  Finding isolated condensation of eigenvalue/vector would be the sign of stable emergent spacetime.

- The matrix model in this talk provides an arena for developing tools to analyze the canonical tensor model.