The triangle relation and beyond

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03 Aug. 2020, @HANOI

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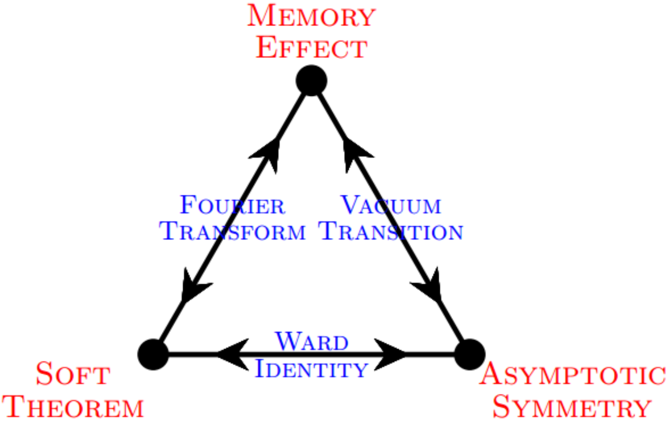
1605.09731 (PRD)
1612.08294 (JHEP)
1703.06588 (PRD)
1704.05740 (PRD)
1707.07118 (PLB)

Outline

The triangle relation



Proposed by A. Strominger and collaborators [Strominger 2017]

Soft theorems

For theories with long-range interactions mediated by a massless spin-s boson ($s=1,2$), when emitting one of these bosons with very low frequency, the tree-level scattering amplitude develops a pole whose residue is given by the universal formula:

$$M_{n+1}\left(p_1,\ldots,p_n,\left\{q;\epsilon^{\pm s}\right\}\right)=S^{(0)}(\epsilon^{\pm s},p_k,q)M_n(p_1,\ldots,p_n)+\mathcal{O}\left(\omega^0\right)\ , \tag{1}$$

where $\omega=q_0$ and $\epsilon^{\pm s}$ are respectively the energy and polarization tensor of the soft boson, and

$$S^{(0)}=\sum_{k=1}^{n}g_k\,\frac{\left(p_k\cdot\epsilon^{\pm}\right)^s}{p_k\cdot q} \tag{2}$$

is called a soft factor, with g_k being the cubic couplings controlling the emission of the soft particle from the external legs. For the case of $s=2$, equation (1) has come to be known as Weinberg's soft graviton theorem.

Asymptotic symmetries as approximate symmetry

- Poincaré symmetry
$$\mathcal{L}_\xi \, \eta_{\mu\nu} = 0, \tag{3}$$
where ξ is a spacetime vector that generates the infinitesimal transformation.

- Asymptotically flat spacetime
$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(\frac{1}{r}). \tag{4}$$

- Asymptotic symmetry
$$\mathcal{L}_\xi \, g_{\mu\nu} = \mathcal{O}(\frac{1}{r}). \tag{5}$$

Asymptotic symmetries in general relativity

A suitable coordinate system $\{u, r, \theta, \phi\}$ with the line element ansatz

$$\mathrm{d}s^2 = \frac{Ve^{2\beta}}{r}\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\mathrm{d}r + r^2 h_{AB}(\mathrm{d}x^A - U^A\mathrm{d}u)(\mathrm{d}x^B - U^B\mathrm{d}u), \quad (6)$$

where

$$\begin{aligned} 2h_{AB}\mathrm{d}x^A\mathrm{d}x^B = &(e^{2\gamma} + e^{2\delta})\mathrm{d}\theta^2 + 4\sin\theta\sinh(\gamma - \delta)\mathrm{d}\theta\mathrm{d}\phi \\ &+ \sin^2\theta(e^{-2\gamma} + e^{-2\delta})\mathrm{d}\phi^2. \end{aligned}$$

Boundary conditions:

- For some choice of u, one can go to the limit $r \to \infty$ along each ray.
- For some choice of θ and ϕ and the above choice of u, $\lim\limits_{r\to\infty} \frac{V}{r} = -1$, $\lim\limits_{r\to\infty} rU^A = \lim\limits_{r\to\infty} \beta = \lim\limits_{r\to\infty} \gamma = \lim\limits_{r\to\infty} \delta = 0$.
- Over the coordinate ranges $u_0 \leq u \leq u_1$, $r_0 \leq r \leq \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\phi$, all metric components can be expanded in the powers of $\frac{1}{r}$ at $r = \infty$.

[Bondi, van der Burg, Metzner 1962] [Sachs 1962]

- The infinitesimal diffeomorphisms preserving those boundary conditions are of the form

$$
\begin{aligned}
\xi^u &= T(z,\bar{z}) + \frac{1}{2} u D_A Y^A, \\
\xi^r &= -\frac{1}{2} r\, D_A Y^A + \mathcal{O}(1), \\
\xi^z &= Y(z) + \mathcal{O}(\frac{1}{r}), \\
\xi^{\bar{z}} &= \bar{Y}(\bar{z}) + \mathcal{O}(\frac{1}{r}),
\end{aligned}
\tag{7}
$$

  where $z = e^{i\phi} \cot \frac{\theta}{2}$ is the standard stereographic coordinates.
- The transformations

$$
\xi^u = T(z,\bar{z}),\ \xi^r = \mathcal{O}(1),\ ,\xi^z = \mathcal{O}(\frac{1}{r}),\ \xi^{\bar{z}} = \mathcal{O}(\frac{1}{r}). \tag{8}
$$

  form the "supertranslation" subgroup. This terminology comes from the fact that the translations in Minkowski space are elements of this subgroup.

Asymptotic symmetries of gauge theory

- In the retarded spherical coordinates

$$ds^2 = -du^2 - 2du\,dr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}\,, \qquad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}\,, \tag{9}$$

the relevant fields will be expanded in the powers of $\frac{1}{r}$ at $r = \infty$ and the theory will be solved order by order respect to certain gauge and boundary conditions.

- For Maxwell theory, a convenient choice is

$$A_r = 0\,, \;\; A_u = \mathcal{O}(r^{-1})\,, \;\; A_z = \mathcal{O}(1)\,, \tag{10}$$

- Asymptotic symmetries of Maxwell theory is the residual (large) $U(1)$ gauge symmetry preserving those conditions

$$\delta_\epsilon A_r = 0\,, \;\; \delta_\epsilon A_u = \mathcal{O}(r^{-1})\,, \;\; \delta_\epsilon A_z = \mathcal{O}(1)\,, \tag{11}$$

which leads to $\epsilon = \epsilon(z, \bar{z})$.

[He, Mitra, Porfyriadis, Strominger 2014]

Asymptotic conserved charges

- The associated asymptotic conserved charge of Maxwell theory is

$$Q_{\epsilon_{\mathrm{out}}} = \int_{r\to\infty} \mathrm{d}z\mathrm{d}\bar{z}\, \gamma_{z\bar{z}}\, \epsilon(z,\bar{z})\, r^2\, F_{ru} \tag{12}$$

- The associated asymptotic conserved charge of linearized gravity theory is

$$Q_{\xi_{\mathrm{out}}} = \int_{r\to\infty} \mathrm{d}z\mathrm{d}\bar{z}\, \gamma_{z\bar{z}}\, r^2 \Big[\xi_\sigma \nabla^r h^{u\sigma} + \xi^r \nabla^u h - \xi^r \nabla_\sigma h^{u\sigma} - h^{r\sigma} \nabla_\sigma \xi^u + \frac{1}{2} h \nabla^r \xi^u \Big]. \tag{13}$$

[Barnich, Brandt 2001]

Gravitational memory

In general relativity, it is important to focus upon coordinate invariant observables. Consider two spatial freely falling particles, located at $z = 0$, and separated on the x-axis by a coordinate distance L_c. The proper distance L between the two particles in the presence of the GW in TT gauge that propagates down the z-axis, $h_{ab}^{TT}(t, z)$, is given by

$$
\begin{aligned}
L &= \int_0^{L_c} dx \sqrt{g_{xx}} = \int_0^{L_c} dx \sqrt{1 + h_{xx}^{TT}(t, z=0)} \\
&\simeq \int_0^{L_c} dx \left(1 + \frac{1}{2} h_{xx}^{TT}\right) \\
&= L_c + \frac{1}{2} h_{xx}^{TT} L_c
\end{aligned}
\tag{14}
$$

[Flanagan, Hughes 2005]

- This proper distance can be generalized to any direction

$$L^i(t) = L_c^i + \frac{1}{2} h_{ij}^{TT}(t) L_c^j. \tag{15}$$

- The permanent change

$$\Delta L^i = \frac{1}{2} \Delta h_{ij}^{TT} L_c^j = \frac{1}{2} h_{ij}^{TT}(t = +\infty) L_c^j - \frac{1}{2} h_{ij}^{TT}(t = -\infty) L_c^j \tag{16}$$

in the detector after the gravitational wave has passed is called the gravitational wave memory.

- Braginsky and Thorne analyzed the possible detection of the gravitational wave memory produced by the collision and scattering of large massive objects such as stars or black holes and found that such collisions resulted in a net difference in the transverse traceless part of the linearized metric in the momentum-space at large distance given by

$$\Delta h_{\mu\nu}^{TT}(\vec{k}) = \frac{1}{r_0}\sqrt{\frac{G}{2\pi}}\left(\sum_{a=1}^{n}\frac{p'_{a\mu}p'_{a\nu}}{k\cdot p'_a} - \sum_{c=1}^{m}\frac{p_{c\mu}p_{c\nu}}{k\cdot p_c}\right)^{TT}, \tag{17}$$

Here we have $n(m)$ incoming (outgoing) objects with asymptotic momenta $p'_{a\mu}(p_{c\mu})$. $k = (1, \vec{k})$ is the null vector pointing from the collision region to detector.
[Braginsky, Thorne 1987]

Gravitational memory and Soft theorem

Weinberg's soft graviton theorem [S. Weinberg (1965)] is a universal relation between $(n \to m+1)$- particle with one final soft graviton and $(n \to m)$-particle quantum field theory scattering amplitudes. The universal result is given by

$$\lim_{\omega \to 0} M_{n+m+1}\left(p_1', \dots p_n'; p_1, \dots p_m, (\omega k, \epsilon^{\mu\nu})\right) = \\ \sqrt{8\pi G} S_{\mu\nu} \epsilon^{\mu\nu} M_{n+m}(p_1', \dots p_n'; p_1, \dots p_m) + O(\omega^0), \quad (18)$$

where

$$S_{\mu\nu} = \frac{1}{\omega} \left(\sum_{a=1}^{m} \frac{p_{a\mu} p_{a\nu}}{k \cdot p_a} - \sum_{c=1}^{n} \frac{p'_{c\mu} p'_{c\nu}}{k \cdot p'_c} \right)^{TT} . \quad (19)$$

In this expression $\omega k = (\omega, \omega \vec{k})$ with $\vec{k}^2 = 1$ is the four-momentum and $\epsilon^{\mu\nu}$ is the transverse-traceless polarization tensor of the graviton. The superscript TT denotes the transverse-traceless projection. [Strominger, Zhiboedov 2014]

Soft theorem as Ward identity of broken symmetry

- In S-matrix language, a symmetry is just a relation between matrix elements $\langle \mathrm{out}'|\mathrm{in}'\rangle = \langle \mathrm{out}|\mathrm{in}\rangle$, where the *in* and *out* states have been transformed as $|\mathrm{in}'/\mathrm{out}'\rangle = \mathrm{U}^{\mathrm{in/out}}|\mathrm{in/out}\rangle$.
- The operators implementing the symmetry must verify $U^{\mathrm{out}\dagger} U^{\mathrm{in}} = 1$.
- If this symmetry is generated by a charge Q (*i.e.* $U^{\mathrm{in/out}} = e^{i\theta\, Q^{\mathrm{in/out}}}$), the associated Ward identity reads as

$$\langle \mathrm{out}|\mathrm{Q}^{\mathrm{out}} - \mathrm{Q}^{\mathrm{in}}|\mathrm{in}\rangle = 0 \ . \tag{20}$$

- The charge for a spontaneously broken symmetry must act non-linearly on the states,otherwise it would annihilate the vacuum.
- The charge can be decomposed into linear and non-linear pieces $Q = Q_{\mathrm{L}} + Q_{\mathrm{NL}}$.
- The Ward identity for a broken charge becomes

$$\langle \mathrm{out}|\mathrm{Q}^{\mathrm{out}}_{\mathrm{NL}} - \mathrm{Q}^{\mathrm{in}}_{\mathrm{NL}}|\mathrm{in}\rangle = -\langle \mathrm{out}|\mathrm{Q}^{\mathrm{out}}_{\mathrm{L}} - \mathrm{Q}^{\mathrm{in}}_{\mathrm{L}}|\mathrm{in}\rangle \, . \tag{21}$$

- Neglecting issues about a proper, non-divergent definition of a broken charge, if Q_{NL} creates zero-energy Goldstone bosons, equation (21) looks very much like (1).

[Strominger 2013]

Soft photon theorem and large gauge transformation

Strominger and collaborators have shown precisely that the residual (large) $U(1)$ gauge symmetry preserving the following condition

$$A_r = 0\ ,\ \ A_u = \mathcal{O}(r^{-1})\ ,\ \ A_z = \mathcal{O}(1), \tag{22}$$

in the retarded spherical coordinates

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\,, \qquad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}\,, \tag{23}$$

with the associated charge

$$Q_{\epsilon_{\mathrm{out}}} = \int_{\mathscr{I}^+_-} \mathrm{d}z\mathrm{d}\bar{z}\,\gamma_{z\bar{z}}\,\epsilon(z,\bar{z})\,r^2\,F_{ru} \tag{24}$$

is the symmetry responsible for the soft photon theorem.
[He, Mitra, Porfyriadis, Strominger 2014]

The solution space for Maxwell theory

In the bulk we have a gauge field A_μ and matter. We will avoid talking about the kind of matter we have (*e.g.* scalar, fermionic) by introducing just a conserved current J_μ. We choose the following gauge and asymptotic conditions for the current

$$J_r = 0 \ , \quad J_u = \mathcal{O}(r^{-2}) \ , \quad J_z = \mathcal{O}(r^{-2}) \ , \tag{25}$$

which is consistent with the gauge choice of the gauge fields (22) and is always achievable using the equivalent class of the current $J^\mu \sim J^\mu + \nabla_\nu k^{[\mu\nu]}$.

Let us assume the following ansatz for the $\frac{1}{r}$-expansion of the gauge field

$$A_u = \frac{A_u^0(u,z,\bar{z})}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad A_{z(\bar{z})} = A^0_{z(\bar{z})}(u,z,\bar{z}) + \sum_{m=1}^{\infty} \frac{A^m_{z(\bar{z})}(u,z,\bar{z})}{r^m}, \tag{26}$$

and the current

$$J_u = \frac{J_u^0(u,z,\bar{z})}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad J_{z(\bar{z})} = \frac{J^0_{z(\bar{z})}(u,z,\bar{z})}{r^2} + \sum_{m=1}^{\infty} \frac{J^m_{z(\bar{z})}(u,z,\bar{z})}{r^{m+2}}. \tag{27}$$

The reason for not specifying further the expansions of the u-components of A_μ and J_μ is that they are determined by the equations of motion.

- The current conservation equation

$$J_u = \frac{J_u^0(u,z,\bar{z})}{r^2} - \frac{1}{r^2}\int_r^\infty \mathrm{d}r' \left[\gamma_{z\bar{z}}^{-1}(\partial_z J_{\bar{z}} + \partial_{\bar{z}} J_z)\right] \ . \tag{28}$$

- The u-component of the Maxwell's equations

$$A_u = \frac{A_u^0(u,z,\bar{z})}{r} + \int_r^\infty \mathrm{d}r' \frac{1}{r'^2}\int_{r'}^\infty \mathrm{d}r'' \left[\gamma_{z\bar{z}}^{-1}(\partial_z\partial_{r''} A_{\bar{z}} + \partial_{\bar{z}}\partial_{r''} A_z)\right] \ . \tag{29}$$

- The $z(\bar{z})$-components of the Maxwell's equations

$$2\partial_u A_z^1 = \partial_z A_u^0 + \partial_z[\gamma_{z\bar{z}}^{-1}(\partial_z A_{\bar{z}}^0 - \partial_{\bar{z}} A_z^0)] + J_z^0 \ , \tag{30}$$

$$\partial_u A_z^m = \frac{J_z^{m-1}}{2m} - \frac{m-1}{2} A_z^{m-1} - \frac{\partial_z[\gamma_{z\bar{z}}^{-1}(\partial_{\bar{z}} A_z^{m-1})]}{m} \ , \tag{31}$$

when $(m \geq 2)$.

- The r-component of the Maxwell's equations

$$\partial_u A_u^0 = \gamma_{z\bar{z}}^{-1}\partial_u(\partial_z A_{\bar{z}}^0 + \partial_{\bar{z}} A_z^0) + J_u^0 \ . \tag{32}$$

- Inserting the solution of the classical Maxwell equations into the charge yields

$$\begin{aligned} Q_\varepsilon &= -\int_{\mathscr{I}^+_-} \mathrm{d}z\mathrm{d}\bar{z}\, \gamma_{z\bar{z}}\, \varepsilon(z,\bar{z})\, A_u^0 \\ &= -\int_{\mathscr{I}^+} \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}u\, \epsilon \left[\partial_u(\partial_z A_{\bar{z}}^0 + \partial_{\bar{z}} A_z^0) + \gamma_{z\bar{z}} J_u^0\right] . \end{aligned} \tag{33}$$

- We decompose the charge into the linear and non-linear pieces as

$$Q_{\mathrm{NL}} = \int_{\mathscr{I}^+} \mathrm{d}u\mathrm{d}^2 z\, \varepsilon\, \partial_u \left(\partial_z A_{\bar{z}}^0 + \partial_{\bar{z}} A_z^0\right) , \; Q_{\mathrm{L}} = \int_{\mathscr{I}^+} \mathrm{d}u\mathrm{d}^2 z\, \gamma_{z\bar{z}}\, \varepsilon\, J_u^0 .$$

- The non-linear pieces of the charges:

$$\langle \mathrm{out}|Q_{\mathrm{NL}}^{(0)}|\mathrm{in}\rangle = \frac{1}{2}\frac{\sqrt{2}}{1+|w|^2}\lim_{\omega_q\to 0}\langle \mathrm{out}|\omega_q\, \mathfrak{a}_+(q)|\mathrm{in}\rangle, \tag{34}$$

with a concrete choice $\varepsilon(z,\bar{z}) = \frac{1}{w-z}$.

- The Fourier relations:

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}u\, \partial_u F(u) &= 2\pi i \lim_{\omega\to 0}\left[\omega \tilde{F}(\omega)\right]\ ,\\ \int_{-\infty}^{\infty} \mathrm{d}u\, u\, \partial_u F(u) &= -2\pi \lim_{\omega\to 0}\left[\partial_\omega \left(\omega \tilde{F}(\omega)\right)\right]\ ,\end{aligned} \tag{35}$$

- Stationary-phase approximation of the gauge-field mode expansion:

$$A^0_{z(\bar{z})}(x) = -\frac{i}{8\pi^2}\frac{\sqrt{2}}{1+z\bar{z}}\int_0^\infty \mathrm{d}\omega_q \Big[\mathfrak{a}_{+(-)}(\omega_q\hat{x})\, e^{-i\omega_q u} - \mathfrak{a}^\dagger_{-(+)}(\omega_q\hat{x})\, e^{i\omega_q u}\Big].$$

- Regarding the linear pieces, restricting ourselves to scalar charged matter, we just need to use the boundary canonical commutation relation

$$\left[\, \bar{\Phi}^0(u, z, \bar{z}), \Phi^0(u', w, \overline{w}) \,\right] = \frac{i}{4}\gamma_{w\bar{w}}\, \Theta(u' - u)\delta^2(z - w)\ , \tag{36}$$

  to obtain that

$$\langle \mathrm{out}|\, Q_{\mathrm{L}}^{(0)} |\mathrm{in}\rangle = \sum_{k=1}^{n} - \frac{e_k}{2(w - w_k)} \langle \mathrm{out}|\mathrm{in}\rangle. \tag{37}$$

- Assembling all these expressions, it is immediate to recover the leading soft theorem in the asymptotic position space

$$\lim_{\omega_q \to 0} \langle \mathrm{out}|\omega_q\, \mathfrak{a}_+(q)|\mathrm{in}\rangle = \frac{1 + |w|^2}{\sqrt{2}} \sum_{k=1}^{n} \frac{e_k}{w - w_k}\, \langle \mathrm{out}|\mathrm{in}\rangle. \tag{38}$$

The sub-leading soft factors

- Soft theorems can then be thought as factorization properties that scattering amplitudes must obey in a low-energy expansion:

$$M_{n+1}(p_1, \ldots, p_n, q) = \left(\frac{S^{(0)}}{\omega_q} + \cdots + \omega_q^{s-1} S^{(s)} \right) M_n(p_1, \ldots, p_n) + \mathcal{O}(\omega_q^s) \,, \quad (39)$$

where ω_q and s are the energy and spin of the emitted boson. For $s = 1, 2$, soft theorems display several orders in the energy ω_q of the emitted boson.

- For gauge theory, the sub-leading soft factor, which involves the angular momentum operator $J^{\mu\nu}$, is

$$S^{(1)} = \sum_{k=1}^{n} e_k \, \frac{q_\mu \epsilon_\nu^{\pm} \, J_k^{\mu\nu}}{p_k \cdot q} \,, \tag{40}$$

- The sub-leading and sub-sub-leading soft factors of gravity are

$$S^{(1)} = \sum_{k=1}^{n} \frac{\epsilon^{\pm} \cdot p_k \, \epsilon_\mu^{\pm} q_\nu \, J_k^{\mu\nu}}{p_k \cdot q}, \quad S^{(2)} = \sum_{k=1}^{n} \frac{\epsilon_\mu^{\pm} \epsilon_\nu^{\pm} q_\rho q_\sigma \, J_k^{\rho\mu} \, J_k^{\sigma\nu}}{\omega_q \, p_k \cdot q}. \tag{41}$$

- How should we understand the sub-leading factors in the context of Ward identity of broken symmetry?

- A pragmatic way is to first recast the sub-leading soft theorems into the Ward identity of some unknown symmetries, then trying to understand if those unknown symmetries can be recovered from asymptotic symmetries with certain adaptations of boundary conditions.
  [Kapec, Lysov, Pasterski, Strominger 2014]
  [Campiglia, Laddha 2014]
- We have argued that the same asymptotic symmetry with the sub-leading pieces of the associated charges (12) or (13) is responsible for this sub-leading soft factors. This idea has been partially succeeded in different theories.
  [Conde, PM 2016]
  [Conde, PM 2016]
  [PM, Wu 2017]

- Is there a new kind of memories related to the sub-leading soft theorems?
- Yes. The first example was reported in gravitational theory. It is the spin memory which is a Fourier transform of the sub-leading soft graviton theorem.
  [Pasterski, Strominger, Zhiboedov 2015]
- Beams on clockwise and counterclockwise orbits acquire a relative delay induced by radiative angular momentum flux.
- We find a type of electromagnetic memory related to the sub-leading soft photon theorem. It is a "magnetic" type, or B mode, radiation memory effect. Rather than a residual velocity, we find a position displacement of a charged particle induced by the B mode radiation with memory.
  [PM, Ouyang, Wu, Wu 2017]

Electromagnetic memories

- Consider a charge q with mass m moving in the presence of electric fields

$$m\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = q\vec{E}. \tag{42}$$

- It follows that once the wave has passed the charge has received a kick given by

$$\Delta\vec{v} = \frac{q}{m}\int_{-\infty}^{\infty}\vec{E}\mathrm{d}t, \tag{43}$$

- This is the electromagnetic analogue of the gravitational memory effect.
  [Bieri, Garfinkle 2013]

In retarded coordinates

$$ds^2 = -du^2 - 2du\,dr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \quad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}. \tag{44}$$

With the following asymptotic behavior

$$A_u^+ = \frac{A_u^{+0}(u,z,\bar{z})}{r^2} + \mathcal{O}(r^{-3}),$$
$$A_z^+ = A_z^{+0}(u,z,\bar{z}) + \frac{A_z^{+1}(u,z,\bar{z})}{r} + \mathcal{O}(r^{-2}),$$
$$j_u^+ = \frac{j_u^{+0}(u,z,\bar{z})}{r^2} + \mathcal{O}(r^{-3}), \quad j_r^+ = \mathcal{O}(r^{-4}), \quad j_z^+ = \frac{j_z^{+0}(u,z,\bar{z})}{r^2} + \mathcal{O}(r^{-3}), \tag{45}$$

the Maxwell's equations $\nabla_\nu F^{\mu\nu} = 4\pi j^\mu$ yield

$$\partial_u A_u^{+0} = \gamma_{z\bar{z}}^{-1}\partial_u(\partial_z A_{\bar{z}}^{+0} + \partial_{\bar{z}} A_z^{+0}) - 4\pi j_u^{+0}\,, \tag{46}$$
$$2\partial_u A_z^{+1} = \partial_z A_u^{+0} + \partial_z[\gamma_{z\bar{z}}^{-1}(\partial_z A_{\bar{z}}^{+0} - \partial_{\bar{z}} A_z^{+0})] - 4\pi j_z^{+0}\,. \tag{47}$$

Kick memory

- Inserting the E mode decomposition $A^{+0}_{z(\bar{z})} = \partial_{z(\bar{z})}\alpha(u, z, \bar{z})$ into (46) leads to

$$D_A D^A \Delta\alpha = \Delta A^{+0}_u + 4\pi \int_{-\infty}^{\infty} \mathrm{d}u\, j^{+0}_u\,, \tag{48}$$

  where D_A is the spherical covariant derivative.
- The angular part of the electric fields are $E_{z(\bar{z})} = F_{z(\bar{z})u}$. They are related to the memory formula as

$$\int_{-\infty}^{\infty} \mathrm{d}u E_{z(\bar{z})} = -\partial_{z(\bar{z})}\Delta\alpha. \tag{49}$$

- From the standard analysis of the motion of a charged particle in the presence of electric fields, (49) will leave a residual velocity to the charged particle (a 'kick'). Following the terminology of Bieri and Garfinkle, the first term on the right hand side of (48) is called ordinary 'kick' and the second one induces a null 'kick'.

Displacement memory

- For the B mode part where we can set $A_z^{+0} = i\partial_z\beta(u, z, \bar{z})$ and $A_{\bar{z}}^{+0} = -i\partial_{\bar{z}}\beta(u, z, \bar{z})$, it was proven that $\beta = 0$ at $u = \pm\infty$ in the case of physically realistic source [Winicour 2014]. Obviously $\Delta\beta = 0$, which means no B mode 'kick' memory though it is mathematically possible.
- Alternatively we will propose a new type of memory which is defined as

$$\Delta\Gamma = \int_{-\infty}^{\infty} \mathrm{d}u\, \beta. \tag{50}$$

- In terms of the electric fields, one will get

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{u} \mathrm{d}u'\, E_z &= -i\partial_z\Delta\Gamma, \\ \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{u} \mathrm{d}u'\, E_{\bar{z}} &= i\partial_{\bar{z}}\Delta\Gamma. \end{aligned} \tag{51}$$

- The new memory formula can be arranged into a nicer way:

$$
\begin{aligned}
iD_A D^A D_B D^B \Delta\Gamma = 2\gamma_{z\bar{z}}^{-1}\Delta(\partial_z A_{\bar{z}}^{+1} - \partial_{\bar{z}} A_z^{+1}) \\
+ 4\pi\gamma_{z\bar{z}}^{-1}\int_{-\infty}^{\infty} \mathrm{d}u\,(\partial_z j_{\bar{z}}^{+0} - \partial_{\bar{z}} j_z^{+0}).
\end{aligned} \tag{52}
$$

- Considering a very slowly moving charged particle, the electric field dominates the motion of the charged particle. Hence the new memory effect (51) with one more integration over u than the 'kick' memory effect (49) will be a displacement of the charged particle.
- Following the terminology of the 'kick' memory, we will refer to the first term on the right hand side of (52) as ordinary displacement and the second one as null displacement.

- The triangle relation in higher dimension is less clear even at the leading order.
- The asymptotic symmetry (*e.g.* the BMS supertranslation in gravitational theory) that is supposed to be responsible for the leading soft theorem seems to be debatable. [Hollands, Ishibashi, Wald 2016]
- The memory effect in higher dimension was suggested to be absent. [Garfinkle, Hollands, Ishibashi, Tolish, Wald 2017]

- we propose that the equivalence of the soft theorems and memories should be understood in the following way: The classical computation arises as a limiting case of the quantum results (soft theorems). [PM, Ouyang 2017]
- In practice, we consider the soft factor in asymptotic position space as a classical field

$$S^{(0)} = \lim_{\omega_q \to 0} \frac{\langle \mathrm{out}|\omega_q\, \mathfrak{a}_+(q)|\mathrm{in}\rangle}{\langle \mathrm{out}|\mathrm{in}\rangle} \tag{53}$$

- One can show that this classical field satisfy the classical equations of motion.
- The memory effects are determined completely by the classical fields in the asymptotic region.

The scalar theory

- Soft scalar theorem

$$M_{N+1}^{\text{scalar}} = \sum_{k=1}^{N} \frac{g_k}{p_k \cdot q} M_N^{\text{scalar}} + \mathcal{O}(\omega^0), \tag{54}$$

where g_k are the coupling constants, q is the momentum of the soft scalar and p_k are the momenta of the hard particles.
[Campiglia, Coito, Mizera 2017]

- The soft factor will be interpreted as the expectation value of the scalar field in the process of scattering in momentum space at the low energy limit

$$\varphi_d(\omega, \vec{q}) = \sum_{k=1}^{N} \frac{g_k}{p_k \cdot q}, \tag{55}$$

where d denotes the dimension of the spacetime.

- Performing a Fourier transformation, one obtains the scalar field in the position space as

$$\varphi_d(x) = \sum_{k=1}^{N} \int \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{1}{2\omega} \frac{\eta_k g_k}{q \cdot p_k} (e^{iq\cdot x} + \mathrm{c.c.}) \tag{56}$$

where $\eta_k = 1$ or -1 for an outgoing or incoming particle.

- Defining the generating function

$$\Phi \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (\pi s)^n \varphi_{4+2n}. \tag{57}$$

- Then $\varphi_d(x)$ can be derived from the generating function easily by taking the limit

$$\varphi_{4+2n}(x) = \lim_{s\to 0} \frac{1}{\pi^n} \frac{d^n}{ds^n} \Phi \tag{58}$$

- Finally the generating function has the following form

$$\Phi = \sum_{k=1}^{N} g_k \Phi_k, \tag{59}$$

where

$$\Phi_k = \frac{\eta_k}{4\pi\sqrt{(x \cdot p_k)^2 - p_k^2 x^2 - p_k^2 s}}\left(\Theta(t-\sqrt{r^2-s}) - \Theta(t+\sqrt{r^2-s})\right). \tag{60}$$

- This is nothing but the radiation field obtained from the solutions of the massless scalar wave equation!

Radiation fields

- The radiation fields were introduced by Dirac to investigate the radiation of electrons. They are defined by the difference of the retarded and advanced solutions.
- In $d = 4 + 2n$ dimensional spacetime, the retarded and advanced Green's functions satisfying the equation

$$-\eta^{\mu\nu}\partial_\mu\partial_\nu G_d(x-x') = (\partial_t^2 - \sum_{i=1}^{3+2n} \partial_{x_i}^2) G_d(x-x') = \delta_d(x-x'). \quad (61)$$

are given by

$$G_{4+2n}^{\mathrm{ret}} = \frac{1}{2\pi^{n+1}} \delta^{(n)}((t-t')^2 - |\mathbf{x}-\mathbf{x}'|^2)\Theta(t-t'), \quad (62)$$

$$G_{4+2n}^{\mathrm{adv}} = \frac{1}{2\pi^{n+1}} \delta^{(n)}((t-t')^2 - |\mathbf{x}-\mathbf{x}'|^2)\Theta(t'-t). \quad (63)$$

Considering the source corresponding to a particle created or destroyed at the origin

$$S_{dk} = \int_0^\infty d\tau \delta^d(x^\mu - \eta_k p_k^\mu \tau), \tag{64}$$

we obtain the retarded solution for the wave function

$$-\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi_{dk}^{\mathrm{ret}} = S_{dk}, \tag{65}$$

as

$$\varphi_{4+2n\,k}^{\mathrm{ret}} = \int_0^\infty \frac{d\tau}{2\pi^{n+1}}\delta^{(n)}(-(x - v_k\tau)^2)\Theta(t - E_k\eta_k\tau). \tag{66}$$

- Introducing the retarded generating function

$$\Phi_k^{\rm ret} \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \varphi_{4+2n\,k}^{\rm ret} (\pi s)^n, \\ = \int_0^{\infty} \frac{d\tau}{2\pi} \delta(-(x - p_k\tau)^2 + s)\Theta(t - E_k\eta_k\tau). \tag{67}$$

- For massive particle source, we have

$$\Phi_k^{\rm ret} = \frac{\Theta(\eta_k(t - \sqrt{r^2 - s}))}{4\pi\sqrt{(x \cdot p_k)^2 - p_k^2 x^2 - p_k^2 s}}. \tag{68}$$

- The generating function for the advanced solution with massive particle source can be derived in a similar way

$$\Phi_k^{\rm adv} = \frac{\Theta(\eta_k(t + \sqrt{r^2 - s}))}{4\pi\sqrt{(x \cdot p_k)^2 - p_k^2 x^2 - p_k^2 s}}. \tag{69}$$

- A general source can be written as a linear superposition of such created and destroyed particles (64), so the solutions can be written as a superposition of individual ones.
- The radiation field for the case of a generic source is obtained as

$$\Phi^{\mathrm{rad}} = \sum_{k=1}^{N} g_k \left(\Phi_k^{\mathrm{ret}} - \Phi_k^{\mathrm{adv}} \right) \tag{70}$$

- In certain gauge, the Maxwell's equations and linearized Einstein equations take the form of a wave function (61) of each components. Hence, the same analysis can be extended to electromagnetic theory and linearized gravity theory easily.

Open questions

- Super Yang-Mills, Supergravity...... and Memories
- Double soft theorem and Memories
- New gravitational memories?

Thanks for your attention!