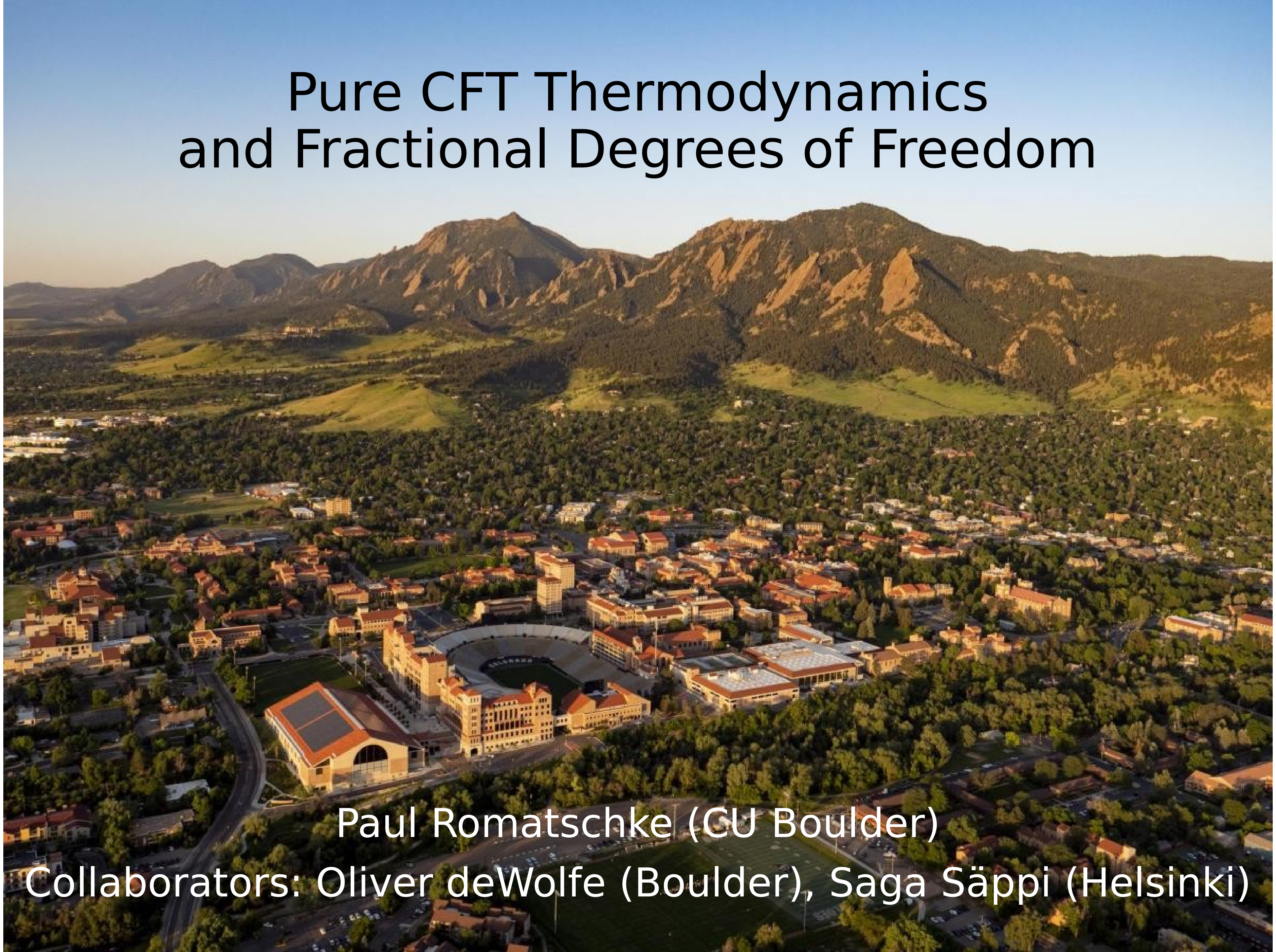
Pure CFT Thermodynamics and Fractional Degrees of Freedom



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Apologies

4th INTERNATIONAL CONFERENCE
on HOLOGRAPHY, STRING THEORY and DISCRETE APPROACH
in HANOI, VIETNAM

My talk is not on holography

My talk is not on string theory

My talk is not on discrete approach

My talk is not in Hanoi

… I sincerely hope that you nevertheless find some of it useful/interesting...

Outline of the Talk

- Motivation
- What are “pure” CFTs?
- Solving pure CFTs: thermodynamics
- CFT transport at strong coupling from field theory side
- Fractional degrees of freedom

Motivation

N=4 SYM

- SU(N) gauge theory+fermions+scalars
- Supersymmetric
- One coupling constant: ‘t Hooft coupling $\lambda = g^2 N$
- Theory exactly conformal for all N and all λ

N=4 SYM at weak coupling (λ=0)

- Can put the theory at finite temperature T (breaks SUSY)
- Calculate entropy density s of the theory at weak coupling
- At large N, find:

$$s_{free}=8\ N^2\ 2\ \pi^2 T^3/45\ (1+7/8)=N^2\ 2\ \pi^2 T^3/3$$

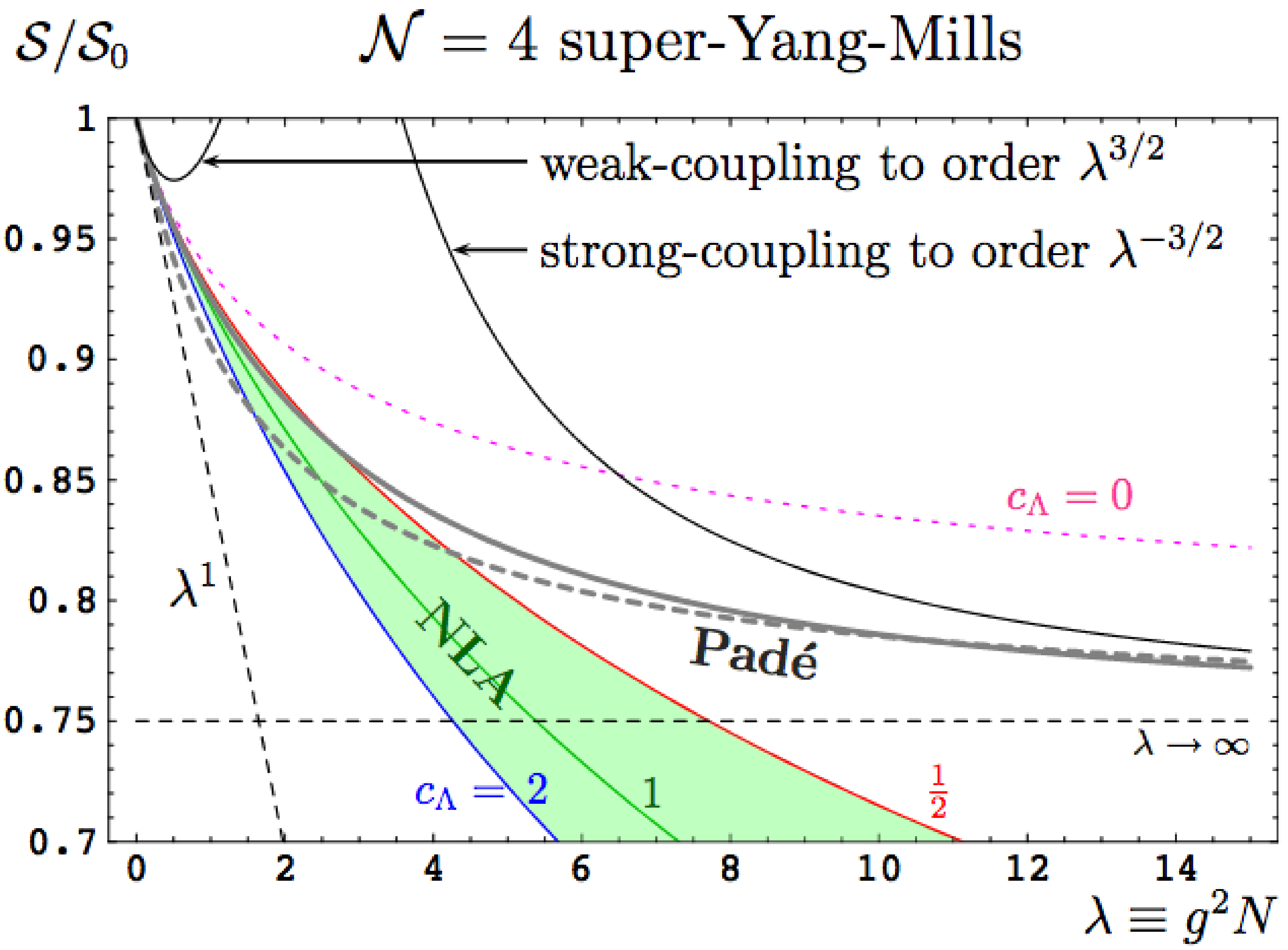
$N=4$ SYM at strong coupling ($\lambda=\infty$, $N=\infty$)

- Handle theory via gravity dual: classical gravity in AdS_5
- Put theory at finite temperature: black brane in AdS_5
- Calculate entropy of the theory at strong coupling (horizon area)
- At large N, find:

$$s_{strong} = N^2 \pi^2 T^3/2 = \tfrac{3}{4}\, s_{free}$$

It's exactly ¾

Why?



[hep-ph/0611393]

Effective degrees of freedom – cosmology

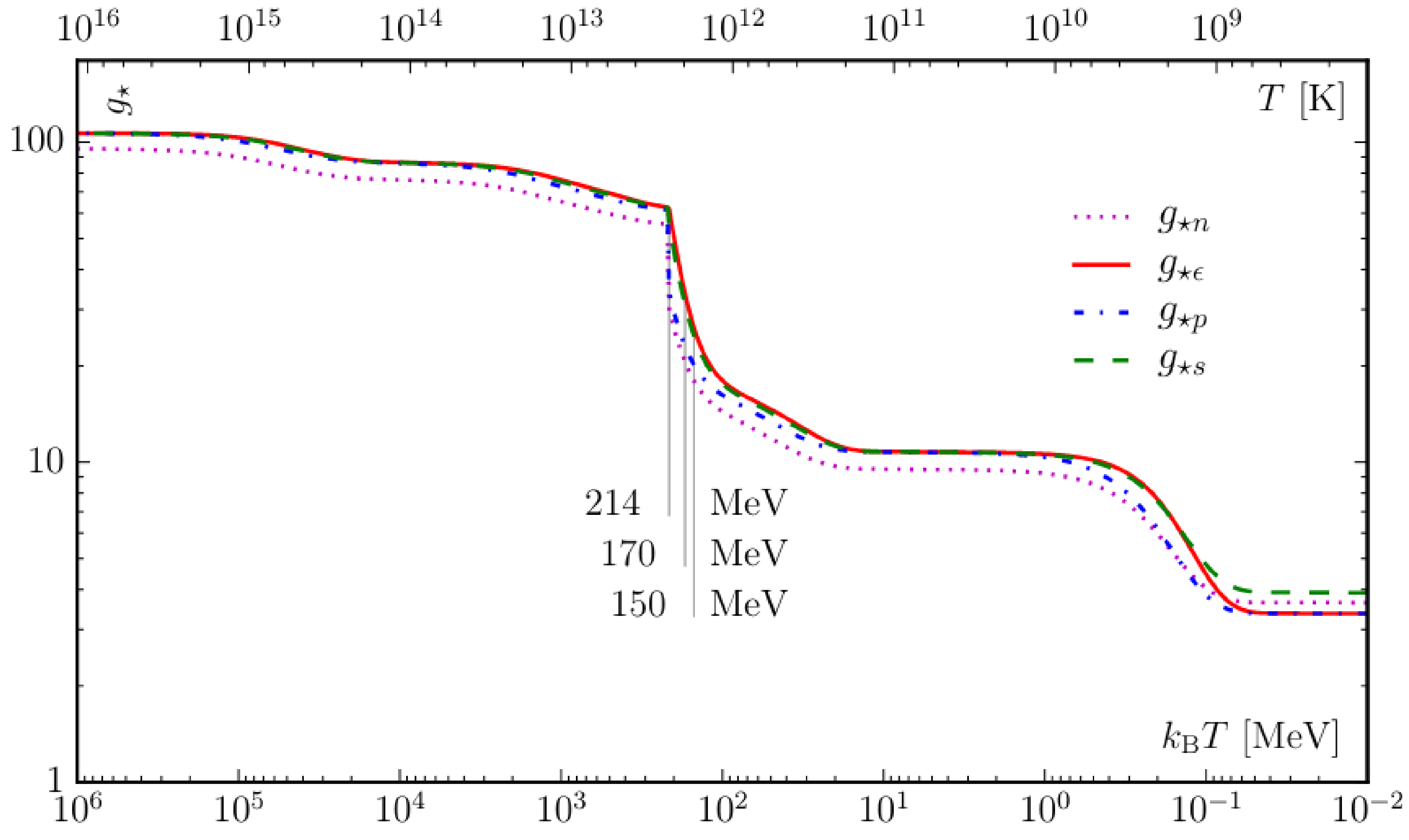


Figure 1. The evolution of the number density ($g_{\star n}$), energy density ($g_{\star \epsilon}$), pressure ($g_{\star p}$), and entropy density ($g_{\star s}$) as functions of temperature.

[1609.04979]

Motivation

- It would be nice to solve QFTs exactly for all values of coupling (even if only at large N)
- At finite temperature, gives effective number of degrees of freedom
- It would be nice to understand why funny fractions such as ¾ appear in N=4

Pure CFTs

What are “Pure” CFTs?

- CFTs are QFTs that respect conformal symmetry

CFTs have

1. Vanishing beta function
2. No (zero-temperature) mass scales
3. Vanishing trace of the energy momentum tensor

Normally, we get a CFT from a QFT by “tuning” parameters, e.g. bare masses and bare couplings, e.g. $\lambda = \lambda_{crit}$

What are “Pure” CFTs?

- “Pure” CFTs are QFTs which respect conformal symmetry at all values of λ, not just λ_{crit}
- Some people call this a “line of fixed points”
- Unsurprisingly, pure CFTs are rare
- Well-known example for a pure CFT: N=4 SYM

Are there any others?

Example for a pure CFT in 2+1d

- Consider vector O(N) model with sextic interactions
- Finite Temperature via imaginary time
- Space-time structure is thermal cylinder ($S^1 x R^2$), 3d euclidean metric
- Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \vec{\phi}\right) \cdot \left(\partial_\mu \vec{\phi}\right) + \frac{1}{2} m^2 \vec{\phi}^2 + \frac{\lambda_2}{N^2}\left(\vec{\phi}^2\right)^3$$

$$\vec{\phi} = \left(\phi_1, \phi_2, \ldots \phi_N\right)$$

I claim that for m=0, this theory is a pure CFT

Example for a pure CFT in 2+1d

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \vec{\phi}\right)\cdot\left(\partial_\mu \vec{\phi}\right) + \frac{1}{2}m^2\vec{\phi}^2 + \frac{\lambda_2}{N^2}\left(\vec{\phi}^2\right)^3$$

- The partition function Z for this theory can be calculated exactly for all λ_2 at large N
- The beta function for λ_2 is zero for all λ_2 when $m=0$
- The energy-momentum tensor is traceless for all λ_2 when $m=0$
- The calculation is straightforward, and only uses standard QFT tools
- I will not bore you with equations, but only give you a “birds-eye” view; details can be found in 1904.09995

Solving pure large N CFTs at any λ – bird's eye view for experts

- Want to calculate thermodynamics of pure CFTs
- Need partition function $Z = \int \mathcal{D}\phi e^{-\int \mathcal{L}}$
- Use auxiliaries (HB-transformation)

$$1 = \int \mathcal{D}\sigma \delta \left(\sigma - \vec{\phi}^2\right) = \int \mathcal{D}\sigma \mathcal{D}\zeta e^{i \int \zeta \left(\sigma - \vec{\phi}^2\right)}$$

- Large N: only zero modes σ_0, ζ_0 contribute (flucs 1/N suppressed)
- ϕ integration becomes trivial: just free massive scalars
- Large N: integral over zero modes exactly given by saddle points for σ_0, ζ_0
- Saddle point condition becomes "gap" equation for scalar mass

Works not only for CFTs!

Note: large N technique does not rely on CFT. It is a handle on exact results at any coupling for large class of QFTs

Solving pure large N CFTs at any λ – results

- Scalar mass is "thermal mass" $z^* = \xi^2 T^2$
- For any coupling, fulfills "gap equation"

$$\frac{4\xi}{\sqrt{6\lambda_2}} = -\frac{\xi}{\pi} - \frac{2}{\pi}\ln\left(1 - e^{-\xi}\right)$$

- Fully non-perturbative, not a weak-coupling concept
- Interesting value for λ_2 -> ∞:

$$\xi \to 2\ln\frac{1+\sqrt{5}}{2}$$

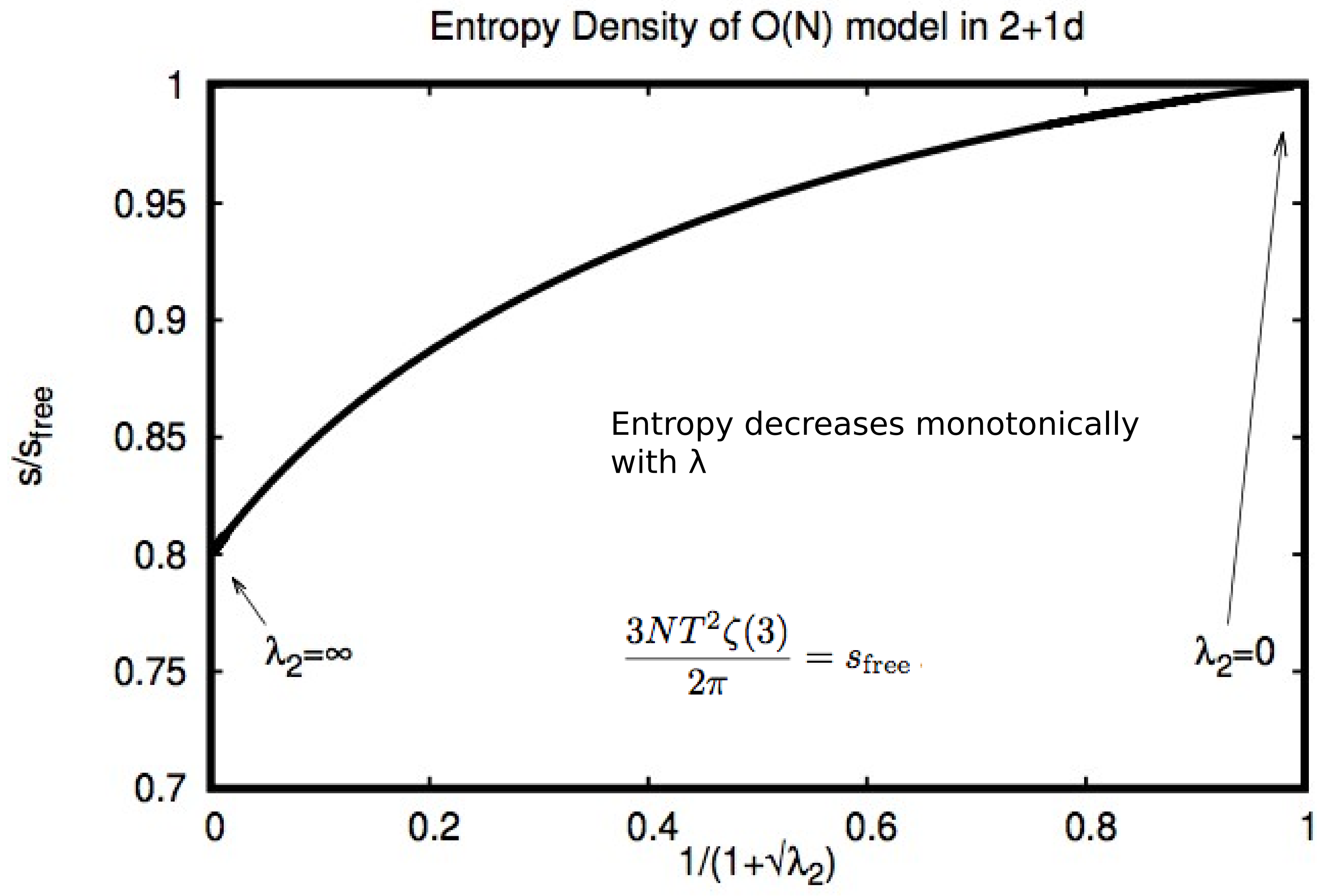
Solving pure large N CFTs at any λ – results

- Thermodynamics found from partition function Z
- E.g. Pressure P and entropy density s

$$P = T\frac{\ln Z}{\partial V} \qquad s = \frac{\partial P}{\partial T}$$

- These depend only on coupling λ_2 through mass $\xi(\lambda_2)$

$$s = \frac{NT^2}{4\pi}\left[\xi^3 + \xi^2 \ln\frac{1-e^{-\xi}}{(1-e^{\xi})^3} - 6\xi \mathrm{Li}_2\left(e^{\xi}\right) + 6\mathrm{Li}_3\left(e^{\xi}\right)\right]$$


Entropy Density of O(N) model in 2+1d
1
0.95
0.9
0.85
0.8
0.75
0.7
0
0.2
0.4
0.6
0.8
1
s/s_{free}
$1/(1+\sqrt{\lambda_2})$
Entropy decreases monotonically with λ
$\lambda_2=\infty$
$\lambda_2=0$
$\frac{3NT^2\zeta(3)}{2\pi} = s_{\mathrm{free}}$

Thermal vs. Entanglement entropy

- For CFT, can calculated entanglement entropy S_{EE} for spherical region from sphere free energy
- Same technique as for thermal entropy, just now S^3 instead of $S^1 x R^2$
- At large N, find $S_{NN}/S_{NN,free}=1$ for all couplings
- EE too bland observable? EE unrelated to d.o.f.s?

Solving pure large N CFTs at any λ – results

- Curious value s/s_{free} for $\lambda_2 \to \infty$

$$s = \frac{NT^2}{4\pi}\left[\xi^3 + \xi^2 \ln\frac{1-e^{-\xi}}{(1-e^{\xi})^3} - 6\xi\mathrm{Li}_2\left(e^{\xi}\right) + 6\mathrm{Li}_3\left(e^{\xi}\right)\right]$$

evaluated at $\xi \to 2\ln\frac{1+\sqrt{5}}{2}$ gives

$$\frac{12NT^2\zeta(3)}{10\pi} = \frac{4}{5}s_{\mathrm{free}}$$

It's exactly 4/5!

[1904.09995], [hep-th/9305131], [hep-ph/9708426]

Large N CFTs at λ = ∞

- For pure CFT in 2+1d found $s/s_{free}=4/5$
- Maybe it's a fluke?
- Check different interactions (*not* pure CFTs), e.g. quartic

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \vec{\phi}\right)\cdot\left(\partial_\mu \vec{\phi}\right) + \frac{1}{2}m^2\vec{\phi}^2 + \frac{\lambda}{N}\left(\vec{\phi}^2\right)^2$$

or why not *any* interaction potential U(x) with single min at x=0

$$\frac{\lambda}{N}\left(\vec{\phi}^2\right)^2 \to N \times U\left(\vec{\phi}^2/N\right)$$

Strong coupling universality

- Suprising (?) outcome: for large class of U(x), strong coupling limit gives

$$\xi \to 2 \ln \frac{1+\sqrt{5}}{2}$$

$s/s_{free}=4/5$

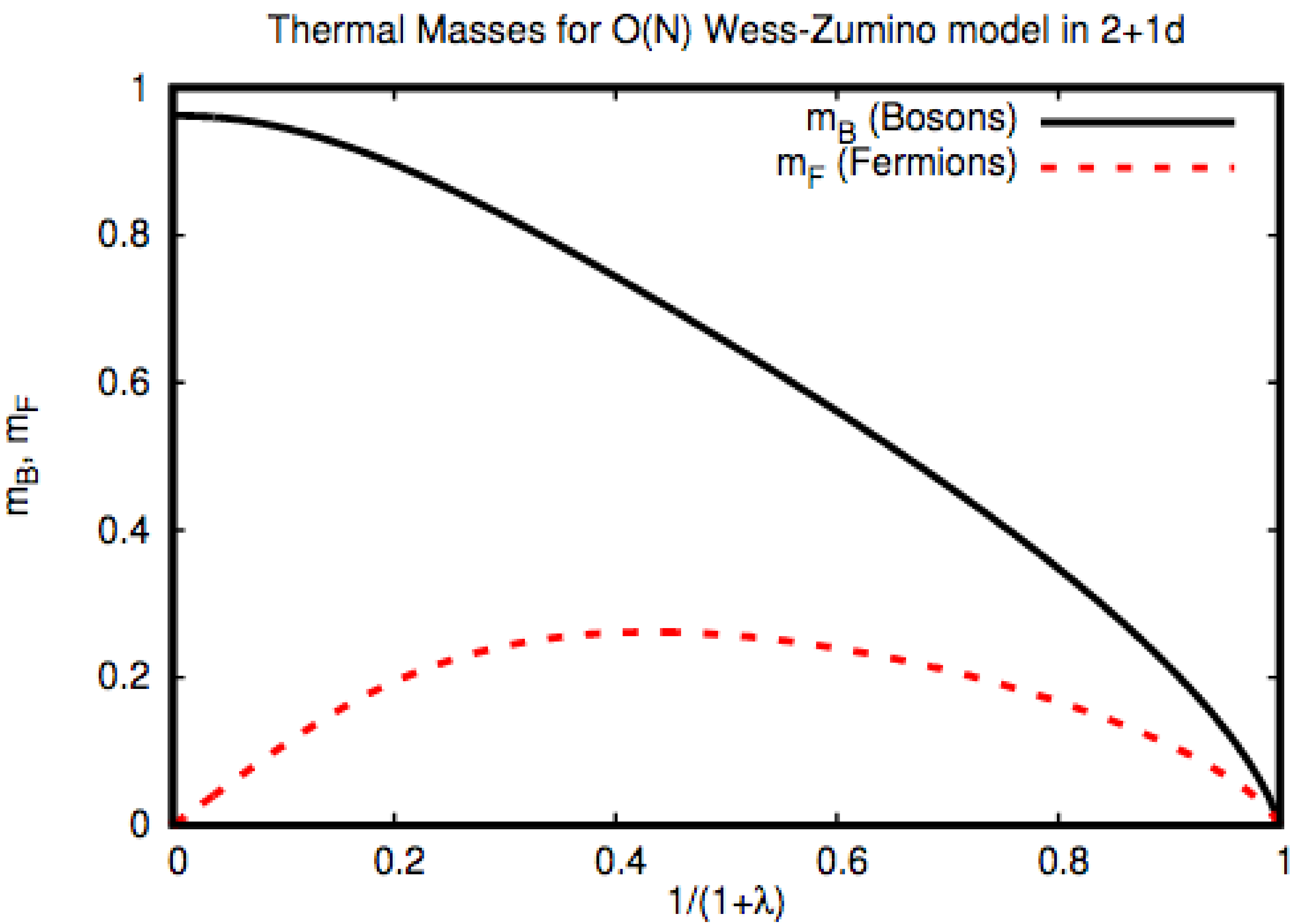
This result is the universal strong coupling limit for interacting bosons at large N

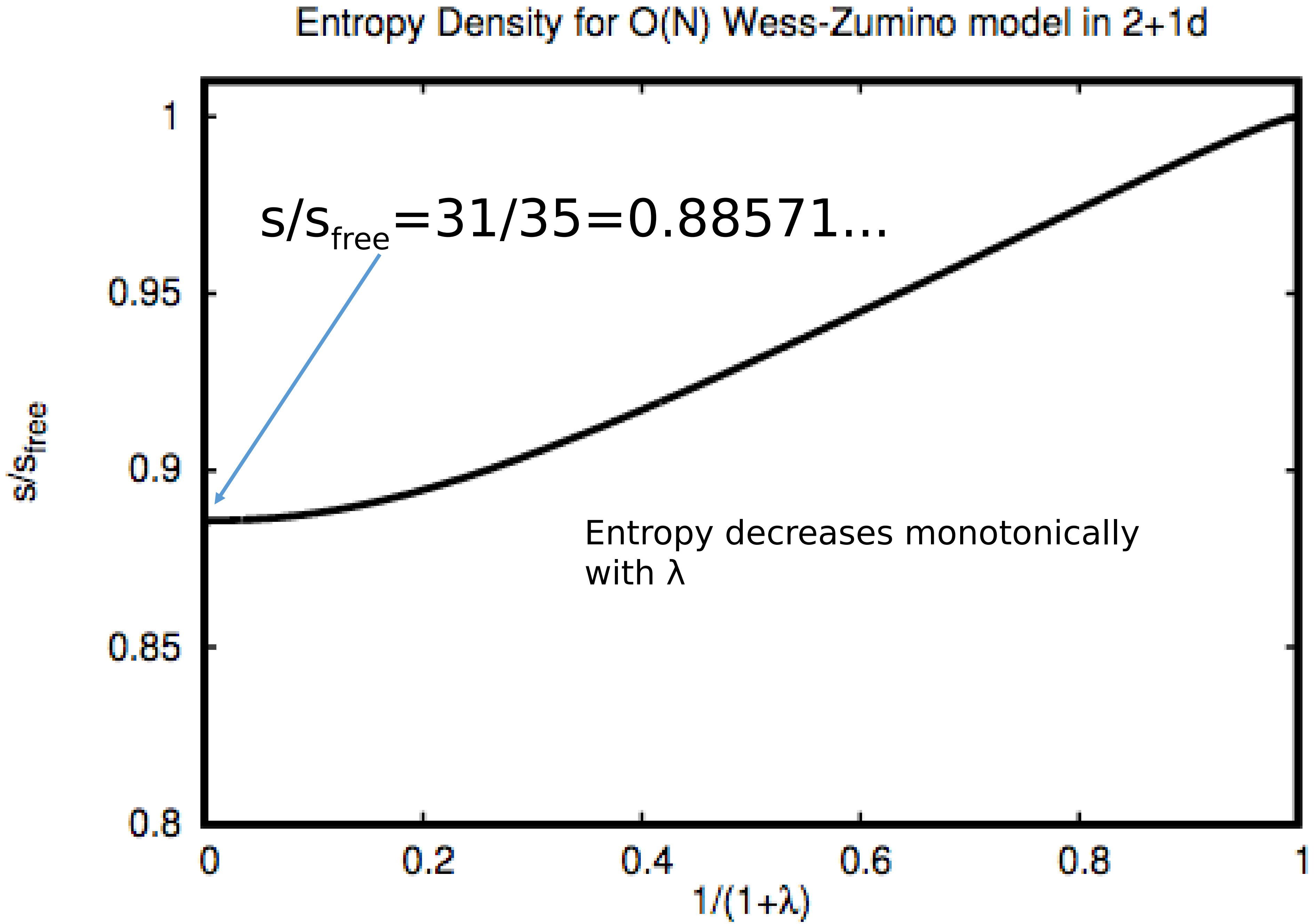
Why?

More pure large N 3d CFTs: Wess-Zumino model

$$S = \int d^3x d\bar{\theta} d\theta \left(\frac{1}{2} \bar{D}\Phi_a D\Phi_a + \frac{2\lambda}{N} (\Phi_a \Phi_a)^2 \right)$$

- Solved in [1905.06355] using same technique
- Upshot: pure CFT with SUSY at T=0
- Same thermal mass for bosons at strong coupling, but zero mass for fermions
- Non-SUSY versions exhibit strong coupling universality


Thermal Masses for O(N) Wess-Zumino model in 2+1d
m_B (Bosons)
m_F (Fermions)
m_B, m_F
$1/(1+\lambda)$
0
0.2
0.4
0.6
0.8
1



Entropy Density for O(N) Wess-Zumino model in 2+1d
s/sfree=31/35=0.88571...
Entropy decreases monotonically
with λ
s/sfree
1
0.95
0.9
0.85
0.8
0
0.2
0.4
0.6
0.8
1
1/(1+λ)

Strong coupling universality

- 3d theory with equal numbers of bosons and fermions
- For a large class of interaction potentials at strong coupling

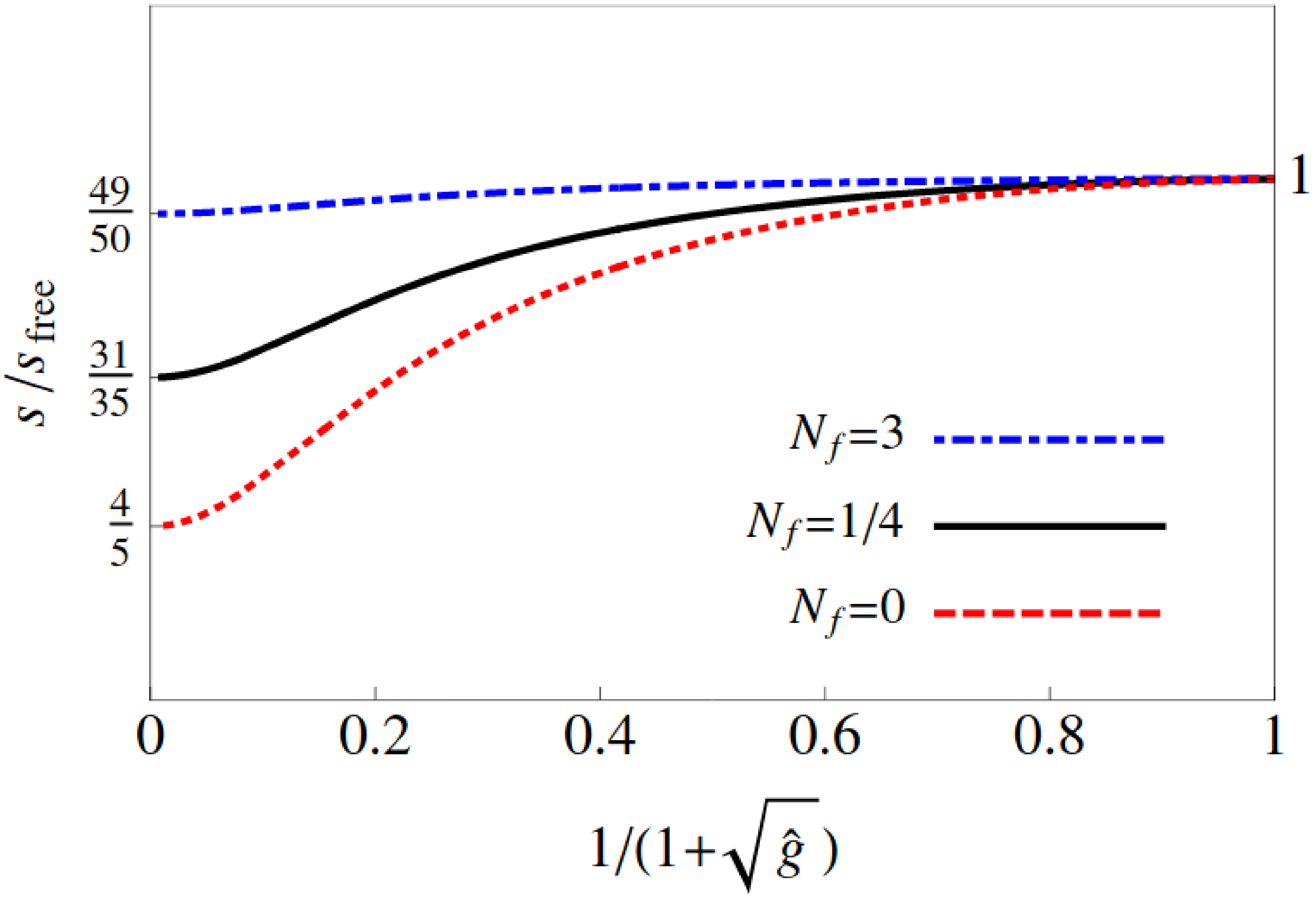
$$\xi \to 2\ln\frac{1+\sqrt{5}}{2} \qquad s/s_{free}=31/35=(4/5+3/4)/(1+3/4)$$

- Unequal numbers of bosons and fermions give the universal bound $s/s_{free}>4/5$

[1905.06355]

Strong coupling universality (2)

- Universality extends to a large class of (non-CFT) Lagrangians, see e.g. [Pinto, 2007.03784]



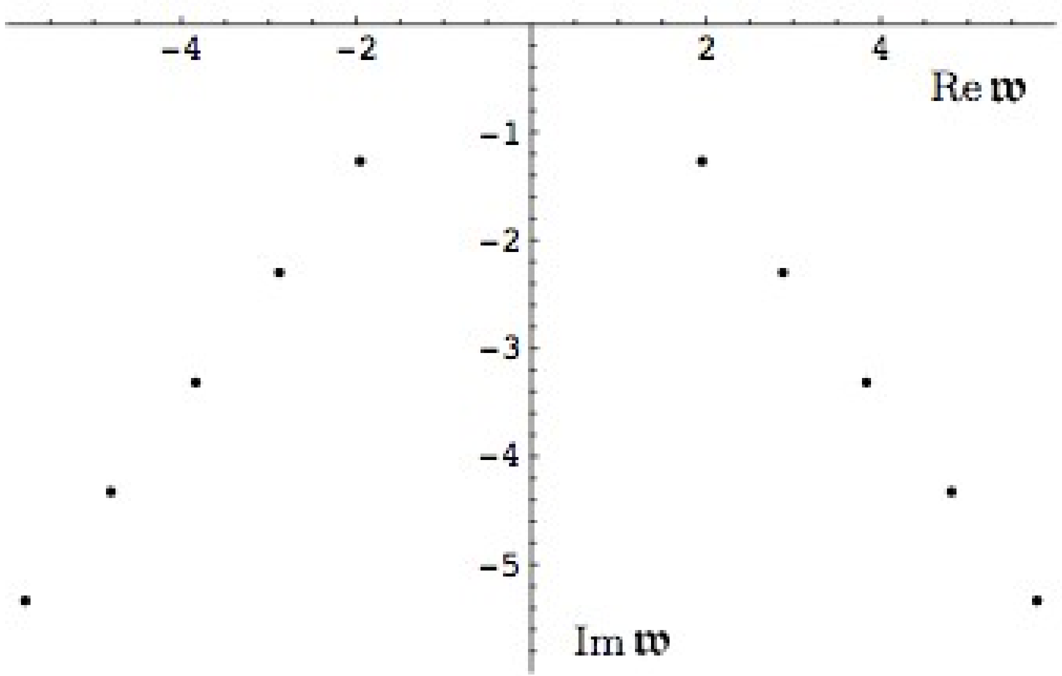
[2007.03784]

Strong Coupling Transport

N=4 SYM

N=4 SYM in 3+1d at λ=∞: Energy-momentum tensor correlators are given by black brane quasinormal modes

$$G_R^{xy,xy}(\breve{\omega}) = -\langle T^{xy}T^{xy}\rangle_E(\omega_m)|_{\omega_m \to -i\omega+0^+}$$



[hep-th/0506184]

O(N) model in 2+1d

- Field theory is exactly solvable at any λ_2
- Can calculate correlator $G_R^{xy,xy}(\omega) = -\left\langle T^{xy} T^{xy} \right\rangle_E (\omega_m)\big|_{\omega_m \to -i\omega + 0^+}$
- Find

$$G_R^{xy,xy}(\omega) = \frac{N}{8\pi T} \sum_n \frac{\ln \frac{\xi^2 - \frac{\omega^2}{4T^2}}{\xi^2 + (2\pi n)^2}}{\frac{\omega^2}{T^2} + 4(2\pi n)^2} \quad \text{+analytic}$$

where $\xi(\lambda)$ is thermal mass, e.g. $\xi(\infty)=0.96\ldots$

- No poles, only branch cuts at $\lambda = \infty$
- Result is universal in strong coupling limit

[1904.09995], cf [1210.4166]

Fractional degrees of freedom

Strong coupling results

- N=4 SYM in 4d $s/s_{free}=3/4$
- O(N) model in 3d $s/s_{free}=4/5$
- Wess-Zumino model in 3d $s/s_{free}=31/35$

Why are all of these simple fractions (rather than rational numbers)?

Yet another solvable large N theory: QED3

$$f_{\text{ghost}} + f_{\text{photon}} = \frac{1}{2} \sum\!\!\!\!\!\!\int \ln \left[\frac{(K^2 + \Pi_A)(K^2 + \Pi_B)}{K^2} \right]$$

- For free theory, $\Pi_{A,B}=0$ and free entropy density for each A,B is

$$s_{\text{free}} = -\frac{\partial}{\partial T} \left[T \int \frac{d^2k}{(2\pi)^2} \ln\left(1 - e^{-k/T}\right) \right] = \frac{3\zeta(3) T^2}{2\pi}$$

[1908.02758]

Yet another solvable large N theory: QED3

$$f_{\text{ghost}} + f_{\text{photon}} = \frac{1}{2} \sum\!\!\!\!\!\!\int \ln \left[\frac{\left(K^2 + \Pi_A\right)\left(K^2 + \Pi_B\right)}{K^2} \right]$$

- At strong coupling $\lambda = e^2 N/T = \infty$, $\Pi_{A,B}$ are given by vacuum bubble

$$\Pi_V(P) = \frac{e^2 N_f}{8} \sqrt{P^2}$$

- At strong coupling, each photon branch gives $s_{\text{strong}} = \frac{s_{\text{free}}}{2}$

Exactly ½ !

Yet another solvable large N theory: QED_3

Exactly ½ ! Why?

- For QED_3, the reason for the ½ is that the photon contribution gets split in half
- Photon dispersion relation is still quadratic, so no photon mass
- What happens is that the photon at strong coupling is not the same as the photon in the free theory
- It's akin to taking "the square root" of the free theory kinetic term

Does this sound familiar?

[1908.02758]

Fractional degrees of freedom

- Observation: in many (all?) CFTs, we find that the entropy at $\lambda = \infty$ is a simple fraction of the free theory result (3/4, 4/5, 31/35, ½,…)
- For QED_3, we observe that the photon itself contributes only ½ “free” degree of freedom
- Can it be that for CFTs at $\lambda = \infty$, what we observe are “fractional” degrees of freedom?
- Seems crazy at first sight (shouldn’t quantum mechanics forbid this?)
- Maybe not so crazy at second sight (we do have the fractional quantum Hall effect as a model)

The End