Krylov Complexity of free Scalar Field Theory

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7th International Conference on Holography and String Theory in Da Nang, August 26, 2024.

joint work with Hai-Qing Zhang

arXiv: 2407.02756

Outline

1. Introduction to the Krylov complexity
2. The Krylov complexity of five-dimensional free scalar field theory
   - Low temperatures $\beta m \gg 1$
   - General temperatures
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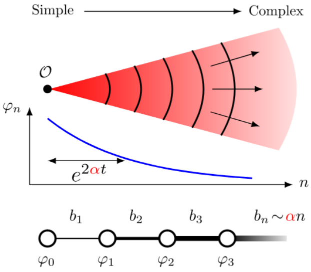
   - Explanation of the oscillations at low temperatures
   - Does the low-temperature approximation apply to the high-temperature regime?
3. Conclusions

Outline

Basics of Krylov complexity

Krylov complexity is a new quantity to describe the operator growth.



[Parker et al., 2019, "A Universal Operator Growth Hypothesis"]

A simple operator will become complex under time evolution. The operator growth can related to the 1D chain [Parker et al., 2019, "A Universal Operator Growth Hypothesis"].

Krylov complexity $=$ Wave-packet position on the chain

operater growth

Heisenberg equation

$$\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)] \equiv i\mathcal{L}\mathcal{O}(t), \qquad \mathcal{L} := [H, \cdot] \qquad \text{(Liouvillian superoperator)} \tag{1}$$

Solution

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} = e^{i\mathcal{L}t}\mathcal{O}(0) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\mathcal{L}^n\mathcal{O}(0) \tag{2}$$

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!}[A^{(n)}, B] \tag{3}$$

$\mathcal{O}(t)$ can be expand by $\{\mathcal{L}^n\mathcal{O}(0)\}$.

We can regard an operator A as a state $|A)$, then

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}\,|\mathcal{O}(0)) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\mathcal{L}^n\,|\mathcal{O}(0)) \tag{4}$$

The space spaned by $\{\mathcal{L}^n\,|\mathcal{O}(0))\}$ is called Krylov space. Define the inner product in Krylov space as the Wightman inner product

$$(A|B) := \left\langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B \right\rangle_\beta \equiv \frac{1}{\mathcal{Z}}\text{tr}\left(e^{-\beta H/2} A^\dagger e^{-\beta H/2} B\right) \tag{5}$$

$$\beta = 1/T, \qquad \mathcal{Z}_\beta = \text{tr}\left(e^{-\beta H}\right) \tag{6}$$

Liouvillian $\mathcal{L}$ is a Hermitian operator. Genera speaking, the basis $\{\mathcal{L}^n\,|\mathcal{O}(0))\}$ is neither normalized nor orthogonal, but we can use $\{\mathcal{L}^n\,|\mathcal{O}(0))\}$ to construct an orthonormal basis called the Krylov basis. Just as quantum mechanics, where an orthonormal basis can be constructed using the Schmidt orthogonalization process.

Gram-Schmidt orthogonalization procedure

The Gram-Schmidt orthogonalization can be express as follow:

Gram-Schmidt orthogonalization procedure

In quantum mechanics [Geroch, 2013, *Quantum field theory: 1971 lecture notes*], if $\xi, \eta_1, \cdots, \eta_n$ are elements of a Hilbert space H, then

$$\xi = \mu_1 \eta_1 + \cdots + \mu_n \eta_n + \tau \qquad (\mu_i \in \mathbb{C}, (\eta_i, \tau) = 0). \tag{7}$$

That is to say, any vector in H can be written as a linear combination of $\eta_1, \cdots, \eta_n$, plus a vector perpendicular to η's.

This method can be use to construct orthonormal basis in Krylov space.

Lanczos algorithm

Let $\{|\mathcal{O}_n)\}$ denote the Krylov basis. Then it satisfies

$$(\mathcal{O}_i|\mathcal{O}_j) = \delta_{ij}. \tag{8}$$

Without loss of genarality, assumption $|\mathcal{O}(0))$ is a normalized vector, then define

$$|\mathcal{O}_0) = |\mathcal{O}(0)) \tag{9}$$

According to Gram-Schmidt orthogonalization procedure, $\mathcal{L}\,|\mathcal{O}_n)$ can be written as a linear combination of $|\mathcal{O}_0)\,, |\mathcal{O}_1)\,, \cdots, |\mathcal{O}_n)$, plus a vector $|A_{n+1}) = b_{n+1}\,|\mathcal{O}_{n+1})$ perpendicular to $|\mathcal{O}_0)\,, |\mathcal{O}_1)\,, \cdots, |\mathcal{O}_n)$,

$$\mathcal{L}\,|\mathcal{O}_n) = \sum_{i=0}^{n} \alpha_i\,|\mathcal{O}_i) + |A_{n+1})\,. \tag{10}$$

Since $(\mathcal{O}_i|\mathcal{O}_j) = \delta_{ij}$, $\alpha_i = (\mathcal{O}_i|\mathcal{L}|\mathcal{O}_n)$.

Lanczos algorithm

We have defined the inner product in Krylov space as

$$(A|B) := \left\langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B \right\rangle_\beta \equiv \frac{1}{\mathcal{Z}} \mathrm{tr} \left(e^{-\beta H/2} A^\dagger e^{-\beta H/2} B \right)$$

Then, for any Hermitian operator A we have

$$(A|\mathcal{L}|A) = 0 \tag{11}$$

So

$$b_1 \left|\mathcal{O}_1\right) = \mathcal{L} \left|\mathcal{O}_0\right) \tag{12}$$

$$b_2 \left|\mathcal{O}_2\right) = \mathcal{L} \left|\mathcal{O}_1\right) - b_1 \left|\mathcal{O}_0\right) \tag{13}$$

$$\cdots \tag{14}$$

$$b_n \left|\mathcal{O}_n\right) = \mathcal{L} \left|\mathcal{O}_{n-1}\right) - b_{n-1} \left|\mathcal{O}_{n-2}\right) \tag{15}$$

These b_n are called the Lanczos coefficients. The above procedure is known as Lanczos algorithm.

Krylov complexity

We can expand $|\mathcal{O}(t))$ in terms of Krylov basis $\{|\mathcal{O}_n)\}$ as

$$|\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \varphi_n(t) \, |\mathcal{O}_n) \,, \qquad \sum_{n=0}^{\infty} |\varphi_n(t)|^2 = 1 \tag{16}$$

A discrete "Schrödinger" equation can be derived from the Heisenberg equation:

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1} \varphi_{n+1}(t), \qquad \varphi_n(0) = \delta_{n0}, \qquad \varphi_{-1}(t) = 0 \tag{17}$$

The operator growth is equivalent to a single-particle hopping problem in one-dimensional Krylov chain. The Krylov complexity of an operator is defined as [Parker et al., 2019, "A Universal Operator Growth Hypothesis"]

$$K(t) = (\mathcal{O}(t)| \, n \, |\mathcal{O}(n)) = \sum_{n=0}^{\infty} n |\varphi_n(t)|^2 \tag{18}$$

The definition used in our work is [Dymarsky and Smolkin, 2021, "Krylov complexity in conformal field theory"]

$$K(t) = 1 + \sum_{n=0}^{\infty} n |\varphi_n(t)|^2 \tag{19}$$

If we know $\varphi_0(t)$ and Lanczos coefficients $\{b_n\}$, we can get $\varphi_n(t)$ from the "Schödinger" equation, and obtain the Krylov complexity from its definition. Here are several important quantitise:

- Wightman 2-point function

$$C(t) := \varphi_0(t) = (\mathcal{O}(t)|\mathcal{O}(0)) = \left\langle \mathcal{O}^\dagger(t - i\beta/2)\mathcal{O}(0)\right\rangle_\beta := \Pi^W(t) \tag{20}$$

- Wightman power spectrum $f^W(\omega)$

$$f^W(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}\Pi^W(t) \tag{21}$$

- moments $\{\mu_{2n}\}$

$$\mu_{2n} := (\mathcal{O}(0)|\,\mathcal{L}^{2n}\,|\mathcal{O}(0)) = \frac{1}{i^{2n}}\frac{d^{2n}}{dt^{2n}}\Pi^W(t)\bigg|_{t=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\omega^{2n} f^W(\omega) \tag{22}$$

Knowing one of these quantities allows us to obtain the Krylov complexity.

The Lanczos coefficients can be get from the moments $\{\mu_{2n}\}$ [Viswanath and Müller, 1994, *The recursion method: application to many body dynamics*]

$$b_n = \sqrt{M_{2n}^{(n)}}, \tag{23}$$

$$M_{2l}^{(j)} = \frac{M_{2l}^{(j-1)}}{b_{j-1}^2} - \frac{M_{2l-2}^{(j-2)}}{b_{j-2}^2}, \qquad \text{with} \qquad l = j, \cdots, n, \tag{24}$$

$$M_{2l}^{(0)} = \mu_{2l}, \qquad b_{-1} \equiv b_0 := 1, \qquad M_{2l}^{(-1)} = 0. \tag{25}$$

Let $\tilde{M}_{lj} \equiv M_{2l}^{(j)}$, we can calculate b_n in the following order:

- $\tilde{M}_{11} \Rightarrow b_1$
- $\tilde{M}_{21} \rightarrow \tilde{M}_{22} \Rightarrow b_2$
- $\cdots$
- $\tilde{M}_{n1} \rightarrow \tilde{M}_{n2} \rightarrow \cdots \rightarrow \tilde{M}_{nn} \Rightarrow b_n$

The discrete "Schrödinger" equation

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t), \qquad \varphi_n(0) = \delta_{n0}, \qquad \varphi_{-1}(t) = 0$$

can be numerically solved using the Runge-Kutta method. Discretize the time:

$$t_i \approx (i-1)\Delta t, \qquad i = 1, 2, \cdots \tag{26}$$

Using Talor expansin, we have

$$\varphi_n(t_{i+1}) \equiv \varphi_n(t_i + \Delta t) \approx \varphi_n(t_i) + \partial_t \varphi_n(t_i)\Delta t = \varphi_n(t_i) + [b_n \varphi_{n-1}(t_i) - b_{n+1}\varphi_{n+1}(t_i)]\Delta t \tag{27}$$

Let $\varphi_n^i = \varphi_n(t_i)$ and $\vec{\varphi}^i = (\varphi_0^i\ \varphi_1^i\ \varphi_2^i \cdots)^{\mathsf{T}}$, the discrete Schrödinger equation can be written in a compact form

$$\vec{\varphi}^{i+1} = \vec{\varphi}^i + A\vec{\varphi}^i \Delta t, \tag{28}$$

where

$$A = \begin{pmatrix} 0 & -b_1 & 0 & 0 & 0 & \cdots \\ b_1 & 0 & -b_2 & 0 & 0 & \cdots \\ 0 & b_2 & 0 & -b_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{29}$$

Fourth-order Runge-Kutta method formula:

$$\vec{\varphi}^{i+1} = \vec{\varphi}^i + \frac{\Delta t}{6}(K_1 + 4K_2 + K_3), \tag{30}$$

where

$$K_1 = A\vec{\varphi}^i, \qquad K_2 = A(\vec{\varphi}^i + K_1 \Delta t/2), \qquad K_3 = A(\vec{\varphi}^i + K_2 \Delta t/2), \qquad K_4 = A(\vec{\varphi}^i + K_3 \Delta t) \tag{31}$$

Outline

The Krylov complexity of five-dimensional free scalar field theory

Lagrangian

$$L_\phi = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2), \qquad \mu = 0, 1, 2, 3, 4, \qquad x^0 = -i\tau \tag{32}$$

The Wightman power spectrum $f^W(\omega)$ can be obtained at the finite temperature $T = \beta^{-1}$

[Camargo et al., 2023, "Krylov complexity in free and interacting scalar field theories with bounded power spectrum"]

$$f^W(\omega) = \tilde{N}(m, \beta)(\omega^2 - m^2)\Theta(|\omega| - m)/\sinh(\beta|\omega|/2) \tag{33}$$

where Θ represents the Heaviside step function and $\tilde{N}(m, \beta)$ is the normalization factor satisfying

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega f^W(\omega) = 1 \tag{34}$$

As mentioned earlier, knowing the Wightman power spectrum allows us to calculate the Wightman 2-point function $\varphi_0(t)$ and moments $\{\mu_{2n}\}$, and subsequently compute the Krylov complexity $K(t)$.

Outline

Low temperatures $\beta m \gg 1$

This situation was first studied in [Camargo et al., 2023, "Krylov complexity in free and interacting scalar field theories with bounded power spectrum"]. In this case, we have

$$f^W(\omega) \approx N(m,\beta) e^{-\beta|\omega|/2}(\omega^2 - m^2)\Theta(|\omega| - m), \qquad \beta m \gg 1 \tag{35}$$

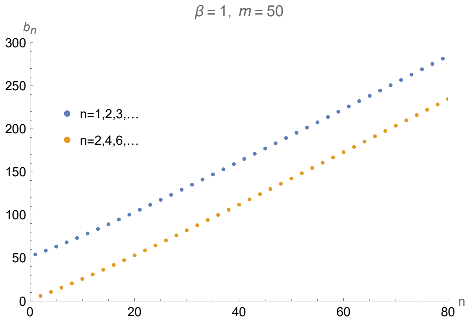
$$\mu_{2n} = \frac{2^{-2} e^{\frac{\beta m}{2}}}{2+\beta m}\left(\frac{2}{\beta}\right)^{2n}\left[4\tilde{\Gamma}\left(3+2n, \frac{\beta m}{2}\right) - \beta^2 m^2 \tilde{\Gamma}\left(1+2n, \frac{\beta m}{2}\right)\right] \tag{36}$$

where $\tilde{\Gamma}(n,z)$ is the incomplete Gamma function

$$\tilde{\Gamma}(n,z) = \int_z^\infty t^{n-1} e^{-t} dt \tag{37}$$

Lanczos coefficients

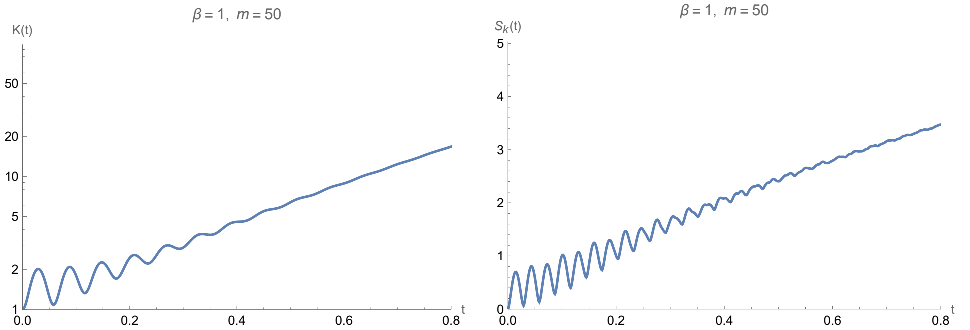
Taking $\beta = 1, m = 50$ as an example:



These coefficients can be separated into two families, one for even n and one for odd n. When n is large enough, these two families of coefficients appear to be linear dependences with n.

Krylov complexity and Krylov entropy

The Krylov entropy is defined by $S_K(t) = -\sum_{n=0}^{\infty} |\varphi_n(t)|^2 \log |\varphi_n(t)|^2$.



The vertical axis in the left plot is logarithmic, so exponential behavior will appear linear in the plot. Initially, Krylov complexity oscillates upward, the oscillations weaken over time, and eventually they appear almost exponential.

Outline

Approximate the Wightman power spectrum

Now we hope to extend the previous discussion to more general situations. Note that

$$f^W(\omega) = \tilde{N}(\beta,m)(\omega^2 - m^2)/\sinh\left(\frac{\beta|\omega|}{2}\right) = 2\tilde{N}(\beta,m)(\omega^2 - m^2)\frac{e^{-\beta|\omega|/2}}{1 - e^{-\beta|\omega|}}, \qquad |\omega| > m. \tag{38}$$

$\exp(-\beta|\omega|)$ is always less than 1. Then we can Use $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ to rewrite $f^W(\omega)$ as

$$f^W(\omega) = 2\sum_{k=0}^{\infty} \tilde{N}(\beta,m)(\omega^2 - m^2)e^{-\beta|\omega|(k+1/2)} \equiv N(\beta,m)\sum_{k=0}^{\infty}(\omega^2 - m^2)e^{-\beta_k|\omega|/2}, \qquad |\omega| > m, \tag{39}$$

where

$$N(\beta,m) = 2\tilde{N}(\beta,m), \qquad \beta_k = \beta(2k+1). \tag{40}$$

Ignoring the summation symbol in (39), (39) is very similar to $f^W(\omega)$ at low temperatures. Obviously, at lower temperatures, we do not need to sum to infinity to approximate $f^W(\omega)$ well. Of course, the higher the temperature, the larger the summation upper limit required to approximate $f^W(\omega)$.

Approximate the Wightman power spectrum

Moments

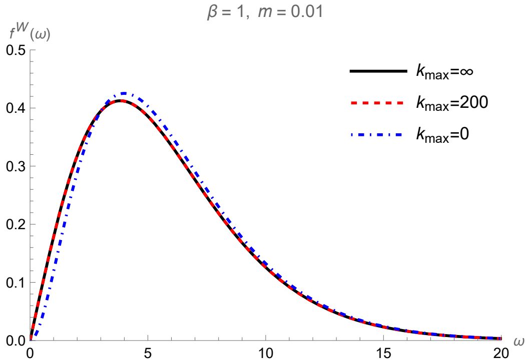
$$\mu_{2n} = N(\beta, m) \sum_{k=0}^{\infty} \frac{2^{2n+1} \beta_k^{-(2n+3)}}{\pi} \left[4\tilde{\Gamma}\left(2n+3, \frac{\beta_k m}{2}\right) - \beta_k^2 m^2 \tilde{\Gamma}\left(2n+1, \frac{\beta_k m}{2}\right) \right]. \quad (41)$$

To obtain numerical results, we need to truncate the infinite series at an appropriate k to ensure computational accuracy and speed. To obtain numerical results, we can only retain a finite number of terms in the summation. To ensure computational speed and accuracy, the number of retained terms should be neither too many nor too few. Assuming the last retained term is the one where $k = k_{\text{max}}$, then

$$f^W(\omega) \approx N(\beta, \omega) \sum_{k=0}^{k_{\text{max}}} (\omega^2 - m^2) e^{-\beta_k \omega/2}, \qquad |\omega| > m. \quad (42)$$

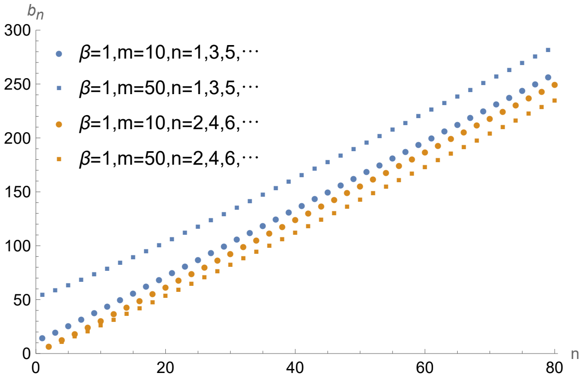
The accurate results can be considered as taking $k = k_{\text{max}}$. In our work, we always set $k_{\text{max}} = 200$.

Compare the Wightman power spectrum for different values of k_{max}



For $\beta m = 0.01$, there is no noticeable difference in $f^W(\omega)$ between $k_{\text{max}} = \infty$ and $k_{\text{max}} = 200$. Therefore, for any $\beta m > 0.01$, there is also no noticeable difference. However, $k_{\text{max}} = 0$ shows a significant deviation at $\beta m = 0.01$, which is expected.

Lanczos coefficients

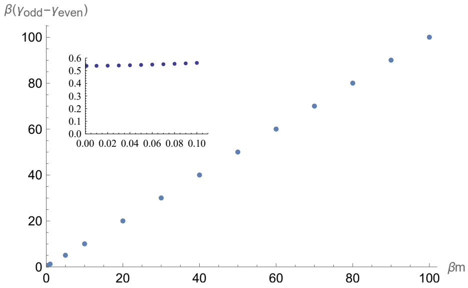


$$b_n \sim \alpha_{\text{odd}} n + \gamma_{\text{odd}}, \qquad \text{odd } n, \tag{43}$$

$$b_n \sim \alpha_{\text{even}} n + \gamma_{\text{even}}, \qquad \text{even } n. \tag{44}$$

Lanczos coefficients

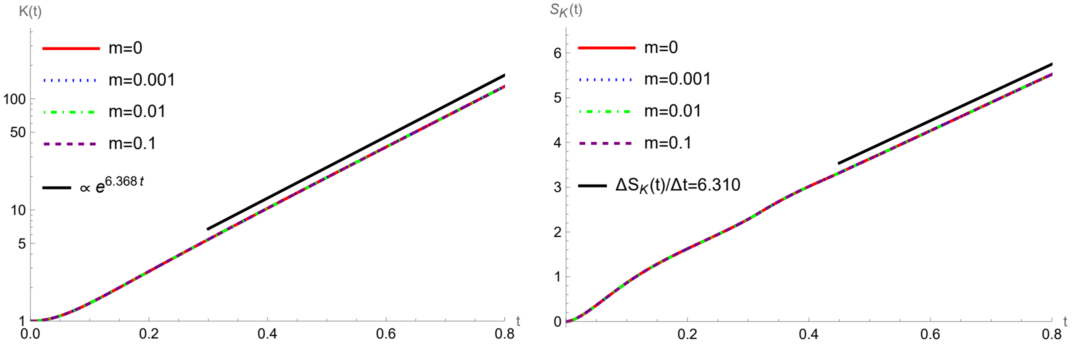
$$b_n \sim \alpha_{\text{odd}} n + \gamma_{\text{odd}}, \qquad \text{odd } n,$$
$$b_n \sim \alpha_{\text{even}} n + \gamma_{\text{even}}, \qquad \text{even } n.$$



Generally, $\beta(\gamma_{\text{odd}} - \gamma_{\text{even}})$ is proportional to βm, but when βm is sufficiently small, i.e., at high temperatures, this relationship no longer holds. This behavior is exactly as expected, because $\beta(\gamma_{\text{odd}} - \gamma_{\text{even}})$ is not equal to zero for CFT.

Outline

$\beta m \ll 1$



As expected, at high temperatures, the behavior of Krylov complexity approaches the results of conformal field theory. A significant difference in the behavior of Krylov complexity at high temperatures compared to low temperatures is the absence of oscillations at small t.

Outline

Explanation of the oscillations at low temperatures

From our previous discussion, we know that the Wightman power spectrum can be expressed as a sum of many modes

$$f^W(\omega) \propto \sum_k f_k^W(\omega). \tag{45}$$

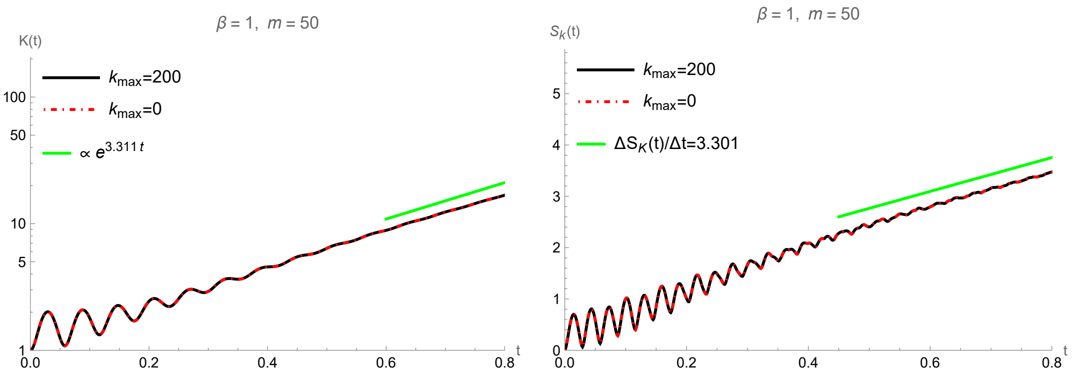
Then, the autocorrelation function can also be expressed as a sum of modes

$$\varphi_0(t) \propto \sum_k \varphi_0^k(t) \tag{46}$$

Calculations show that $\varphi_0^k(t)$ might exhibit significant oscillations. However, at higher temperatures, i.e., when βm is small, many modes are active, and their oscillations cancel each other out, resulting in no significant oscillations in $\varphi_0(t)$ at high temperatures. In contrast, at low temperatures, fewer modes are active, leading to significant oscillations in $\varphi_0(t)$.

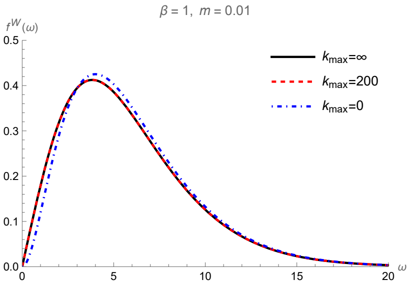
Explanation of the oscillations at low temperatures

For $K(t)$ with oscillatory behavior, the number of active $\varphi_n(t)$ increases with time. These $\varphi_n(t)$ also cancel out each other's oscillations, so the oscillations disappear as time increases.



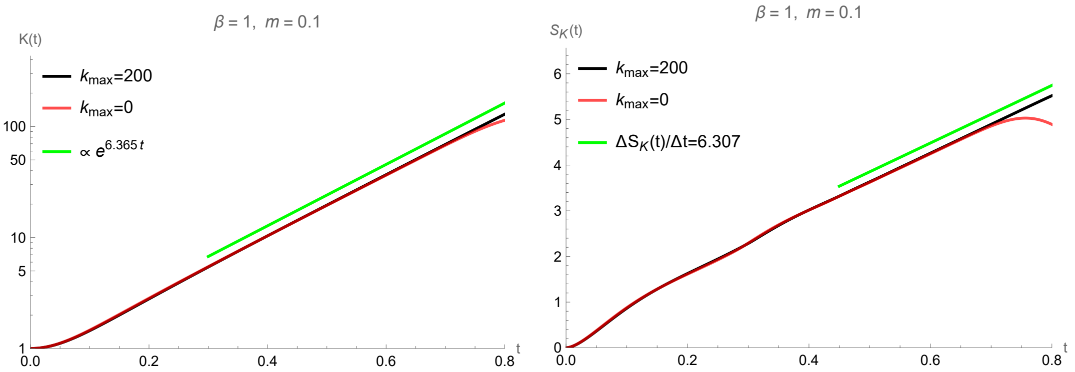
Outline

Does the low-temperature approximation apply to the high-temperature regime?



Even in the high-temperature regime, the low-temperature approximation of $f^W(\omega)$ does not seem to differ significantly from the exact $f^W(\omega)$. Therefore, a natural question is whether using the low-temperature approximation to obtain Krylov complexity in the high-temperature regime would also not deviate too much.

Does the low-temperature approximation apply to the high-temperature regime?



At shorter times, the low-temperature approximation does not seem to deviate significantly. However, at longer times, there is a noticeable deviation.

Does the low-temperature approximation apply to the high-temperature regime?

- In the high-temperature regime, each moment obtained using the low-temperature approximation will, in principle, have a deviation from the actual one.
- Using moments to calculate b_n will also cause b_n to deviate from its true value. The b_n calculated from the moments also deviates from the true value, but the deviation is small
- The low-temperature approximation of $\varphi_n(t)$ will also be close to the true value when n is small, but the deviation will increase as n grows.
- Since, when t is small, the value of Krylov complexity $K(t)$ is mainly contributed by $\varphi_n(t)$ with smaller n, the behavior of Krylov complexity can be obtained using the low-temperature approximation method even in the high-temperature region when t is small. However, as t increases, there will be significant deviations.

This suggests that if we want to obtain the Krylov complexity for large t, we should take k_{max} to be sufficiently large.

Outline

Conclusions

- The Wightman power spectrum of scalar field theory can be expressed in the form of a series expansion. Appropriately retaining a finite number of terms in the series can provide a good approximation of the Wightman power spectrum.
- For $\beta m \ll 1$, $\beta(\gamma_{\text{odd}} - \gamma_{\text{even}})$ is no longer proportional to βm.
- At low temperatures, fewer modes are active, causing the Krylov complexity to exhibit oscillatory behavior. At high temperatures, more modes are active, and the oscillations disappear.
- In the high-temperature region, to obtain the Krylov complexity for sufficiently long times, we need to take k_{max} to be sufficiently large. However, even if k_{max} is small, we can still obtain the Krylov complexity for shorter times in the high-temperature region.

Thank you!