The Double Copy of a Point Charge

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based on 1912.02177

with Kim, Lee (Daejeon), Nicholson (Edinburgh), Peinador Veiga (QMUL)

and also on previous work with O'Connell (Edinburgh), White (QMUL)

Outline

Part I Brief Introduction to the Double Copy

Part II Detailed Example: Double Copy of Coulomb Solution

Part I

Brief Introduction to the Double Copy

Perturbative gravity is hard!

Feynman rules: expand Einstein-Hilbert Lagrangian $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ [DeWitt '66]

$$\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \longrightarrow$$

$$\begin{aligned}
&\mathrm{Sym}\big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda})-\tfrac{1}{4}P_6(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda})+\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda})+\tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda})+P_3(p^\sigma p^\lambda\eta^{\mu\nu}\eta^{\tau\rho})\\
&-\tfrac{1}{2}P_3(p^\tau p'^\mu\eta^{\nu\sigma}\eta^{\rho\lambda})+\tfrac{1}{2}P_3(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+\tfrac{1}{2}P_6(p^\rho p^\lambda\eta^{\mu\sigma}\eta^{\nu\tau})+P_6(p^\sigma p'^\lambda\eta^{\tau\mu}\eta^{\nu\rho})+P_3(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\big],
\end{aligned}$$

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&\mathrm{Sym}\big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{8}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa})-\tfrac{1}{4}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})+\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\
&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{4}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa})+\tfrac{1}{2}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})-\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\
&+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{24}(p^\sigma p^\tau\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa})+\tfrac{1}{4}P_{12}(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa})+\tfrac{1}{2}P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa})\\
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&-P_{12}(p^\sigma p^\tau\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})-P_{12}(p^\rho p'^\lambda\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu})-P_{24}(p_\sigma p'^\rho\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda})-P_{12}(p^\rho p'^\iota\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\
&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu})-P_{12}(p^\sigma p^\rho\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda})-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa})-P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\
&\qquad\quad-P_6(p^\rho p'^\iota\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau})-P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda})-P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu})+2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\big].
\end{aligned}$$

$+$ infinite number of higher-point vertices…

Gravity $\sim$ (Yang-Mills)^2 in Scattering Amplitudes

Asymptotic states

- Yang-Mills theory: gluon $A_\mu = e^{ik\cdot x}\,\epsilon_\mu\,T^a$
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Scattering amplitudes

- "Factorisation" of ϵ^μ, $\tilde{\epsilon}^\nu$ preserved by interactions!
- **Double copy** $\boxed{\mathcal{A}_{\text{grav}}(\varepsilon_i^{\mu\nu}) \sim (\text{prop})^{-1}\,\mathcal{A}_{\text{YM}}(\epsilon_i^\mu) \times \mathcal{A}_{\text{YM}}(\tilde{\epsilon}_i^\mu)\,\big|_{\text{colour stripped}}}$
- Famous application: supergravity UV behaviour. [Bern,Carrasco,Johansson,Roiban,...]

String theory origin

QFT from string theory as $\alpha' = \ell_s^2 \to 0$: alternative to Feynman expansion.

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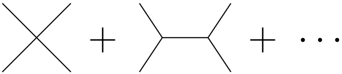
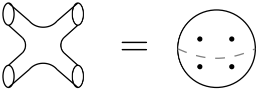


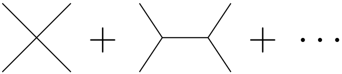
particle scattering
(many Feynman diagrams)

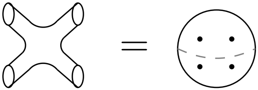


string scattering
(one "world-sheet", 2D CFT)

String theory origin

QFT from string theory as $\alpha' = \ell_s^2 \to 0$: alternative to Feynman expansion.





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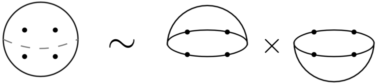
Gravity (closed strings) vs. gauge theory (open strings):

Asymptotic states (vertex operators): $V_{\text{closed}}(\varepsilon^{\mu\nu} = \epsilon^{\mu}\tilde{\epsilon}^{\nu}) \sim V_{\text{open}}(\epsilon^{\mu})\,\bar{V}_{\text{open}}(\tilde{\epsilon}^{\nu})$

Scattering amplitudes:

$\boxed{\text{KLT relations}}$

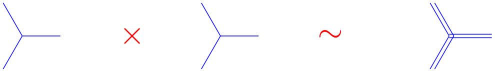
[Kawai, Lewellen, Tye 86]



Field theory limit: **Gravity $\sim$ (Yang-Mills)^2** (KLT, BCJ, CHY, ...)

Why simpler?

Basic example: 3-pt interactions.



Gauge theory field A^a_μ

3-pt vertex: $\boxed{f^{abc}\; V^{\mu\nu\lambda}\; A^a_\mu(p_1)\, A^b_\nu(p_2)\, A^c_\lambda(p_3)}$

$$V^{\mu\nu\lambda} = (p_1 - p_2)^\lambda \eta^{\mu\nu} + (p_2 - p_3)^\mu \eta^{\nu\lambda} + (p_3 - p_1)^\nu \eta^{\lambda\mu}$$

Gravity field $H_{\mu\mu'} \sim$ graviton + dilaton + B-field 'fat graviton'

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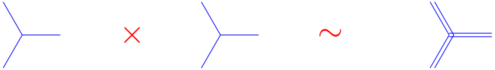

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Great simplification: **index factorisation**, c.f. $\sim$100 terms in GR 3-pt vertex!

Powerful implementation: colour-kinematics duality. [Bern, Carrasco, Johansson '08] [...]

New directions in (classical) perturbative gravity

Generically, double copy applies in perturbation theory.

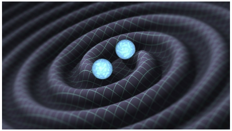
- Double-copy-like field theory for gravity.

  [Goldberger et al] [Luna et al] [Cheung et al] [Plefka et al] [Borsten et al] [...]

- Gauge-invariant approach: classical physics from scattering amplitudes.

  [Neill et al] [Bjerrum-Bohr et al] [Kosower et al] [Guevara et al] [Huang et al] [Arkani-Hamed et al] [...]

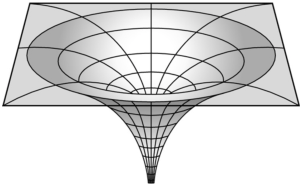
- Beyond Minkowski: amplitudes on plane wave backgrounds. [Adamo et al] [...]



- Highlight: new G^3 (3PM) corrections to 2-body potential. [Bern et al]

Beyond perturbation theory

Question: is a black hole a double copy of something?



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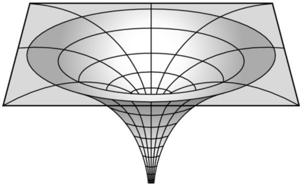
- What is “graviton” in exact solution?
- Non-perturbative double copy?



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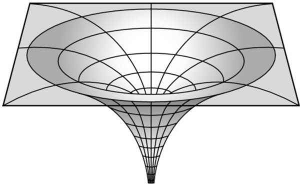

Still...

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  Examples: Schwarzchild [Duff 73; Neill, Rothstein 13], shockwave [Saotome, Akhoury '12].

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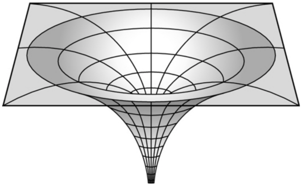

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Stationary Kerr-Schild spacetimes

[RM, O'Connell, White 14]

"Exact perturbation" $\boxed{g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu}$

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Einstein equations linearise:

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Stationary vacuum case (take $k_0 = 1$): $R^0{}_0 = \frac{1}{2}\nabla^2\phi = 0$

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$$0 = D_\mu F^{a\mu\nu} = c^a \begin{cases} -\nabla^2\phi & \nu = 0 \\ -\partial_\ell \left[\partial^i \left(\phi k^\ell\right) - \partial^\ell \left(\phi k^i\right)\right] & \nu = i \end{cases} \qquad \checkmark$$

Simplest example: point charge

Check spherically symmetric solutions sourced by point charge.

Einstein theory: Schwarzschild solution

- $$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu, \qquad \phi(r) = \frac{2M}{r}, \qquad k = dt + dr}$$

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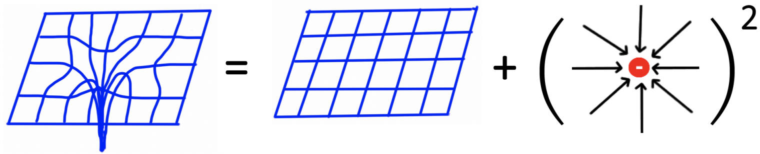
$$\boxed{\text{Schwarzschild} \sim (\text{Coulomb})^2}$$



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Rotation

- Kerr black hole: (M, a), $a = J/M$.
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  Other black holes families? Black rings, etc...

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Cosmological constant $\leftrightarrow$ constant charge density. [Luna, RM, O'Connell, White '15]
[Bahjat-Abbas, Luna, White '17; Carrillo-Gonzalez, Penco, Trodden 17]

NUT charge $\leftrightarrow$ magnetic monopole: multi-Kerr-Schild [Luna, RM, O'Connell, White '15]

$$g_{\mu\nu}^{\text{(Taub-NUT)}} = \eta_{\mu\nu} + \phi\, k_\mu k_\nu + \psi\, \ell_\mu \ell_\nu\,, \quad \phi \propto M\,, \quad \psi \propto N \quad \Rightarrow \quad A_\mu^{\text{(dyon)}} = \phi\, k_\mu + \psi\, \ell_\mu$$

Radiation from accelerated particle: correct Bremsstrahlung.
[Luna, RM, Nicholson, O'Connell, White '16]

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Weyl (spinorial) double copy for vacuum type D metrics and pp-waves.
[Luna, RM, Nicholson, O'Connell '18]

Many others – today's talk by Tucker Manton

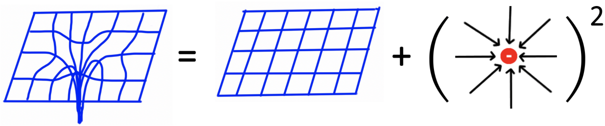
Part II

Double Copy of Coulomb Solution – Revisited

[Kim, Lee, RM, Nicholson, Peinador Veiga '19]

Part II Double Copy of Coulomb Solution

Simplest example: (Coulomb)^2 $\sim$ Schwarzschild.



But (YM)^2 $\sim$ Einstein $h_{\mu\nu}$ $+$ dilaton Φ $+$ B-field $B_{\mu\nu}$.

Other fields?

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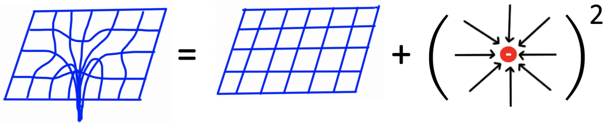

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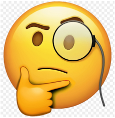
Double-copy structure of Einstein equations?

double copy

Gravity $=$ YM $\times$ YM



double field theory

doubled geometry $(x^\mu, \tilde{x}_\mu)$

Double copy of Coulomb: perturbative approach

Double copy for Coulomb?

Linearised "fat graviton": [Luna, RM, Nicholson, Ochirov, O'Connell, White, Westerberg 16]

$$H_{\mu\nu} \;=\; \text{graviton } h_{\mu\nu} + \text{dilaton } \phi + \text{B-field } B_{\mu\nu}$$

Double copy for Coulomb?

Linearised "fat graviton": [Luna, RM, Nicholson, Ochirov, O'Connell, White, Westerberg 16]

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Kerr-Schild double copy.

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Both consistent with other double copy approaches,

e.g., 'convolution' [Anastasiou et al 14] [Luna, Nagy, White 20] .

Double copy for Coulomb: not unique!

Clue from momentum states: take polarisations ϵ_μ , $\tilde{\epsilon}_\mu$. $\epsilon \cdot k = \tilde{\epsilon} \cdot k = 0$

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$$\varepsilon_{\mu\nu} = C^{(h)} \left(\epsilon_{(\mu}\tilde{\epsilon}_{\nu)} - \frac{\Delta_{\mu\nu}}{D-2}\, \epsilon \cdot \tilde{\epsilon} \right) + C^{(B)}\, \epsilon_{[\mu}\tilde{\epsilon}_{\nu]} + C^{(\phi)}\, \frac{\Delta_{\mu\nu}}{D-2}\, \epsilon \cdot \tilde{\epsilon} \, .$$

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Linearised (Coulomb)^2: no B-field, $M \sim C^{(h)}$ graviton, $Y \sim C^{(\phi)}$ dilaton.

$$\boxed{H_{\mu\nu} = M \left(\frac{u_\mu u_\nu}{r} - P_{\mu\nu}\left[\frac{u^2}{r}\right] \right) + Y\, P_{\mu\nu}\left[\frac{u^2}{r}\right]} \qquad P_{\mu\nu}\left[\tfrac{u^2}{r}\right] = \tfrac{-1}{2\,r}(\eta_{\mu\nu} - q_\mu l_\nu - q_\nu l_\mu)$$

$Y = 0$: linearised Schwarzschild. Any Y: linearised JNW [Janis, Newman, Winicour '68].

Can correct order-by-order in double-copy-like way.

General point charge: JNW solution

Unique static, spherically symmetric, asymp. flat solution
of Einstein + minimally coupled scalar.

Two parameters (M, Y) or (ρ_0, γ). Found by Janis, Newman, Winicour '68:

$$ds^2 = -\left(1 - \frac{\rho_0}{\rho}\right)^{\gamma} dt^2 + \left(1 - \frac{\rho_0}{\rho}\right)^{-\gamma} d\rho^2 + \left(1 - \frac{\rho_0}{\rho}\right)^{1-\gamma} \rho^2 d\Omega^2$$

$$\phi = \frac{Y}{\rho_0} \log\left(1 - \frac{\rho_0}{\rho}\right) \qquad \rho_0 = 2\sqrt{M^2 + Y^2} \qquad \gamma = \frac{M}{\sqrt{M^2 + Y^2}}$$

- $Y = 0$: vacuum gravity $\rightarrow$ Schwarzschild (usual coords)
- $Y \neq 0$: naked singularity at origin $\rho = \rho_0$, cf. no-hair theorems

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Solution above is in Einstein frame. In string frame, $\quad g^S_{\mu\nu} = e^{2\phi}\, g^E_{\mu\nu}\,.$

Double copy of Coulomb:
exact map with double field theory

Double Field Theory

[Siegel '93] [Hull, Zwiebach '09 + Hohm '10]

For us: fancy formulation of 'product gravity' (massless level closed string).

Motivation: low-energy effective theory of closed string exhibiting T-duality.

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$$\Lambda^M{}_N \in O(D,D): \ (\Lambda)^T(\mathcal{J})(\Lambda) = (\mathcal{J}). \quad \mathcal{J}_{MN} = \begin{pmatrix} 0 & \delta^\mu{}_\nu \\ \delta_\mu{}^\nu & 0 \end{pmatrix} \text{ is } O(D,D) \text{ metric.}$$

- T-duality manifest: $O(D,D)$ covariance.
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Fields packaged as tensor and scalar wrt to $O(D,D)$.

- Generalised metric: $\mathcal{H}_{MN} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}B_{\rho\nu} \\ B_{\mu\rho}g^{\rho\nu} & g_{\mu\nu} - B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} \end{pmatrix} \in O(D,D)$.
- DFT dilaton d : $e^{-2d} = \sqrt{-g}\, e^{-2\phi}$.

Kerr-Schild-inspired ansatz

Recall Kerr-Schild ansatz: $\boxed{g_{\mu\nu} = \eta_{\mu\nu} + \varphi\, k_\mu k_\nu}$ k_μ null and geodesic.

DFT version: take $\mathcal{H}_{0MN} = \mathcal{H}_{MN}\,(g_{\mu\nu} = \eta_{\mu\nu}, B_{\mu\nu} = 0)\,,$ [Lee 18] [Cho, Lee 19] [Kim, Lee, RM, Nicholson, Veiga 19]

$$\boxed{\mathcal{H}_{MN} = \mathcal{H}_{0MN} + \varphi(K_M \bar{K}_N + K_N \bar{K}_M) - \frac{1}{2}\varphi^2 \bar{K}^2 K_M K_N}$$

$$K_M = \frac{1}{\sqrt{2}} \begin{pmatrix} k^\mu \\ \eta_{\mu\nu} k^\nu \end{pmatrix} \qquad \bar{K}_M = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{k}^\mu \\ -\eta_{\mu\nu} \bar{k}^\nu \end{pmatrix}$$

where k_μ and $\bar{k}_\mu$ are null and satisfy diff. contraints (or $\bar{k}_\mu$ not null).

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$$B_{\mu\nu} = \frac{\varphi}{1 + \frac{\varphi}{2}(k \cdot \bar{k})}\, k_{[\mu} \bar{k}_{\nu]}\,.$$

First examples of exact double copy with dilaton and B-field. [Lee 18]

JNW solution: fits ansatz, B-field is pure gauge.

Double Field Theory versus Double Copy

Generalised metric $\mathcal{H}^M{}_N$ induces chirality:

$$P_M{}^N = \frac{1}{2}\left(\delta_M{}^N + \mathcal{H}_M{}^N\right), \qquad \bar{P}_M{}^N = \frac{1}{2}\left(\delta_M{}^N - \mathcal{H}_M{}^N\right).$$

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$=$ left and right moving sectors! (pullback to worldsheet)

Kerr-Schild-like ansatz $\boxed{\mathcal{H}_{MN} = \mathcal{H}_{0MN} + \varphi(K_M\bar{K}_N + K_N\bar{K}_M) + \ldots}$

Satisfy definite chiralities: $(P_0)_M{}^N K_N = K_M\,, \qquad (\bar{P}_0)_M{}^N \bar{K}_N = \bar{K}_M\,.$

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KLT picture of Kerr-Schild double copy!

DFT equations of motion

As in Kerr-Schild double copy, gravity e.o.m. $\rightsquigarrow$ gauge theory e.o.m.

$$\text{'left'} \qquad 4e^{-2d}\,\mathcal{R}_{\mu 0} = \partial^{\nu} F_{\nu\mu} = 0\,, \quad F = dA\,, \quad A_{\mu} = e^{-2d}\varphi\, k_{\mu} + C_{\mu}$$

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General relation: A_μ and $\bar{A}_\mu$ independent

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Our example: $A_\mu = \bar{A}_\mu$ up to gauge $=$ Coulomb

JNW $\sim$ ('left-moving' Coulomb) $\times$ ('right-moving' Coulomb)

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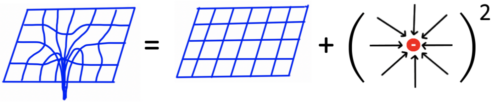
- Double copy of classical solutions possible.



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  Exact case: fully non-linear but not generic.
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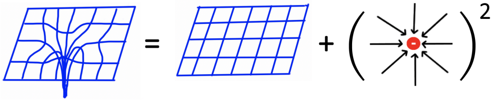

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Much more to explore

- Larger classes of solutions, duality transf., asymptotic symmetries, . . .
  [e.g., Godazgar et al, Huang et al, Alawadhi et al, Banerjee et al, Moynihan et al]
- **Aim:** general formulation of fully non-linear double copy.