|| Sri Sainath ||

New asymptotic conservation laws for electromagnetism

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Contents

- Introduction :
  Asymtotic conservation laws and Soft theorems.
- Q_m-conservation laws in presence of long range forces.
- Classical soft theorems for $m = 2, 3$.
- Conclusion

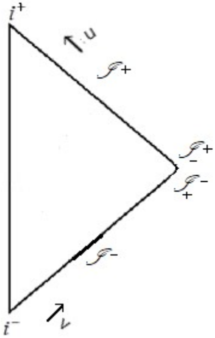
Asymptotic Conservation laws

Asymptotic conservation laws :

$$Q^+[\epsilon^+]\,|_{\mathcal{I}^+_-} \quad = \quad Q^-[\epsilon^-]\,|_{\mathcal{I}^-_+}.$$

$$\epsilon^+(\hat{x}) = \epsilon^-(-\hat{x}).$$

$\mathcal{I}^+_-$ is the $u \to -\infty$ sphere of $\mathcal{I}^+$.
$\mathcal{I}^-_+$ is the $v \to \infty$ sphere of $\mathcal{I}^-$.



Asymptotic Conservation law

- Classical conservation law :

$$Q_0^+[\epsilon^+] \mid_{\mathcal{I}^+_-} \;=\; Q_0^-[\epsilon^-] \mid_{\mathcal{I}^-_+}.$$

Q_0 is defined in terms of radial component of electric field.

- At quantum level, S-matrix has to satisfy the Ward identity :

$$< \mathrm{out}| \; Q_0^+ \; S \; - \; S \; Q_0^- \; |\mathrm{in} > \; = \; 0.$$

Asymptotic Conservation law

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- At quantum level,

$$< \text{out}| \; Q_0^+ \; S \; - \; S \; Q_0^- \; |\text{in} > \; = \; 0.$$

This Ward identity is equivalent to leading soft photon theorem. [He, Mitra, Porfyriadis and Strominger,1407.3789; 1703.05448]

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- Soft expansion of loop amplitudes :

$$\lim_{\omega \to 0} \mathrm{Amp}_{n+1}(p_i, k) = \left[\frac{S_0}{\omega} + S_1 \log \omega + ... \right] \mathrm{Amp}_n(p_i).$$

$k = \omega(1, \vec{q})$ is the soft momentum.
n is number of hard particles in the scattering process.
$(n+1)^{th}$ particle is the soft photon.

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n is number of hard particles in the scattering process.
$(n+1)^{th}$ particle is the soft photon.

- Soft factors are universal.

$$S_0 = \sum_{i=1}^{n} e_i \frac{\varepsilon . p_i}{p_i . q}.$$

Subleading term for loop amplitudes

Soft expansion of loop amplitudes is given by :

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The Sahoo-Sen soft theorem (arxiv:1808.03288) is equivalent to a new asymptotic conservation law.
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We will discuss the proof of above asymptotic conservation law in this talk.

Long range forces

- $\log\omega$ term in the soft expansion is related to the long range forces ($V \sim \frac{1}{r}$).
- Asymptotically particles are not free. They radiate and and this produces the $\log\omega$ term.
- We will incorporate the effect of long range forces on asymptotic dynamics.
- This leads to new asymptotic conservation laws.

Plan

(Based on arxiv:2007.03627)

- Part 1 : New asymptotic conservation laws

  We discuss Q_m-conservation laws that are directly related to long range electromagnetic force.

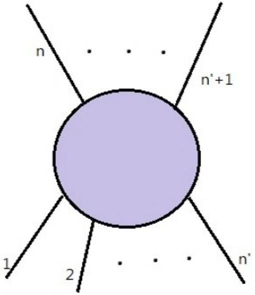
- Part 2 : Classical soft theorems for $m = 2, 3$

  We discuss the soft theorems related to above Q_m charges.

Part 1

Asymptotic conservation laws for classical scattering.

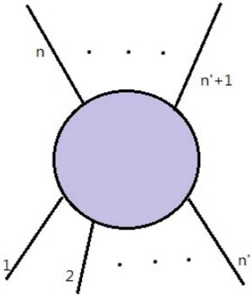
Scattering process



Let us consider scattering of charged particles where n' number of particles come in and interact in a finite region say a sphere of radius L around $r = 0$. At the end, they produce $(n - n')$ number of outgoing particles.

This interaction could be of any sort or of any strength.

Scattering process



For $r > L$, the particles are apart enough so that only possible interactions between them would be the long range forces.

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- Hence an incoming particle has the trajectory :

$$x_i^\mu = [V_i^\mu \tau + d_i]\Theta(-T - \tau).$$

Similarly, an outgoing particle has the trajectory :

$$x_j^\mu = [V_j^\mu \tau + d_j]\Theta(\tau - T).$$

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$$x_j^\mu = [V_j^\mu \tau + d_j]\Theta(\tau - T).$$

$$j_\sigma(x') = \int d\tau \Big[\sum_{i=n'+1}^{n} e_i V_{i\sigma}\ \delta^4(x' - x_i)\ \Theta(\tau - T) + \text{ in } \Big].$$

$F_{\mu\nu}$ at $\mathcal{O}(e)$

- We get at $\mathcal{I}^+$:

$$F_{\mu\nu}|_{\mathcal{I}^+_-} \sim \sum_{\substack{m,n \\ m<n}} \frac{u^m}{r^n} + \cdots \ .$$

'$\cdots$' represent terms that are atleast exponentially suppressed.

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- In particular, we have :

$$F^2_{rA}|_{\mathcal{I}^+_-} = u \ F^{(2,-1)}_{rA} + u^0 \ F^{(2,0)}_{rA} + \cdots \ .$$

A denotes S^2 co-ordinates.
The coefficients are a function of $\hat{x}$.

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- Long range forces lead to new logarithmic terms in field strength.

Asymptotic dynamics including subleading term

$$m_i \frac{\partial^2 x_i^\mu}{\partial \tau^2} = e_i \; F^{\mu\nu}(\tau) \; V_{i\nu}.$$

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Substitute $\mathcal{O}(e)$ solution of $F^{\mu\nu}$ in above equation to get

$$m_i \frac{\partial^2 x_i^\mu}{\partial \tau^2} = \mathcal{O}(\frac{e^2}{\tau^2}).$$

Hence we get :

$$x_i^\mu = V_i^\mu \; \tau + c_i^\mu \log \tau + d_i + \mathcal{O}(\frac{1}{\tau}).$$

Asymptotic dynamics including subleading term

We have :

$$x_i^\mu = V_i^\mu\ \tau + c_i^\mu \log\tau + d_i + \mathcal{O}(\frac{1}{\tau}).$$

$$c_i^\mu = -\frac{1}{4\pi}\sum_{\substack{j=n'+1,\\ j\neq i}}^{n} e_i e_j \frac{p_i.p_j\ m_j^2 p_i^\mu + m_i^2 m_j^2 p_j^\mu}{[(p_i.p_j)^2 - m_i^2 m_j^2]^{3/2}}.$$

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• Above expression represents effect of other particles j on the i^{th} particle.

• c_i's for an outgoing particle includes contribution only from outgoing particles.

$\mathcal{O}(e^3)$ fall offs in the field strength

- Using $\Box A_\mu = -j_\mu$,

$$A_\sigma(x) = \frac{1}{2\pi}\int d\tau\ \delta(\ [x - x'(\tau)]^2)\ j_\sigma(\tau).$$

  Now j_σ includes $\mathcal{O}(e^3)$ terms.

- The fields admit new fall offs :

$$F^2_{rA}|_{\mathcal{I}^+_-} \quad = \quad u\ F^{(2,-1)}_{rA} + \log u\ F^{(2,\log)}_{rA} + u^0\ F^{(2,0)}_{rA} + \cdots\ .$$

$\mathcal{O}(e^3)$ fall offs in the field strength

- At future :

$$F_{rA}|_{u\to-\infty} = \frac{1}{r^2}\left[u\ F_{rA}^{(2,-1)} + \log u\ F_{rA}^{(2,\log)} + ...\right] + \mathcal{O}(\frac{1}{r^3})\ . \tag{1}$$

- We repeat the similar calculation at past null infinity (4).

$$F_{rA}|_{v\to\infty} = \frac{\log r}{r^2}\ v^0\ F_{rA}^{(\log,0)} + \mathcal{O}(\frac{1}{r^2})\ .$$

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$$F_{rA}|_{v\to\infty} = \frac{\log r}{r^2}\ v^0\ F_{rA}^{(\log,0)} + \mathcal{O}(\frac{1}{r^2})\ .$$

- We show that :

$$F_{rA}^{(2,\log)}(\hat{x})\ |_{\mathcal{I}^+_-} = F_{rA}^{(\log,0)}(-\hat{x})\ |_{\mathcal{I}^-_+}\ .$$

This law was suggested by Campiglia and Laddha. We proved it. This is the $m = 1$ conservation law.

Asymptotic dynamics including subsubleading term

- Asymptotically particles accelerate under long range force and radiate.
- This radiation backreacts on the particles and corrects the trajectory of matter particles.

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- This radiation backreacts on the particles and corrects the trajectory of matter particles.

$$m_i \frac{\partial^2 x_i^\mu}{\partial \tau^2} = e_i \; F^{\mu\nu}(\tau) \; V_{i\nu}^{\text{cor}}(\tau).$$

Substituting $F_{\mu\nu} \sim \mathcal{O}(e^3)$:

$$m_i \frac{\partial^2 x_i^\mu}{\partial \tau^2} \sim \frac{e^2}{\tau^2} + e^4 \frac{\log \tau}{\tau^3} + \cdots .$$

Thus, asymptotic trajectories of the particles are corrected to :

$$x_i^\mu = V_i^\mu \; \tau + c_i^\mu \log \tau + d_i + f_{i\sigma} \frac{\log \tau}{\tau}. \tag{2}$$

(5)

$\mathcal{O}(e^5)$ fall offs in the field strength

- Including the $\mathcal{O}(e^5)$ terms around future null infinity :

$$F^3_{rA}|_{u\to-\infty} = u^2\ F^{(3,-2)}_{rA}\ + u\log u\ F^{(3,-1)}_{rA} + (\log u)^2\ F^{(3,\log^2)}_{rA}\ + \ldots\ .$$

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- Expansion around the past null infinity is given by :

$$F_{rA}|_{v\to\infty} = \frac{\log r}{r^2}\ v^0\ F^{(\log,0)}_{rA}\ + \frac{(\log r)^2}{r^3}\ v^0\ F^{(\log^2,0)}_{rA}\ + \mathcal{O}(\frac{1}{r^2})\ .$$

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- We proved following $\mathcal{O}(e^5)$ conservation law :

$$F_{rA}^{(3,\log^2)}(\hat{x})|_{\mathcal{I}^+_-} = F_{rA}^{(\log^2,0)}(-\hat{x})|_{\mathcal{I}^-_+} .$$

This is the $m=2$ conservation law.

Summary so far

We have established conservation laws for following modes of F_{rA} :

$$\mathcal{O}(e^3): \quad \frac{\log u}{r^2} \text{ and } \frac{\log r}{r^2}.$$

$$\mathcal{O}(e^5): \quad \frac{(\log u)^2}{r^3} \text{ and } \frac{(\log r)^2}{r^3}.$$

m^{th} order asymptotic conservation laws

- We expect these conservation laws to exist for all m-modes of F_{rA} :

$$\mathcal{O}(e^{2m+1}) : \quad \frac{(\log u)^m}{r^{m+1}} \text{ and } \frac{(\log r)^m}{r^{m+1}}.$$

Proved for $m = 1, 2, 3$.

Concluding Part 1

- Classical theory admits a set of conservation laws ($m = 1, 2, 3$) :

$$Q^{+}_{m}[Y^{+}_{m}] \mid_{\mathcal{I}^{+}_{-}} \quad = \quad Q^{-}_{m}[Y^{-}_{m}] \mid_{\mathcal{I}^{-}_{+}}.$$

- Strominger and his colloborators have established a correspondence between asymptotic conservation laws and soft theorems.

Concluding Part 1

- Classical theory admits a set of conservation laws ($m = 1, 2, 3$) :

$$Q_m^+[Y_m^+] \mid_{\mathcal{I}_-^+} \quad = \quad Q_m^-[Y_m^-] \mid_{\mathcal{I}_+^-} .$$

- Strominger and his colloborators have established a correspondence between asymptotic conservation laws and soft theorems.
- As we discussed earlier, Q_1-conservation law is equivalent to the Sahoo-Sen $\log\omega$ soft theorem.

$$\left[\ Q_1\ ,\ S\ \right] = 0$$

(Campiglia and Laddha, arxiv:1903.09133; SB, arxiv:1912.10229.)

For $m > 1$

- It can be argued that Q_m-conservation laws are related to following terms in soft expansion of loop amplitudes :

$$\lim_{\omega \to 0} \mathrm{Amp}_{n+1}(p_i, k) = \Big[\frac{S_0}{\omega} + \sum_m S_m \, \omega^{m-1} \log \omega^m + \ldots\Big].$$

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- These m^{th} level soft photon theorems for quantum amplitudes have not been explored for $m > 1$.
- Let us discuss the classical version of soft theorems for $m = 2, 3$.

Part 2

Classical Soft theorems.

Classical Soft theorems

- Laddha and Sen (arXiv:1801.07719) have discussed the classical limit of soft theorems.

$$\lim_{\omega \to 0} \mathrm{Amp}_{n+1}(p_i, k) = \left[\frac{S_0}{\omega} + S_1 \log \omega + \ldots\right] \mathrm{Amp}_n(p_i).$$

- Soft theorems control the classical radiative field in low energy limit. Classical limit of soft theorems :

$$\lim_{\omega \to 0} \epsilon^\mu \tilde{A}_\mu(\omega) \quad = \quad \left[\frac{S_0}{\omega} + S_1^{\mathrm{class}} \log \omega + \ldots\right].$$

- S_1^{class} was derived classically by Saha, Sahoo and Sen (arXiv:1912.06413).

Classical Soft theorems

Next goal : derive the subsubleading universal terms.

$$\begin{aligned} & \lim_{\omega \to 0} \epsilon^\mu \tilde{A}_\mu(\omega) \\ = \quad & \left[\frac{S_0}{\omega} + S_1^{\text{class}} \log \omega + S_2^{\text{class}} \omega (\log \omega)^2 + S_3^{\text{class}} \omega^2 (\log \omega)^3 + \ldots \right]. \end{aligned}$$

Late time radiative field

Steps :

- We consider a generic classical scattering process of charged bodies we had discussed earlier.
- We calculate the radiative component of the electromagnetic field emitted at late times.
- This will get related to soft field via a Fourier transform.

Late time radiative field

- Let us derive the universal terms that appear upto $\mathcal{O}(e^5)$.

  Using $\Box A_\mu = -j_\mu$,

$$A_\sigma(x) = \frac{1}{2\pi} \int d\tau \; \delta(\, [x - x'(\tau)]^2) \; j_\sigma(\tau).$$

Late time radiative field

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$$A_\sigma(x) = \frac{1}{2\pi} \int d\tau \; \delta(\; [x - x'(\tau)]^2) \; j_\sigma(\tau).$$

- We need to retain the subsubleading terms in asymptotic trajectories of the particles (5):

$$x_i'^\mu = V_i^\mu \; \tau + c_i^\mu \log \tau + d_i + f_{i\sigma} \frac{\log \tau}{\tau}. \tag{3}$$

so that j_σ includes $\mathcal{O}(e^5)$ terms.

Universal terms in late time radiative field upto $\mathcal{O}(e^5)$

$$A_\mu|_{\mathcal{I}^+} = \frac{1}{4\pi r}\Big[\; a_\mu^{\pm} u^0 \;+\; \frac{[b_\mu^{(0)}]^{\pm}}{u} \;+\; [b_\mu^{(1)}]^{\pm}\frac{\log u}{u^2}\;\Big] + \ldots, \quad u \to \pm\infty.$$

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- u^0 term is the 'memory term'. Rederiving the leading soft factor :

$$a_\mu^+ = \sum_{\text{out}} \frac{e_i p_{i\mu}}{q.p_i}$$

$$a_\mu^- = -\sum_{\text{in}} \frac{e_i p_{i\mu}}{q.p_i}$$

$$S_0 = \int_{\mathcal{I}^+} du\ \partial_u\ \epsilon^\mu A_\mu = \sum_i e_i \frac{\epsilon.p_i}{q.p_i}$$

- The next term is the tail term discussed by Saha, Sahoo, Sen. It is related to the $m = 1$ classical soft theorem via Fourier transform.

Universal terms in late time radiative field upto $\mathcal{O}(e^5)$

$$A_\mu|_{\mathcal{I}^+} = \frac{1}{4\pi r}\Big[\ a_\mu^{\pm} u^0 \ + \ \frac{[b_\mu^{(0)}]^{\pm}}{u} \ + \ [b_\mu^{(1)}]^{\pm}\frac{\log u}{u^2}\ \Big] + \ldots, \quad u \to \pm\infty.$$

We derive the subsubleading term.

$$[b_\sigma^{(1)}]^{+} = \sum_{\text{out}} e_i \big[q.c_i \ c_{i\sigma} - p_{i\sigma}\frac{(q.c_i)^2}{q.p_i} + \frac{p_{i\sigma}}{m_i} f_i.q - \frac{q.p_i}{m_i} f_{i\sigma}\big]$$

$$[b_\sigma^{(1)}]^{-} = \sum_{\text{in}} e_i \big[q.c_i \ c_{i\sigma} - p_{i\sigma}\frac{(q.c_i)^2}{q.p_i} + \frac{p_{i\sigma}}{m_i} f_i.q - \frac{q.p_i}{m_i} f_{i\sigma}\big]$$

Universal terms in late time radiative field upto $\mathcal{O}(e^5)$

$$A_\mu|_{\mathcal{I}^+} = \frac{1}{4\pi r}\Big[\ a_\mu^\pm u^0 \ + \ \frac{[b_\mu^{(0)}]^\pm}{u} \ + \ [b_\mu^{(1)}]^\pm \frac{\log u}{u^2}\ \Big] + \ldots, \quad u \to \pm\infty.$$

The subsubleading term is interesting because

- It is universal : completely independent of the details of the scattering process.
- It is related to the Q_2-conservation law, hence tied to a soft theorem.

Classical soft theorem for $m = 2$

$$\tilde{A}_\mu(\omega) = \frac{e^{i\omega r}}{4\pi i r}\left[\ \frac{1}{\omega}\ S_0\ + S_1^{\text{class}}\ \log\omega + S_2^{\text{class}}\ \omega(\log\omega)^2 + \ldots\right] \text{ as } \omega \to 0.$$

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The coefficient of $\omega(\log\omega)^2$:

$$S_2^{\text{class}} = -\frac{1}{2}\sum_{i=1}^{n} e_i\Big[q.c_i\ \left[p_i.\epsilon\frac{q.c_i}{p_i.q} - c_i.\epsilon\right] - \epsilon_\mu q_\nu\ [p_i^\mu f_i^\nu - p_i^\nu f_i^\mu]\ \Big].$$

recalling (5)

$$x_i^\mu = V_i^\mu\ \tau + c_i^\mu \log\tau + d_i + f_{i\sigma}\frac{\log\tau}{\tau}.$$

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recalling (5)

$$x_i^\mu = V_i^\mu\ \tau + c_i^\mu \log\tau + d_i + f_{i\sigma}\frac{\log\tau}{\tau}.$$

- This term is $\mathcal{O}(e^5)$, hence in the quantum amplitudes it will appear at 2-loop order.

Higher order universal terms in the soft radiative field

- We carry out similar calculations at higher orders in e.

- The asymptotic part of the trajectory admits following universal corrections that are governed by the long range electromagnetic force. (5)

$$x_i^\sigma = V_{i\sigma}\tau + c_{i\sigma}\log\tau + d_{i\sigma} + \sum_{m=1}^{\infty} (\eta_i)^m f_{i\sigma}^{(m)} \frac{(\log\tau)^m}{\tau^m} + \dots \, .$$

We will find the resultant radiative field.

Classical soft theorems

- We show for $m = 2, 3, 4$:

$$\tilde{A}_\mu|_{\omega\to 0} = \frac{e^{i\omega r}}{4\pi i r}\Big[\ \frac{1}{\omega}\ S_0\ + \sum_{m=1}^{\infty} S_m^{\text{class}}\ \omega^{m-1}(\log\omega)^m + ...\Big].$$

The form of S_m^{class} is given by :

$$S_m^{\text{class}} = \frac{(i)^m}{m!}\sum_{i=1}^{n}\Big[e_i\ [\frac{\epsilon.p_i}{q.p_i}(q.c_i) - \epsilon.c_i]\ (q.c_i)^{m-1} + \ \epsilon_\mu\ q_{\nu_1}...q_{\nu_{m-1}}\ \mathcal{F}_i^{\mu\nu_1...\nu_{m-1}}\ \Big].$$

$$\mathcal{F}_{i\mu\nu_1\cdots\nu_m} = m\sum_{m'=2}^{m+1} e_i c_{i\nu_1}\cdots c_{i\nu_{m+1-m'}}\frac{p_{i\nu_{m+2-m'}}}{m_i}\cdots\frac{1}{m_i}p_{i[\nu_m}\ f_{i\mu]}^{(m'-1)}.$$

Conclusion

- A new set of Q_m-conservation laws ($m = 1, 2, 3$). Charges are $\mathcal{O}(e^{2m+1})$ and tied to long range forces.
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- A new set of Q_m-conservation laws ($m = 1, 2, 3$). Charges are $\mathcal{O}(e^{2m+1})$ and tied to long range forces.
- Expected to be related to soft theorems for loop amplitudes.
- We derived these soft theorems for classical field for $m = 2, 3, 4$.
- There is compelling evidence that this structure holds for all m's. These interesting questions need to be explored.

THANK YOU !

Comparison between future and past solutions (1)

Using $\Box A_\mu = -j_\mu$,

$$A_\sigma(x) = \frac{1}{2\pi}\int d\tau\ \delta(\ (x - x'(\tau))^2)\ j_\sigma(\tau).$$

- The retarded root of the delta function is

$$\tau_0 = -V_i.(x - d_i) - \left[\ (V_i.x - V_i.d_i)^2 + (x - d_i)^2\ \right]^{1/2}. \qquad (4)$$

- Around future :

$$\tau_0|_{\mathcal{I}^+} = \frac{u}{(V_i^0 - \hat{x}.V_i)} + \mathcal{O}(\frac{1}{r}).$$

- Around past :

$$\tau_0|_{\mathcal{I}^-} = -2r\ (V_i^0 + \hat{x}.V_i) + \mathcal{O}(r^0).$$

Asymptotic dynamics including subsubleading term

Thus, asymptotic trajectories of the particles are corrected to :

$$x_i^\mu = V_i^\mu\ \tau + c_i^\mu \log\tau + d_i + f_{i\sigma}\frac{\log\tau}{\tau},$$

where

$$c_i^\mu = -\frac{1}{4\pi}\sum_{\substack{j=n'+1,\\ j\neq i}}^{n} e_i e_j \frac{p_i.p_j\ m_j^2 p_i^\mu + m_i^2 m_j^2 p_j^\mu}{[(p_i.p_j)^2 - m_i^2 m_j^2]^{3/2}}.$$

$$f_i^\mu = -\sum_{\substack{i=n'+1,\\ i\neq j}}^{n} m_i m_j^2 \frac{Q_i Q_j}{2}\ \Big[3 m_i m_j p_j.c_i\ \frac{(p_i.p_j\ p_i^\mu + m_i^2 p_j^\mu)}{[(p_i.p_j)^2 - m_i^2 m_j^2]^{5/2}} + \frac{[p_i.p_j\ c_i^\mu - (p_i.p_j\ c_j^\mu - p_j.c_i\ p_j^\mu)]}{[(p_i.p_j)^2 - m_i^2 m_j^2]^{3/2}}\Big]. \quad (5)$$

(2)(3)

Log ω soft theorem

- $S_{\log}$ has 2 parts. A part that survives in the classical limit.

$$S_{\log}^{\text{class}} = \frac{i}{4\pi} \sum_{\substack{\eta_i\eta_j=1 \\ i\neq j}} e_i^2 e_j \frac{\epsilon^\mu q^\nu}{(q.p_i)} \; m_i^2 m_j^2 \; p_{i[\mu}\partial_{i\nu]} \; \frac{p_i.p_j}{\sqrt{(p_i.p_j)^2 - m_i^2 m_j^2}}$$

- This term appears in the soft radiation emitted in a classical scattering.
- Important to note that this term vanishes for massless particles.

Log ω soft theorem

• The other part is purely quantum and does not appear in classical physics.

$$S_{\text{log}}^{\text{quan}} = -\frac{1}{8\pi^2} \sum_{i \neq j} e_i^2 e_j \frac{\epsilon^\mu q^\nu}{(q.p_i)} \; p_{i[\mu} \partial_{i\nu]} \; \frac{f(p_i, p_j)}{\sqrt{(p_i.p_j)^2 - m_i^2 m_j^2}}$$

• The exact form of this expression is not important for us. Interesting to note the relative factor of i between two terms.