Chaos and Instability in Black Hole Thermodynamics: Unravelling the Temperature Mystery

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Motivation

Lets start with Black hole and Event horizon

- In 1915 Einstein came up with his stunning idea of *General Relativity*.
- **Einstein's equation**

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

- However, essential singularities exist in the solution of Einstein's equation – **Black holes**.
- An **event horizon** is that region in space-time beyond which events cannot affect an outside observer.

And this one-way membrane has some fascinating characteristics....

Black hole thermodynamics

- In the early 70's Bekenstein and Hawking demonstrated that BH horizon shows thermodynamic phenomena. [*Bekenstein: PRD 1970, Hawking: Nature 1974*]
- Hawking showed Temperature associated with the horizon is observer dependent and a pure quantum quantity [*Hawking: 1974, 1975*]

$$T = \frac{\hbar\kappa}{2\pi} \tag{1}$$

  where κ is the surface gravity of BH.
- However, BH thermodynamics originates through an analogy between the laws of BH and those of usual thermodynamical systems

$$\underbrace{dM}_{dE} = \underbrace{\Theta d\alpha}_{TdS} + \underbrace{\Omega dJ + \Phi dQ}_{-PdV} \tag{2}$$

So the question: why thermodynamical quantities are associated with the horizon?

- Systems start showing chaotic dynamics whenever it comes under the influence of horizon [*Cardoso: 2008, Hashimoto: 2017,2018, Cubrovic: 2019, and lots of papers...*].
- Here, I will particularly address the massless particle case.
- Study of the massless particles that follow null geodesics is an interesting one as photons come out as Hawking quanta from BH.
- However, the reason behind this chaos is not fully understood.

So another question emerges: Why horizon creates chaos?

Is there any connection between chaos and thermality?

- Whenever there is chaos - - there is **Lyapunov exponent (LE)**.
- According to Sachdev-Ye-Kitaev (SYK) model the universal upper bound of LE is the surface gravity [*Maldacena, Shenker, Stanford: JHEP 2016*].
- Interestingly, this upper bound is dependent on temperature [*Shenker and Stanford: JHEP 2014*].
- Also, surface gravity is connected to Hawking temperature.

That means: Is there any relationship between chaos and thermality?

There are some connections indeed!

Some evidences show that there is a link ....

- Chaotic systems has an intrinsic connection with Quantum thermalization through *Eigenstate thermalization hypothesis* [*M. Srednicki: PRE 1994*].
- An unstable classical mode with fixed LE cannot have zero temperature in the quantum scale [*T. Morita: PRL 2019*].
- Inverse Harmonic Oscillator (IHO) causes temperature to rise under quantization and temperature is proportional to LE [*Hegde et al: PRL 2019*].

Therefore, can we explain the horizon thermality through instability?

Chaos near Event horizon

Massless particle moving near a Static Spherically Symmetric black hole horizon

- Starting with a static, spherically symmetric blackhole background

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

  where the horizon $r = r_H$ is determined by $f(r = r_H) = 0$.

- In Painleve coordinate transformation:

$$dt \to dt - \frac{\sqrt{1 - f(r)}}{f(r)} dr$$

  the metric becomes

$$ds^2 = -f(r)dt^2 + 2\sqrt{1 - f(r)}dt\,dr + dr^2 + r^2 d\Omega^2$$

- Now using the covariant form of the dispersion relation $g^{ab}p_a p_b = -m^2$ and expanding this under the background, we obtain

$$\boxed{E^2 + 2\sqrt{1 - f(r)}\; p_r E - \left(f(r)p_r^2 + \frac{p_\theta^2}{r^2}\right) = m^2} \qquad (3)$$

Near Horizon approximation

- Now for a massless particle Eq.(3) becomes

$$\boxed{E = -\sqrt{1-f(r)}\ p_r + \sqrt{p_r^2 + \frac{p_\theta^2}{r^2}}} \quad \text{[for outgoing particle]} \tag{4}$$

- Now the radial motion of the particle very near to the horizon (taking $p_\theta = 0$) for the energy (4) is

$$\dot{r} = \frac{\partial E}{\partial p_r} = -\sqrt{1-f(r)} + 1 \tag{5}$$

Making an expansion of $f(r)$ near the horizon as

$$f(r) \simeq 2\kappa(r - r_H) \quad \text{[where } \kappa = f'(r_H)/2 \text{]} \tag{6}$$

- Now the solution of Eq.(3) $\rightarrow r = r_H + C\, r_H\, e^{\kappa\lambda} \rightarrow$ Exponential Growth $\rightarrow$ Signature of Chaos !!

Our system

- In order to trap the particle near the horizon we introduce some external potential.
- Here, we trap the particle using a harmonic potential.
- Our aim is to study how blackhole horizon influences an integrable system and affects the motion of the particle in it.
- We consider the potential form of the system
  $V(r,\theta) = (1/2)K_r(r - r_c)^2 + (1/2)K_\theta(y - y_c)^2$ where $y = r_H\theta$
- Corresponding total energy

$$E = -\sqrt{1 - f(r)}\, p_r + \sqrt{p_r^2 + \frac{p_\theta^2}{r^2}} + \frac{1}{2}K_r(r - r_c)^2 + \frac{1}{2}K_\theta(y - y_c)^2$$

Poincaré Sections of the motion of the particle near the SSS BH horizon for different energies

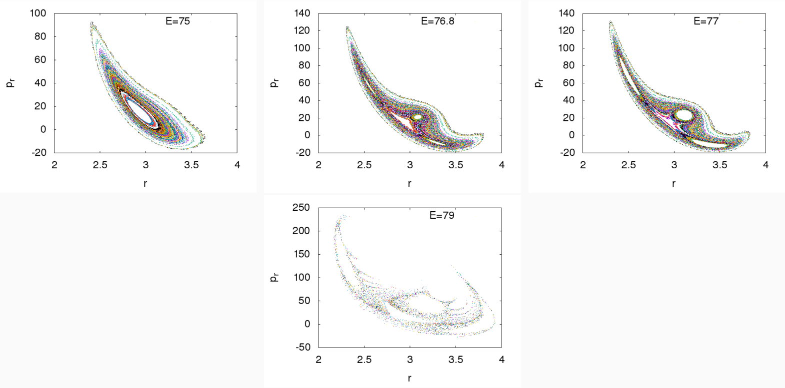

Figure 1: The Poincaré sections in the $(r,\ p_r)$ plane with $\theta = 0$ and $p_\theta > 0$ at different energies for the SSS black hole. The energies are $E = 75$, $E = 76.8$, $E = 77$, and $E = 79$. The other parameters are $r_H = 2.0$, $\kappa = 0.25$, $r_c = 3.2$, $\theta_c = 0$, $K_r = 100$ and $K_\theta = 25$. For large energy the KAM Tori break and the entire region gets filled with the scattered points indicating the presence of chaos.

Poincaré Sections of the motion of the particle near the Kerr BH horizon for different energies

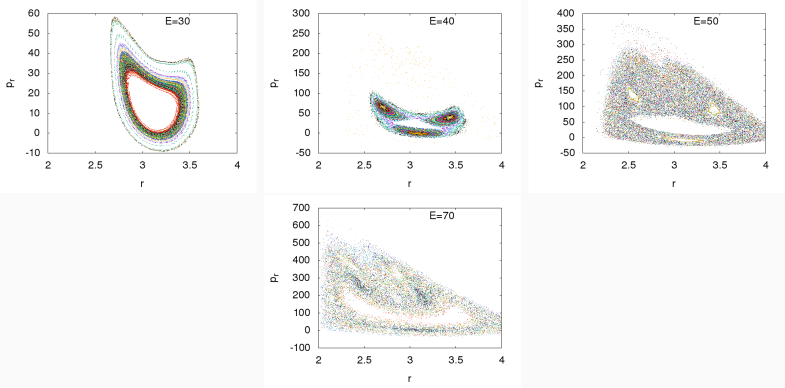

Figure 2: The Poincaré sections in the $(r,\ p_r)$ plane with $\theta = 0$ and $p_\theta > 0$ for different energy for the Kerr black hole model at fixed rotation parameter $a = 0.9$. The other parameters are same as in Fig. 1. For large energy the KAM Tori break and the entire region is filled with the scattered points indicating the chaotic trajectory of the particles.

Poincaré Sections of the motion of the particle near the Kerr BH horizon for different rotation parameters (a)

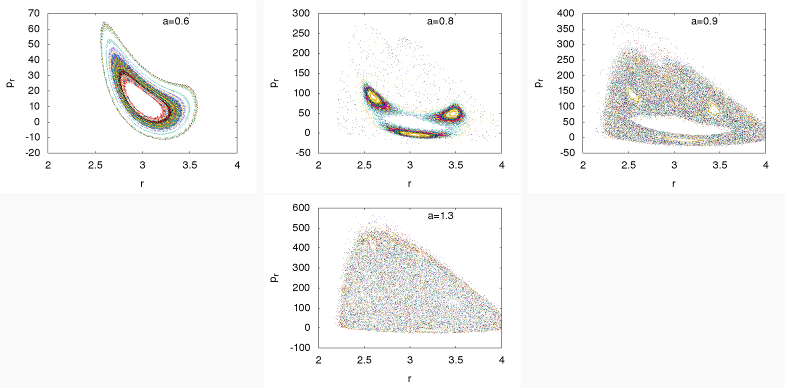

Figure 3: The Poincaré sections in the $(r,\ p_r)$ plane with $\theta = 0$ and $p_\theta > 0$ for fixed energy E=50 with different rotation parameter a. The rotation parameter $a = 0.6$, $a = 0.8$, $a = 0.9$, and $a = 1.3$, respectively from the top to the bottom. The other parameters are $r_H = 2.0$, and $K_r = 100$ and $K_\theta = 25$. Increase in the rotation induces the chaos in the particle dynamics.

Numerical analysis of the Lyapunov Exponent λ_L

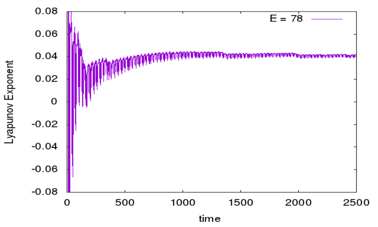

Figure 4: Largest Lyapunov exponent for the SSS black hole at the energy value $E = 78$. The exponent settles at positive value ~ 0.04.

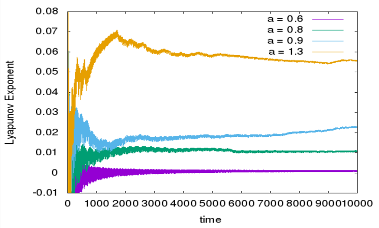

Figure 5: Largest Lyapunov exponents for the Kerr black hole for different values of the rotation parameter $a = 0.6$, 0.8, 0.9 and 1.3 at constant energy $E = 50$. The exponents increases on increase of a.

Why chaos is happening?

Outgoing path of the massless particle in EF

- Static Spherically Symmetric BH metric in E-F coordinates (t, r, θ, ϕ)

  $$ds^2 = -f(r)dt^2 + 2\Big(1 - f(r)\Big)dtdr + \Big(2 - f(r)\Big)dr^2 + r^2\Big(d\theta^2 + \sin^2\theta d\phi^2\Big) \ .$$

  where $t = v - r = t_s + r_* - r$ and $r_* = \int \frac{dr}{f(r)}$.
- Particle motion along the normal to the surface $U = constant = K$.
- Since we are interested in the near horizon region, at the end, the limit ($u \to \infty$, i.e. $t_s \to \infty$) $U = K \to 1$ will be taken to achieve our goal.
- The null normal components to the $U = constant$ surface are (taking the gradient of U)

  $$l^a = \left(1, \frac{f(r)}{2 - f(r)}, 0, 0\right) \ . \tag{7}$$

- Now the integral curves $x^a(\mu) = (t, r, \theta, \phi)$ of l^a, characterized by

  $$\frac{dx^a(\mu)}{d\mu} = l^a(x(\mu)) \ , \tag{8}$$

  where μ is the parameter which fixes the particle position at a particular moment

Radial behaviour: Instability near horizon

- From the time component of l^a

$$\frac{dt}{d\mu} = 1 \Rightarrow \mu = t\ . \tag{9}$$

- From the radial component of l^a

$$\frac{dr}{dt} = \frac{f(r)}{2 - f(r)}\ . \tag{10}$$

- Since we are interested in the near horizon region, therefore

$$f(r) \simeq 2\kappa(r - r_H) \tag{11}$$

- Substituting this in (10) and then keeping upto the relevant leading order ($\mathcal{O}(r - r_H)$), we obtain

$$\begin{aligned} \frac{dr}{dt} &\simeq \frac{2\kappa(r - r_H)}{2 - 2\kappa(r - r_H)} \\ &\simeq \kappa(r - r_H)\ . \end{aligned} \tag{12}$$

The solution of it is

$$r - r_H = \frac{1}{\kappa} e^{\kappa t} \tag{13}$$

Covariant realisation of local instability

- What happens to the family of null geodesics in the near horizon region? Expansion parameter (Θ) can be an important quantity to answer that.
- Raychaudhuri equation for null geodesics:

$$\frac{d\Theta}{d\mu} = \tilde{\kappa}\Theta - \frac{1}{2}\Theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}l^a l^b \ . \tag{14}$$

- We have found the shear parameter $\sigma_{ab} = 0$, since l_a is hypersurface orthonormal, we have the rotation parameter $\omega_{ab} = 0$. In the near horizon region we have also $R_{ab}l^a l^b \sim \mathcal{O}(r - r_H)^2$ and $\Theta \sim \mathcal{O}(r - r_H)$ and $\tilde{\kappa} = \kappa + \mathcal{O}(r - r_H)$.
- Therefore, keeping the leading order terms the right hand side of Raychaudhuri equation becomes

$$\frac{d\Theta}{d\mu} = \kappa\Theta \Rightarrow \frac{d\Theta}{dt} = \kappa\Theta \tag{15}$$

and the solution of it is

$$\Theta = \kappa e^{\kappa t} \tag{16}$$

Hamiltonian from trajectories:

- Near horizon radial motion from before (here $x = r - r_H$)

$$\dot{x} = \frac{\partial H}{\partial p} = \kappa x \ . \tag{17}$$

Solution of this $H = \kappa x p + f_1(x)$, where $f_1(x)$ is an arbitrary function.

- Lagrangian for this comes out to be

$$L = p\dot{x} - H = -f_1(x) \ . \tag{18}$$

- As we know for massless particle the Lagrangian must vanish. So, we must choose $f_1(x) = 0$. Thus the Hamiltonian in the near horizon region is given by

$$\boxed{H = \kappa x p} \tag{19}$$

xp in disguise of Inverse harmonic oscillator!!

Emergence of Thermality

- Following the classical picture we next proceed for the quantum calculation in the present chapter.
- What are the quantum consequences of the near-horizon Hamiltonian $H \sim xp$?
- We shall apply quantum tunneling method in the near horizon region to see how particles behave in the quantum scale near the horizon.

Quantum Tunneling

- Classically nothing can escape from BHs but the quantum probability of escaping from barrier of the horizon can be different.
- Tunneling formalism is mainly used to study the Hawking effect. Adopting the concept of the mechanism, here we shall calculate the tunneling probability of the particle.
- The main essence of tunneling mechanism is to calculate the imaginary part of the classical action of the outgoing particle.
- There are two methods to calculate the action - -
  (a) Hamilton-Jacobi method [*Srinivasan and Padmanabhan: PRD 1999*]
  (b) Radial Null Geodesic method [*Parikh and Wilczek: PRL 2000*]
- Here, we shall apply HJ method.

Methodology for the calculation of tunneling

For our present situation, both the outgoing and the ingoing particles are just outside the horizon but very near to the horizon.

$\downarrow$

Our aim is to calculate the emission probability of the outgoing particle while the absorption probability for the ingoing one.

$\downarrow$

The ratio of both the probabilities will give us the **tunneling probability**.

$\downarrow$

But before that we have to calculate the HJ action for both outgoing and the ingoing particle.

- Standard ansatz for wave function for a particle

$$\psi(x) \sim \exp\left[-\frac{i}{\hbar}S(x)\right] \tag{20}$$

where $S(x)$ is the Hamilton-Jacobi action for the particle defined as

$$S(x) = \int p \; dx \tag{21}$$

- Now, the outgoing and the ingoing trajectories corresponds to

$$\frac{\partial S}{\partial x} < 0 \;\; \text{(outgoing)} \;\; \text{and} \quad \frac{\partial S}{\partial x} > 0 \;\; \text{(ingoing)} \tag{22}$$

Calculation of the HJ action for outgoing particle

- The energy of the outgoing particle $E = \kappa x p$ (from (52)). Therefore

$$\begin{aligned} S[\text{Emission}] &= \frac{E}{\kappa}\int_{-\epsilon}^{\epsilon} \frac{dx}{x} \\ &= -\frac{i\pi E}{\kappa} + (\text{real part})\ . \end{aligned} \tag{23}$$

  where $\epsilon > 0$ and the limit $x = -\epsilon$ to $x = \epsilon$ means from just inside the horizon to just outside.
- $x = 0$ is the pole of the integrand. To evaluate it, the upper complex plane is being considered.
- Since the particle starts from inside the black hole where $x < 0$ and we have $\partial S/\partial x < 0$ which is consistent with the definition of the outgoing nature of the particle.

Calculation of the HJ action for the ingoing particle

- For the ingoing particle the energy comes out to be

$$E = -p \tag{24}$$

- Therefore, the "absorption" action for the ingoing particle comes out to be a real quantity.
- This is trivial because the the limits of the integration for the "absorption" action never includes the horizon singularity.

Tunneling probability and Hawking temperature

- The probability of absorption for the outgoing particle

$$P[\text{Emission}] \sim \left| e^{-\frac{i}{\hbar} S[\text{Emission}]} \right|^2 \propto \exp\left(-\frac{2\pi E}{\hbar\kappa}\right) . \quad (25)$$

- The probability of emission of the ingoing particle turns out to be

$$P[Absorption] = 1 . \quad (26)$$

- Hence the tunneling probability is evaluated as

$$\Gamma = \frac{P[\text{Emission}]}{P[\text{Absorption}]} \sim \exp\left(-\frac{2\pi E}{\hbar\kappa}\right) . \quad (27)$$

- Note that the above one is thermal in nature. The temperature is identified as

$$\boxed{T = \frac{\hbar\kappa}{2\pi}} . \quad (28)$$

This temperature exactly matches with the standard Hawking expression for black hole.

Other methods

- In order to get a distinct idea about the observer and the relevant vacuum state we performed the detector response where the detector follows the null trajectory in EF coordinates in the near horizon regime. The vacuum is chosen to be Boulware vacuum and it showed that the detector will see this vacuum as thermal bath.
- We also studied the scattering phenomena in the presence of this unstable xp kind near horizon Hamiltonian. Identifying the “in” and “out” states we obtained the transition probability which yielded the thermal nature again.
- Our last approach was to find out the thermality using a perturbative approach in presence of xp Hamiltonian. The motive of this approach was to construct a simple quantum mechanical model which mimics the near horizon characteristics.

Generic Null Hyper-Surface

- Dynamical properties of a generic null surface are known to have a thermodynamic interpretation.
- Such an interpretation is completely based on an analogy between the usual law of thermodynamics and structure of gravitational field equation on the surface.
- Therefore, in this chapter we want to materialise this analogy and show that assigning a temperature on the null surface for a local observer is indeed physically relevant.

Outline

- The aim is to understand whether this unstable nature is a general feature to any generic null hypersurface or not. Therefore, the same formalism will be used for generic null-hypersurface.
- Adapting a coordinate system called Gaussian Null Coordinates (GNC) the metric to any null surface takes the following form

$$ds^2 = -2r\alpha dv^2 + 2dvdr - 2r\beta_A dvdx^A + q_{AB}dx^A dx^B \tag{29}$$

where $r = 0$ corresponds to the null surface. The metric components α, β_A and q_{AB} are the functions of all the coordinates (v, r, x^A)

Field description

- Considering the massless real scalar mode ϕ, from the Klein-Gordon (KG) equation $\Box\phi = 0$ under the background of metric (29) yields

$$\partial_v\left(\sqrt{\mu}\partial_r\phi\right) + \partial_r\left(\sqrt{\mu}\partial_v\phi\right) + \partial_r\left[\sqrt{\mu}\left(2r\alpha + r^2\beta^2\right)\partial_r\phi\right] + \partial_r\left(\sqrt{\mu}r\beta^A\partial_A\phi\right)$$
$$+\partial_A\left(\sqrt{\mu}\; r\beta^A\partial_r\phi\right) + \partial_A\left(\sqrt{\mu}\mu^{AB}\partial_B\phi\right) = 0\ , \quad (30)$$

where μ is the determinant of the induced metric μ_{AB}.

- Now, we start with the standard ansatz for the scalar mode as

$$\phi = \mathcal{A}(v, r, x^A)e^{-\frac{i}{\hbar}S(v,r,x^A)}\ , \quad (31)$$

where $S(v, r, x^A)$ is the HJ action and with respect to the HJ action we define the four-momentum as

$$\frac{\partial S}{\partial x^a} = p_a\ . \quad (32)$$

- Now, expanding $S(v, r, x^A)$ in the powers of $\hbar$ we find,

$$S(v, r, x^A) \quad = \quad S_0(v, r, x^A) + \hbar S_1(v, r, x^A) + \hbar^2 S_2(v, r, x^A) + ... \tag{33}$$

- We define $-\partial S_0/\partial v = -p_v = H$, where H is the (semi-classical) Hamiltonian of the system.
- In the semi-classical limit (i.e. $\hbar \to 0$),

$$2(\partial_v S_0)(\partial_r S_0) + (2r\alpha + r^2\beta^2)(\partial_r S_0)^2 + 2r\beta^A(\partial_A S_0)(\partial_r S_0) + \mu^{AB}(\partial_B S_0)(\partial_A S_0) = 0 \ .$$

Here we see from Eq. (34) that $\partial_r S_0$ has two solutions which are

$$\partial_r S_0 = -\frac{\partial_v S_0 + r\beta^A(\partial_A S_0)}{2r\alpha + r^2\beta^2} \pm \left[\left(\frac{\partial_v S_0 + r\beta^A(\partial_A S_0)}{2r\alpha + r^2\beta^2} \right)^2 - \frac{\mu^{AB}(\partial_A S_0)(\partial_B S_0)}{2r\alpha + r^2\beta^2} \right]^{\frac{1}{2}} \ .$$

Among these two solutions, one corresponds to the outgoing mode, and the other one corresponds to the ingoing one.

System Hamiltonian

- The Hamiltonian for the outgoing mode comes out to be

$$H = \alpha^{(0)}(v, x^A) r p_{r_{out}} \tag{34}$$

where $p_{r_{out}}$ is the outgoing momentum in r direction.

- The Hamiltonian for the ingoing one is

$$H = \frac{1}{2}\frac{\mu^{(0)AB} p_A p_B}{p_{r_{in}}} \quad , \tag{35}$$

where $p_{r_{in}}$ is the ingoing momentum in r direction.

Tunneling probability and Temperature

- The probability for the outgoing object crossing the null hypersurface turns out to be

$$P_{out} \sim \left| e^{-\frac{i}{\hbar} S_{out}} \right|^2 \propto \exp\left(-\frac{2\pi \bar{E}}{\hbar \bar{\alpha}(v)} \right) , \quad (36)$$

whereas the probability of crossing the null hypersurface for the ingoing one is $P_{in} \sim 1$. Therefore, the tunneling probability comes out to be

$$\Gamma(v) = \frac{P_{out}}{P_{in}} \sim \exp\left(-\frac{2\pi \bar{E}}{\hbar \bar{\alpha}(v)} \right) . \quad (37)$$

where $\bar{\alpha}(v) r p_{r_{out}} \equiv \bar{E}$ and $\bar{\alpha}(v) = \frac{\int \alpha^{(0)}(v, x^A) \sqrt{\mu} d^2 x^A}{\int \sqrt{\mu} d^2 x^A}$.

- Therefore, the temperature of the system is identified as

$$T(v) = \frac{\hbar \bar{\alpha}(v)}{2\pi} . \quad (38)$$

We see that the temperature is a function of the timilike coordinate (v), unlike the case of SSS and Kerr.

Therefore..

- We applied our previous methodology and show that assigning a temperature on the null surface for a local observer is indeed physically relevant.
- We found that for a local frame, chosen as outgoing massless chargeless particle (or field mode), perceives a " *local unstable Hamiltonian*" very near to the surface and the Hamiltonian is $H \sim xp$ kind once more.
- Due to this xp Hamiltonian it has finite quantum probability to escape through acausal null path which is given by Maxwell-Boltzmann like distribution, thereby providing a temperature on the surface.

Here we basically generalise our conjecture for a generic null hypersurface by showing the connection between local instability and thermality.

Conclusions

- We observed that whenever a system comes under the influence of the horizon, it starts showing chaotic dynamics.
- We tried to find out the reason behind this chaos and we obtained that the near-horizon Hamiltonian ($H \sim xp$) which provides the local instability is responsible for that.
- We investigated the quantum consequences of this local instability in the near-horizon region and we found out that this local instability is the cause of thermalization in the system.
- We can also show that one can assign temperature on a null surface by building the connection with local instability.

Concluding remarks:

Horizon creates a "local instability" which acts as the source of the quantum temperature of the black hole.

THANK YOU FOR LISTENING!

Backup slides

For the Kerr Black hole (Chapter 2)

In the Painleve coordinate the Kerr metric can be written

$$d\hat{s}^2 = -fdt^2 + gdr^2 + 2hdtdr + kd\theta^2$$

where,

$$\begin{aligned} f &= \frac{\triangle\Sigma}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} \ ; \\ g &= \frac{\Sigma}{r^2+a^2} \ ; \\ h &= \frac{\sqrt{2Mr(r^2+a^2)}\Sigma}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} \ ; \\ k &= \Sigma = r^2 + a^2\cos^2\theta \ , \end{aligned}$$

and $\Delta = r^2 + a^2 - 2Mr$. M is the mass of the BH and $a = J/M$ is the angular momentum per unit mass$\rightarrow$ rotation parameter.

The event horizon is given by $\Delta = 0$. It leads to the location of horizon as

$$r_H = M + \sqrt{M^2 - a^2} \ .$$

Using $p_a p^a = 0$ we find the energy of the particle which is moving through the two harmonic potentials is

$$E = -\frac{h}{g}p_r + \sqrt{\frac{h^2}{g^2}p_r^2 + \frac{f}{g}p_r^2 + (\frac{1}{k}\frac{fg + h^2}{g})p_\theta^2} + \frac{1}{2}K_r(r - r_c)^2 + \frac{1}{2}K_\theta(y - y_c)^2$$

Using near horizon approximation i.e expanding $f(r)$ upto the first order

$$\begin{aligned} f(r) &= f(r_H) + (r - r_H)f'(r_H) \\ &= -\frac{a^2 \sin^2\theta}{r_H^2 + a^2\cos^2\theta} + \frac{r_H^4 - a^4\cos^2\theta + r_H^2 a^2 \sin^2\theta}{r_H(r_H^2 + a^2\cos^2\theta)^2}(r - r_H) \end{aligned}$$

Chaos near Rindler horizon

Motivation of the chapter

- *"Presence of horizon may produce chaos in an integrable system"*- - the whole idea is studied from a *uniformly accelerated frame* in the present chapter.
- Rindler frame provides the horizon without any intrinsic curvature to the system.
- It means if we can find chaos near the horizon in this case, then we can claim that - -

"The mere presence of a horizon, rather than the inherent curvature is enough to cause chaos in the particle's motion."

- The frame of a uniformly accelerated observer is given by the Rindler metric. In $(1+3)$ dimensions it is of the form:

$$ds^2 = -2ax\ dt^2 + \frac{dx^2}{2ax} + dy^2 + dz^2\ . \tag{39}$$

- In Painleve coordinate the metric becomes

$$ds^2 = -2ax\ dt^2 + 2\sqrt{1-2ax}\ dtdx + dx^2 + dy^2 + dz^2\ . \tag{40}$$

- Now, in the similar approach after the introduction of the harmonic potential we obtain the total energy of the system (for the outgoing particle)

$$E = -\sqrt{1-2ax}\ p_x + \sqrt{p_x^2 + p_y^2} + \frac{1}{2}K_x(x-x_c)^2 + \frac{1}{2}K_y(y-y_c)^2\ , \tag{41}$$

Poincaré Sections of the motion of the particle near Rindler horizon for different energy values

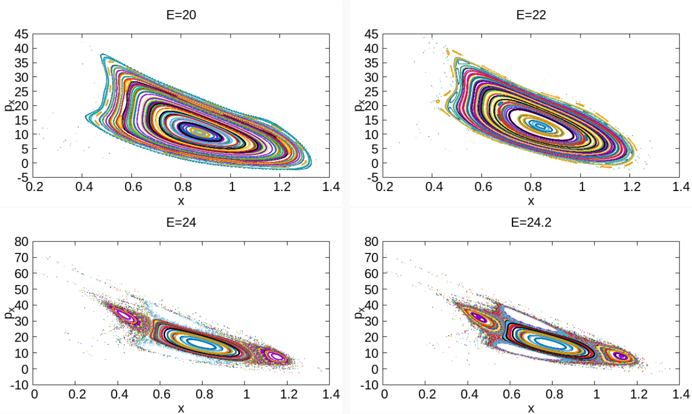

Figure 6: The Poincaré sections in the (x, p_x) plane with $y = 1.0$ and $p_y > 0$ at different values of energy of the system but for fixed acceleration ($a = 0.35$). The energies are $E = 20, 22, 24$ and 24.2. The other parameters are $K_x = 26.75$, $K_y = 26.75$, $x_c = 1.1$ and $y_c = 1.0$. For large value of energy the KAM Tori break and the regions filled with scattered points which indicates the presence of chaotic motion in the particle dynamics.

Poincaré Sections of the motion of the particle near Rindler horizon for different values of acceleration

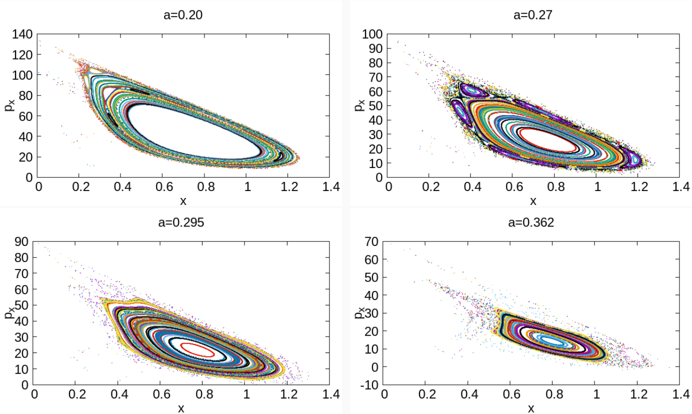

Figure 7: The Poincaré sections in the (x, p_x) plane with $y = 1.0$ and $p_y > 0$ at different values of acceleration of the system for fixed energy ($E = 24.0$). The values of accelerations are $a = 0.20, 0.27, 0.295$ and 0.362. The other parameters are $K_x = 26.75$, $K_y = 26.75$, $x_c = 1.1$ and $y_c = 1.0$. For large value of acceleration the KAM Tori break and the scattered points emerge which indicates the onset of chaotic dynamics.

PSD (for $a = 0.35$)

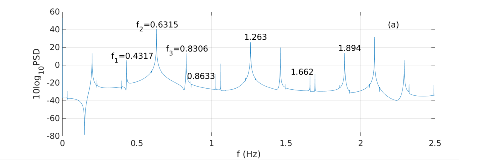

Figure 8: $E = 20$

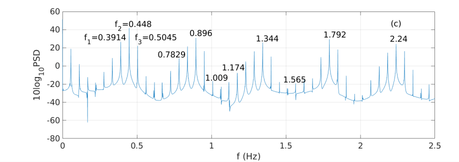

Figure 10: $E = 24$

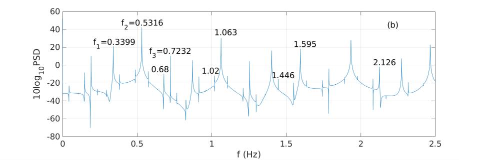

Figure 9: $E = 22$

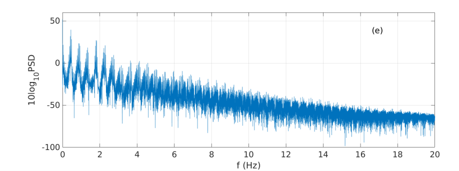

Figure 11: $E = 24.2$

PSD (for $E = 24$)

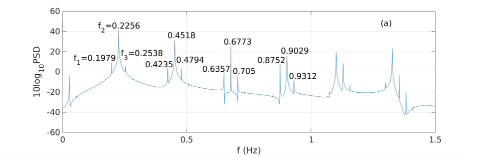

Figure 12: $a = 0.20$

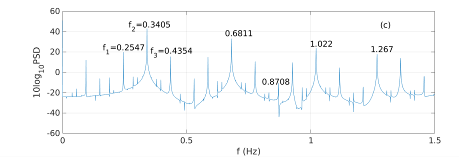

Figure 14: $a = 0.295$

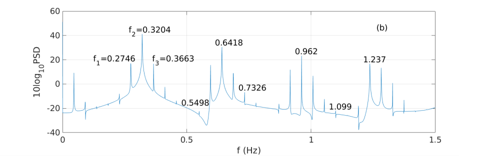

Figure 13: $a = 0.27$

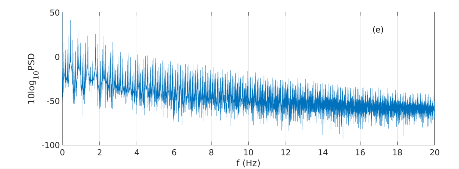

Figure 15: $a = 0.362$

largest Lyapunov expnents

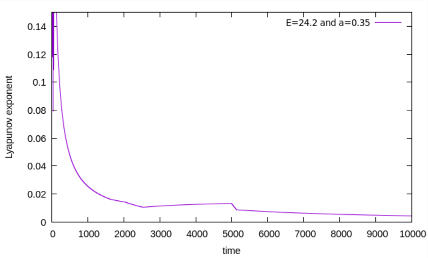

Figure 16: Largest LE for $E = 24.2$ and $a = 0.35$. The exponent settles at positive value ~ 0.01.

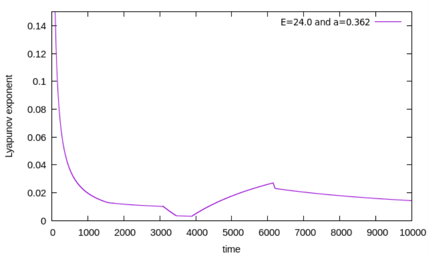

Figure 17: Largest LE for $E = 24$ and $a = 0.362$. The exponent settles at ~ 0.02

Conclusions of the chapter

- We have studied the motion of a massless and chargeless particle in an accelerated frame in the flat space–time background.
- We found that particle trapped in harmonic potential shows chaotic dynamics as it approaches nearer to the horizon.
- Our system satisfies the maximal value of LE which is a (acceleration of the particle).

Therefore, we draw an important conclusion that - -

"mere presence of horizon is enough to make the particle motion chaotic".

The corresponding equations of motion are

$$\dot{x} = \frac{\partial E}{\partial p_x} = -\sqrt{1 - 2ax} + \frac{p_x}{\sqrt{p_x^2 + p_y^2}} \ ; \tag{42}$$

$$\dot{p}_x = -\frac{\partial E}{\partial x} = -\frac{a}{\sqrt{1 - 2ax}} p_x - K_x(x - x_c) \ ; \tag{43}$$

$$\dot{y} = \frac{\partial E}{\partial p_y} = \frac{p_y}{\sqrt{p_x^2 + p_y^2}} \ ; \tag{44}$$

$$\dot{p}_y = -\frac{\partial E}{\partial y} = -K_y(y - y_c) \ . \tag{45}$$

- The model consists of a massless and chargeless particle moving very near to the horizon of Static Spherically Symmetric BH

$$ds^2 = -f(r)dt_s^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{46}$$

This coordinate system is singular at $f(r_H) = 0$.

- We want the particle to follow the outgoing null trajectory. So, Kruskal-Szekeras (KS) coordinates (U, V, θ, ϕ) in the null-null form will be relevant one in this context.
- Since, the paths will be outgoing ones, we consider the particle propagation along the normal to $U =$ constant surface

$$U = \exp(-\kappa u) + 1 \tag{47}$$

and $u = t_s - r_*$.

- However, KS coordinates cover the whole spacetime adapted to freely falling observer. In order to realize the horizon we adopt EF coordinates (t, r, θ, ϕ). EF coordinates are well behaved in the vicinity of the horizon.

From Θ to radial behaviour Chap 4

- General form of expansion parameter (Θ) in terms of l^a

$$\Theta = \nabla_a l^a - \tilde{\kappa} \tag{48}$$

where $\tilde{\kappa}$ is the non-affinity coefficient and its expression in this case

$$\tilde{\kappa} = \frac{2f'(r)}{(f(r)-2)^2} \tag{49}$$

- For the SSSBH metric in E-F coordinates the value of expansion parameter (at the leading order) becomes

$$\Theta \simeq \frac{2\kappa}{r_H}(r - r_H) \tag{50}$$

- Therefore, substitution of this in the solution (16) we obtain

$$\boxed{r - r_H \simeq (r_H/2)e^{\kappa t}} \tag{51}$$

Important: Although the instability is an observer independent feature, this particular radial character with time is related to EF observer.

Hamiltonian from the dispersion relation:

- SSSBH metric in EF coordinates had a timelike Killing vector $\xi'^a = (1, 0, 0, 0)$ and the energy of the particle moving under the this background is $E = -\xi'^a p_a = -p_t$ where $p_a = (p_t, p_r, 0, 0)$.
- From dispersion relation $g^{ab} p_a p_b = -m^2$ (here $m = 0$) we obtain

$$E = \frac{(f(r) - 1)p_r \mp p_r}{2 - f(r)} \ , \tag{52}$$

where $+ve$ for outgoing particle and $-ve$ for the ingoing one.

- The energy of the outgoing particle (taking $+ve$ solution)

$$E = \frac{(f(r) - 1)p_r + p_r}{2 - f(r)} \tag{53}$$

- In the near horizon region it becomes (putting $f(r) \simeq 2\kappa(r - r_H)$)

$$\boxed{E \simeq \kappa(r - r_H)p_r} \tag{54}$$

- Kerr metric in Boyer-Lindquist (BL) coordinates $(t_{BL}, r, \theta, \phi_{BL})$:

$$\begin{aligned} ds^2 = \quad & -\left(1 - \frac{2mr}{\rho^2}\right) dt_{BL}^2 - \frac{4mar\sin^2\theta}{\rho^2} dt_{BL} d\phi_{BL} + \frac{\rho^2}{\Delta} dr^2 \\ & + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\rho^2}\right) \sin^2\theta \; d\phi_{BL}^2 \;, \end{aligned} \tag{55}$$

  where m is the mass and a is the angular momentum per unit mass of BH. $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2mr + a^2$.

- However BL coordinates are not regular at $r_H = m + \sqrt{m^2 - a^2}$. So, spheroidal version of EF coordinates (t, r, θ, ϕ) are adapted which is related to BL as

$$dt = dt_{BL} + \frac{dr}{\frac{r^2+a^2}{2mr} - 1} \;, \tag{56}$$

$$d\phi = d\phi_{BL} + \frac{a\; dr}{r^2 - 2mr + a^2} \;. \tag{57}$$

Outgoing path of the massless particle in Kerr spacetime

- Kerr metric in spheroidal EF coordinates turns out to be

$$\begin{aligned} ds^2 = \quad & -\left(1-\frac{2mr}{\rho^2}\right)dt^2 + \frac{4mr}{\rho^2}dt\ dr - \frac{4amr}{\rho^2}\sin^2\theta\ dtd\phi \\ & +\left(1+\frac{2mr}{\rho^2}\right)dr^2 - 2a\sin^2\theta\left(1+\frac{2mr}{\rho^2}\right)drd\phi \\ & +\rho^2 d\theta^2 + \left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2\ . \end{aligned} \tag{58}$$

- Killing vectors $\rightarrow \xi^a_{(t)} = (1,0,0,0)$ and $\xi^a_{(\phi)} = (0,0,0,1)$ and hence $E = -p_a\xi^a_{(t)} = -p_t$ and $L_z = p_a\xi^a_{(\phi)} = p_\phi$. Therefore, the conserved quantity $K = -\xi^a p_a = E - \Omega_H p_\phi$.
- The null normal components to $\mathcal{H}$

$$l^a = \left(1, \frac{\sqrt{A}-2mr}{\rho^2+2mr}, 0, \frac{a}{\sqrt{A}}\right)\ . \tag{59}$$

  where $A = (r^2+a^2)^2 - (r^2-2mr+a^2)a^2\sin^2\theta$.
- Then, the integral curves $x^a(\mu) = (t,r,\theta,\phi)$ of l^a, are $\frac{dx^a(\mu)}{d\mu} = l^a(x^i(\mu))$, $\mu \rightarrow$ parameter which fixes the particle position at a particular moment.

Radial behaviour: Instability near the Kerr horizon

- from the time component of l^a

$$\frac{dt}{d\mu} = 1 \Rightarrow \mu = t \ . \tag{60}$$

- From the radial part

$$\frac{dr}{dt} = \frac{\sqrt{A} - 2mr}{\rho^2 + 2mr} = f_{kerr}(r). \tag{61}$$

- It can be checked that $f_{kerr}(r_H) = 0$. Therefore, in the near horizon region $r \to r_H$ we have

$$f_{kerr}(r) \simeq \kappa_{kerr}(r - r_H) \tag{62}$$

where $\kappa_{kerr} = f'_{kerr}(r_H)$ and $\kappa_{kerr} = \frac{r_H - m}{2mr_H} = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}$

- Therefore, we have the radial part in the near horizon region

$$\frac{dr}{dt} \simeq \kappa_{kerr}(r - r_H) \tag{63}$$

and the solution again shows

$$r - r_H = \frac{1}{\kappa_{kerr}} e^{\kappa_{kerr} t} \tag{64}$$

- From θ component of l^a

$$\frac{d\theta}{dt} = 0 \Rightarrow \theta = \text{constant} . \tag{65}$$

- From the azimuthal component of l^a

$$\begin{aligned} \frac{d\phi}{dt} &\simeq \frac{a}{2mr_H} = \Omega_H \\ \Rightarrow \phi &= \Omega_H t \end{aligned} \tag{66}$$

- Using Raychaudhuri equation in the near horizon region for null geodesics one can show

$$\frac{d\Theta}{d\mu} \simeq \kappa_{kerr}\Theta \tag{67}$$

and we obtain $\Theta \simeq \kappa_{kerr} e^{\kappa_{kerr}\mu} \rightarrow$ instability just like the SSS case.

- The **Hamiltonian** in the near horizon region

$$\boxed{H = \kappa_{kerr}(r - r_H)p_r + \Omega_H p_\phi} \tag{68}$$

Instability in the near Kerr horizon region

- Just like from the earlier methodology, from the Hamiltonian $H = \kappa_{kerr}(r - r_H)p_r + \Omega_H p_\phi$ we obtain the radial momentum variation as

$$p_r \sim e^{-\kappa_{kerr} t} \ . \qquad (69)$$

- Therefore, in the near horizon region, i.e. in the limit $t \to -\infty$, p_r diverges.
- This is the indication of instability in the near-horizon region just like the earlier case.

Temperature

- In the similar way using the tunneling approach we obtain the tunneling probability

$$\Gamma_{kerr} = \frac{P[Emission]}{P[Absorption]} \sim \exp\left[-\frac{2\pi(E - \Omega_H p_\phi)}{\hbar\kappa_{kerr}}\right] \tag{70}$$

and the temperature of the system as

$$\boxed{T = \frac{\hbar\kappa_{kerr}}{2\pi}} \tag{71}$$

- Like the earlier analysis of SSS case, this time, in Kerr spacetime we obtain that local instability in the near horizon region provides temperature to the system in quantum scale.
- **Importance of this result:** It turns out that the extension of our proposed conjecture is applicable to much more general black holes also.

Hence, the generality of this conjecture is evident here and it may become one of the leading candidates to explain the horizon thermodynamics.