Cubic String Field Interactions and Polyakov String Path Integrals

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Contents

History of Cubic String Interactions (Light-Cone String Field Theories)

1. Cubic string field Interactions in light-cone gauge:
   M. Kaku and K. Kikkawa (1974)
   Define cubic string field interaction, using overlapping functions.
   Difficulties arise when we convert the cubic string field interaction to Fock space representation. One has to deal with inversions of $\infty \times \infty$ matrices.
2. Evaluation of Polyakov string path integrals, using Green's functions on string world sheets:
   S. Mandelstam (1973, 1974), E. Cremmer and J. L. Gervais (1974, 1975), M. B. Green and J. H. Schwarz (1982).

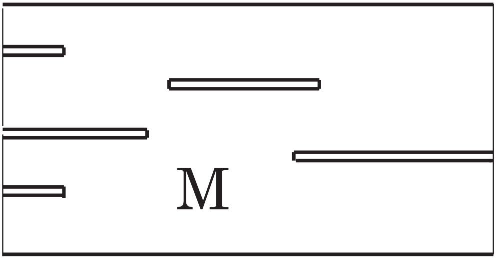
History of Cubic String Interactions (Covariant String Field Theories)

1. BRST invariant string field theory (HIKKO)
   H. Hata, K. Itoh, T. Kugo, H. Kunitomo, and K. Ogawa (1986)
   Covariantize the old light-cone string field theory. This theory contains unphysical length parameters, analogues of the lihjt-cone momenta.
2. Witten's cubic string field theory
   E. Witten (1986)
   Non-linear extension of BRST invariance, Chern-Simons like action. Difficulties associated with conical singlarities on string world sheets.
3. Polyakov string path integrals in the proper-time gauge
   TL (1988, 2017, 2018, 2019,2020)
   Direct conversion of Polyakov string path integrals into Fock space representations (Schwinger-Feynman proper-time representations of covariant string field theory). Deformable to Witten's cubic string field theory.

Polyakov String Path Integral on String World Sheet

Polyakov string path integral and String Scattering Amplitude

$$\begin{aligned} \mathcal{L} &= \frac{1}{4\pi\alpha'}\sqrt{-h}h^{\alpha\beta}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}, \\ \mathcal{A}_M &= \int D[X]D[h]\exp\left[-i\int_M d\tau d\sigma\mathcal{L}\right]. \end{aligned}$$



Polyakov String Path Integral on Cylindrical Surface (Free String Propagator)

Proper-time gauge for closed string: Expanding $N^\alpha = \sum N^\alpha_n e^{in\sigma}$,

$$\dot{N}_{10} = 0, \quad N_{1n} = 0, \quad N_{2n} = 0, \quad n \neq 0.$$

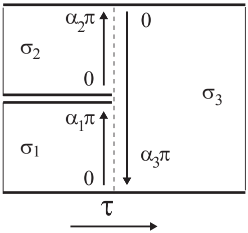
String propagator:

$$\begin{aligned}
& G[X^\mu_1, c_1, d_1; X^\nu_2, c_2, d_2] \\
= \; & \int_0^\infty ds \langle X^\mu_1, c_i, d_1 | \exp\left[-is\left(L_0 + L_{gh} - i\epsilon\right)\right] | X^\nu_2, c_2, d_2 \rangle \\
= \; & \langle X^\mu_1, c_1, d_1 | \frac{1}{L_0 + L^{gh}_0 - i\epsilon} | X^\nu_2, c_2, d_2 \rangle,
\end{aligned}$$

where $s = Tn_1$ is the proper-time.

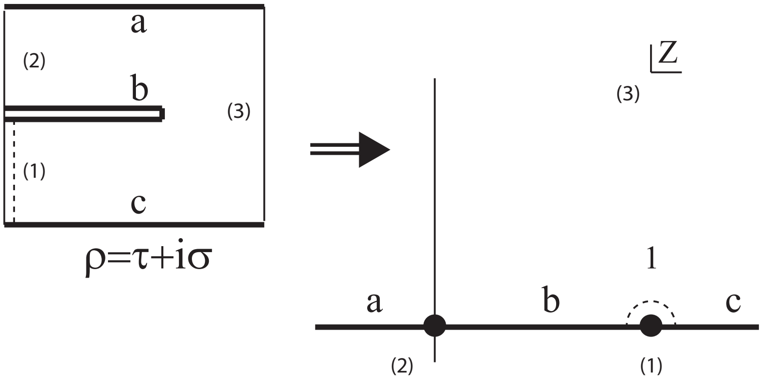
Polyakov String Path Integral in the Proper-Time Gauge: Three-String-Vertex

$$
\begin{aligned}
W[N] &= \int \prod_r DX^{(r)} \exp\left\{ i \sum_{r=1}^{N} \int P^{(r)}(\sigma) \cdot X^{(r)}(\sigma, \tau_r) d\sigma - \int d\tau d\sigma \mathcal{L} \right\} \\
&= \langle \mathbf{P} | \exp\left(\sum_r \xi_r L_0^{(r)} \right) | V[N] \rangle .
\end{aligned}
$$



Three-String-Vertex

Mapping of the three-string vertex diagram onto the upper half complex plane



Three-String-Vertex

- Schwarz-Christoffel trnasformation (Mandelstam1973, Cremmer74, Cremmer75)

$$\rho = \ln(z-1) + \ln z.$$

- Green's function on the upper half plane:

$$N(z,z') = \ln|z-z'| + \ln|z-z'^*|.$$

- The local coordinates $\zeta_r = \xi_r + i\eta_r$, $r = 1,2,3$:

$$\begin{aligned} e^{-\zeta_1} &= e^{\tau_0}\frac{1}{z(z-1)}, \\ e^{-\zeta_2} &= -e^{\tau_0}\frac{1}{z(z-1)} \\ e^{-\zeta_3} &= -e^{-\frac{\tau_0}{2}}\sqrt{z(z-1)} \end{aligned}$$

Neumann Functions:

$$\begin{aligned}
N(z_r, z'_s) &= 2\pi G(z_r, z'_s) \\
&= \ln|z_r - {z'}_s| + \ln|z_r - z'^*_s| \\
&= -\delta_{rs}\left\{\sum_{n\geq 1}\frac{2}{n}e^{-n|\xi_r-\xi'_s|}\cos(n\eta_r)\cos(n\eta'_s) - 2\max\left(\xi_r, \xi'_s\right)\right\} \\
&\quad +2\sum_{n,m\geq 0}\bar{N}^{rs}_{nm}e^{n\xi_r+m\xi'_s}\cos(n\eta_r)\cos(m\eta'_s)
\end{aligned}$$

where z_r and z'_s lie in the region of the r-th and s-th string patches respectively.

Three-String-Vertex

$$
\begin{aligned}
|\boldsymbol{V}_{[3]}\rangle &= (2\pi)^d \delta\left(\sum_{r=1}^{3} p_r\right) \exp\{(E[1,2;3]\}|0\rangle, \\
E[1,2;3] &= \frac{1}{2}\sum_{n,m=1}^{\infty}\sum_{r,s=1}^{3} \bar{N}^{rs}_{nm}\alpha^{(r)}_{-n}\cdot\alpha^{(s)}_{-m} + \sum_{n=1}^{\infty}\sum_{r=1}^{3} \bar{N}^{r}_{n}\alpha^{(r)}_{-n}\cdot\mathbb{P} \\
&\quad + \sum_{r=1}^{3}\frac{\tau_0}{\alpha_r}\frac{(p^{(r)})^2}{2} - \sum_{r=1}^{3}\frac{\tau_0}{\alpha_r},
\end{aligned}
$$

where $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = -2$ and $\mathbb{P} = p^{(2)} - p^{(1)}$.

Three-Gauge Field Vertex from Three-String Vertex

- External string state in the zero-slope limit

$$|\Phi^{(r)}\rangle = A^{\mu}(p_r) a_1^{(r)\dagger}{}_{\mu}|0\rangle + A^{a\mu}(p_r) T^a a_1^{(r)\dagger}{}_{\mu}|0\rangle.$$

- Neumann functions in the proper-time gauge:

$$\begin{aligned}
\bar{N}^{11}_{11} &= \frac{1}{2^4}, \quad \bar{N}^{22}_{11} = \frac{1}{2^4}, \quad \bar{N}^{33}_{11} = 2^2, \\
\bar{N}^{12}_{11} &= \bar{N}^{21}_{11} = \frac{1}{2^4}, \quad \bar{N}^{23}_{11} = \bar{N}^{32}_{11} = \frac{1}{2}, \quad \bar{N}^{31}_{11} = \bar{N}^{13}_{11} = \frac{1}{2}, \\
\bar{N}^{1}_{1} &= \bar{N}^{2}_{1} = \frac{1}{4}, \quad \bar{N}^{3}_{1} = -1, \quad \tau_0 = -2\ln 2.
\end{aligned}$$

Three-Gauge Field Vertex from Three-String Vertex

- Three-string vertex in the zero-slope limit

$$
\begin{aligned}
\boldsymbol{V}_{[3]} &= \frac{i}{2} g_{YM} \int \prod_{r=1}^{3} \frac{d\,p^{(r)}}{(2\pi)^d} (2\pi)^d \delta\left(\sum_{r=1}^{3} p^{(r)}\right) \\
&\quad f^{abc} A^{\mu}{}_a(p_1) A^{\nu}{}_b(p_2) A^{\lambda}{}_c(p_3) \eta_{\mu\nu}(p_2 - p_1)_\lambda, \\
g_{YM} &= (\alpha')^{d/4-1} g.
\end{aligned}
$$

- The three-gauge field interaction term follows from it:

$$
\boldsymbol{V}_{[3]\mathrm{Gauge}} = \frac{g_{YM}}{2} \int d^d x\, f^{abc} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) A^{\mu b} A^{\nu c}.
$$

Witten's Cubic String Field Theory and Polyakov String Path Integral

BRST gauge invaraint action for the string fields on the multiple *Dp*-branes as

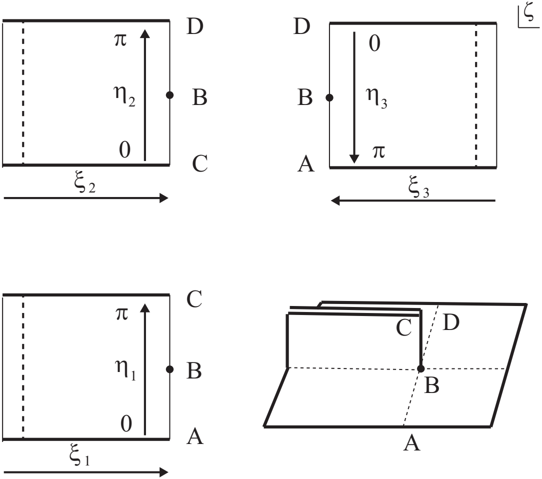
$$\mathcal{S} \;=\; \int \mathrm{tr}\left(\Psi * Q\Psi + \frac{2g}{3}\Psi * \Psi * \Psi\right),$$

where the star product between the string field operators is defined as

$$(\Psi_1 * \Psi_2)\,[X(\sigma)] = \int \prod_{\frac{\pi}{2}\le\sigma\le\pi} DX^{(1)}(\sigma) \prod_{0\le\sigma\le\frac{\pi}{2}} DX^{(2)}(\sigma)$$

$$\prod_{\frac{\pi}{2}\le\sigma\le\pi} \delta\left[X^{(1)}(\sigma) - X^{(2)}(\pi-\sigma)\right] \Psi_1[X^{(1)}(\sigma)]\Psi_2[X^{(2)}(\sigma)],$$

$$X(\sigma) \;=\; \begin{cases} X^{(1)}(\sigma) & \text{for} \quad 0 \le \sigma \le \frac{\pi}{2}, \\ X^{(2)}(\sigma) & \text{for} \quad \frac{\pi}{2} \le \sigma \le \pi. \end{cases}$$


D
π
η2
B
0
C
ξ2
D
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B
η3
A
π
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ζ
C
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B
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A
ξ1
C
D
B
A

CS Mapping of Conincal Surface onto Upper Half Complex Plane

1. Mapping the world sheet of Witten's SFT onto a unit disk by a conformal transformation:

$$\omega_1 = e^{\frac{2\pi i}{3}} \left(\frac{1+ie^{\zeta_1}}{1-ie^{\zeta_1}}\right)^{\frac{2}{3}}, \quad \omega_2 = \left(\frac{1+ie^{\zeta_2}}{1-ie^{\zeta_2}}\right)^{\frac{2}{3}},$$

$$\omega_3 = e^{-\frac{2\pi i}{3}} \left(\frac{1+ie^{\zeta_3}}{1-ie^{\zeta_3}}\right)^{\frac{2}{3}}.$$

2. Mapping each local coordinate patch on the unit disk onto the upper half plane:

$$z = -i\,\frac{\omega_r - 1}{\omega_r + 1}, \quad \frac{\pi}{3} \le \arg \omega_r \le \frac{2\pi}{3}, \quad r = 1,\ 2,\ 3.$$

The external strings are mapped to three points on the real line

$$Z_1 = \sqrt{3}, \quad Z_2 = 0, \quad Z_3 = -\sqrt{3}.$$

Neumann Functions for Witten's Cubic SFT

$$
\begin{aligned}
\bar{N}^{rs}_{00} &= \ln|Z_r - Z_s| \;\text{ for } r \neq s, \quad \bar{N}^{11}_{00} = \bar{N}^{33}_{00} = \ln\frac{8}{3}, \quad \bar{N}^{22}_{00} = \ln\frac{2}{3}, \\
\bar{N}^{12}_{10} &= \frac{8}{3\sqrt{3}}, \quad \bar{N}^{13}_{10} = \frac{4}{3\sqrt{3}}, \quad \bar{N}^{21}_{10} = -\frac{2}{3\sqrt{3}}, \\
\bar{N}^{23}_{10} &= \frac{2}{3\sqrt{3}}, \quad \bar{N}^{31}_{10} = -\frac{4}{3\sqrt{3}}, \quad \bar{N}^{32}_{10} = -\frac{8}{3\sqrt{3}}, \\
\bar{N}^{rs}_{11} &= \frac{a_r a_s}{(Z_r - Z_s)^2} = \frac{2^4}{3^3} \;\text{ for } r \neq s, \quad \bar{N}^{rs}_{11} = 0.
\end{aligned}
$$

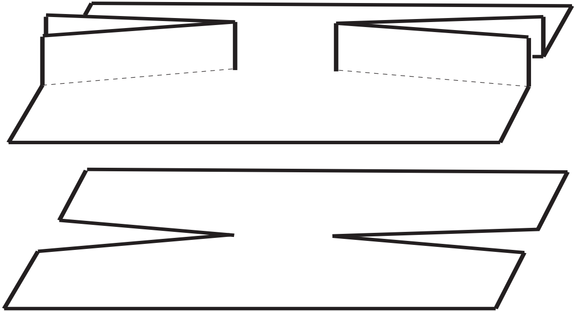
Choosing the external string state as

$$|\Psi^{(1)},\Psi^{(2)},\Psi^{(3)}\rangle = \prod_{r=1}^{3} A(p^{(r)})\cdot a_1^{(r)\dagger}|0\rangle,$$

$$\begin{aligned}
S_{Gauge} &= \frac{2g}{3}\langle\Psi^{(1)},\Psi^{(2)},\Psi^{(3)}|E_{[3]}|0\rangle \\
&= \frac{2g}{3}\int\prod_{i=1}^{3}\frac{dp^{(i)}}{(2\pi)^d}(2\pi)^d\delta\left(\sum_{i=1}^{3}p^{(i)}\right)\,\mathrm{tr}\Big\langle 0\Big|\left\{\prod_{i=1}^{3}A(p^{(i)})\cdot a_1^{(i)}\right\} \\
&\quad \exp\left\{\frac{1}{2}\sum_{r,s}\bar{N}_{11}^{rs}a_1^{(r)\dagger}\cdot a_1^{(s)\dagger}+\sum_{r,s}\bar{N}_{10}^{rs}a^{(r)\dagger}\cdot p^{(s)}-\sum_{r}\tau^{(r)}\right\} \\
&= \frac{g}{6\sqrt{3}}\int\prod_{i=1}^{3}dp^{(i)}\delta\left(\sum_{i=1}^{3}p^{(i)}\right) \\
&\quad \left(p_\mu^{(1)}-p_\mu^{(2)}\right)\mathrm{tr}\,\left(A(p_1)\cdot A(p_2)A(p_3)^\mu\right).
\end{aligned}$$

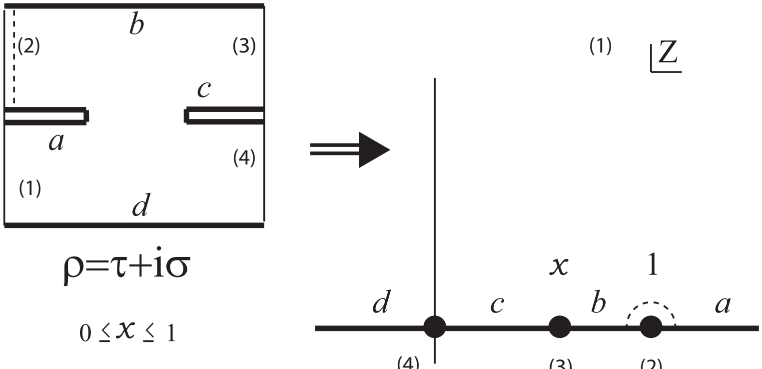
Four-String-Interaction: Deformed Cubic SFT

Four-string-interaction is generated perturbatively by the cubic string intraction.
Four-String Vertex of the Witten's SFT deformed to Planar Diagram (Equivalent to four-string-vertex in the proper-time gauge).



Four-String Vertex: Schwarz-Christoffel Mapping

$$\rho = \ln(z-1) - \ln(z-x) - \ln z.$$
$$Z_1 = \infty, \quad Z_2 = 1, \quad Z_3 = x, \quad Z_4 = 0.$$



Four-String Vertex: Four-Gauge-Interaction

- External string state

$$|\Phi^{(r)}\rangle = A^\mu(p_r) a_1^{(r)\dagger}{}_\mu |0\rangle + A^{a\mu}(p_r) T^a a_1^{(r)\dagger}{}_\mu |0\rangle.$$

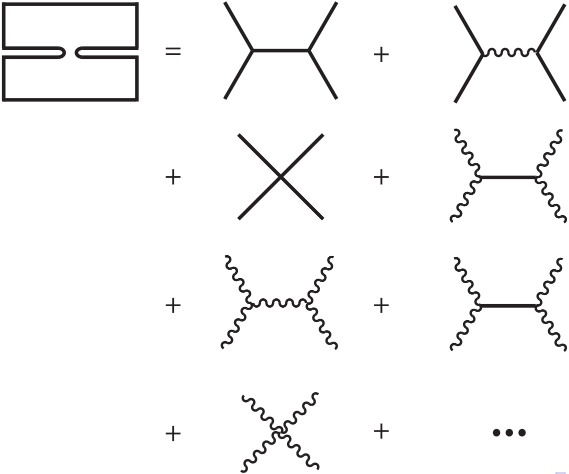
- Four-Gauge interaction from four-string-vertex

$$\begin{aligned} S_{[4]} &= g_{YM}^2 \int \prod_{r=1}^{4} dp^{(r)} \delta\Big(\sum_{r=1}^{4} p^{(r)}\Big) \mathrm{tr}\, \Big(A^\mu(p_1) A^\nu(p_2) A_\mu(p_3) A_\nu(p_4) \\ & \quad + \frac{2u}{s} A^\mu(p_1) A_\mu(p_2) A^\nu(p_3) A_\nu(p_4) \Big) \end{aligned}$$

where $s = -(p_1+p_2)^2, \quad t = -(p_1+p_4)^2, \quad u = -(p_1+p_3)^2.$

- **It contains the contact quartic gauge term (local) and the effective four-gauge field interaction term mediated by the massless gauge field (non-local).**

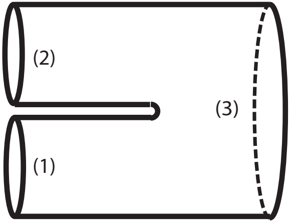
String Scattering Amplitude and Feynman Diagrams

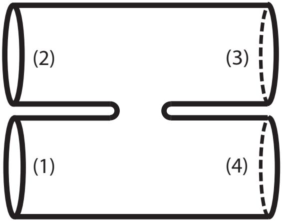


Cubic Closed String Interaction in the Proper-Time Gauge

Closed string theory in the proper-time gauge

We will construct a covariant interacting closed string theory and calculate three-string scattering amplitude in the proper-time gauge.





Closed String Theory: Review

Canonical commutation relations

$$\begin{aligned}[x_L, p_L] &= [x_R, p_R] = i\sqrt{\frac{\alpha'}{2}}, \\ [\alpha_m, \alpha_n] &= [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta(m+n).\end{aligned}$$

Momentum eigentate with eigenvalue $P_n, n \neq 0$:

$$|P_n\rangle = \sqrt{\frac{1}{\pi n}} \exp\left\{\left(\frac{P_n \alpha_{-n}}{n} + \frac{P_{-n}\tilde{\alpha}_{-n}}{n} - \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n} - \frac{P_n \cdot P_{-n}}{4n}\right)\right\}|0\rangle.$$

Mapping from cylindrical surface onto the complex plane

$$z = e^{\rho} = e^{\xi + i\eta}, \qquad -\pi \leq \eta \leq \pi.$$

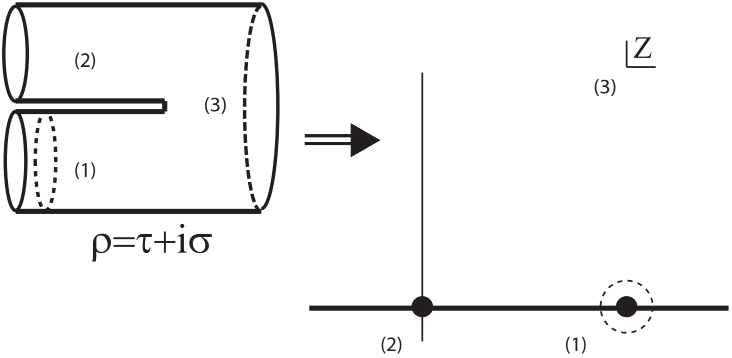
Green's function on complex plane ($\xi > \xi'$), $\Delta = |\xi - \xi'|$,

$$\begin{aligned}G_C(z, z') &= \ln|z - z'| \\ &= \max(\xi, \xi') - \frac{1}{2}\sum_{n=1} \frac{e^{-n\Delta}}{n}\left(e^{in(\eta' - \eta)} + e^{-in(\eta' - \eta)}\right).\end{aligned}$$

Three-Closed-String Interaction and Schwarz-Christoffel Mapping

CS mapping from the world sheet of three closed string scattering onto the complex plane. For the three-string vertex in the proper-time gauge,

$$\rho = \ln(z-1) + \ln z.$$



Closed String Interaction in the Proper Time Gauge

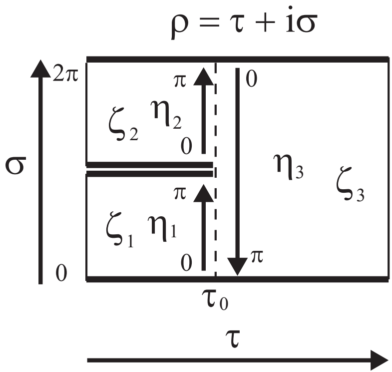
The local coordinates $\zeta_r = \xi_r + i\eta_r$, $r = 1, 2, 3$ defined on individual string world sheet patches are related to z as follows:

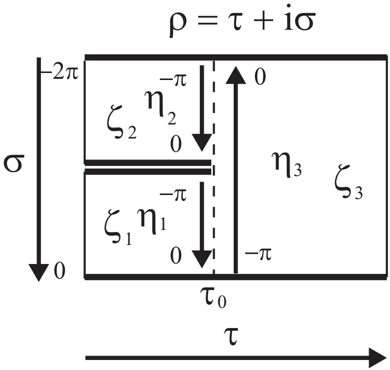
$$
\begin{aligned}
e^{-\zeta_1} &= e^{\tau_0} \frac{1}{z(z-1)}, \\
e^{-\zeta_2} &= -e^{\tau_0} \frac{1}{z(z-1)}, \\
e^{-\zeta_3} &= -e^{-\frac{\tau_0}{2}} \sqrt{z(z-1)}.
\end{aligned}
$$

For convenience we choose, by using $SL(2, C)$ invariance

$$Z_1 = 0, \quad Z_2 = 1, \quad Z_3 = \infty.$$

Local Coordinates for Three-Closed String Scattering





Fourier components of the Green's function on complex plane

$$\begin{aligned} G_C(\rho_r, \rho'_s) &= \ln |z_r - {z'}_s| \\ &= -\delta_{rs} \left\{ \sum_{n=1} \frac{e^{-n\Delta}}{2n} \left(e^{in(\eta'_s - \eta_r)} + e^{-in(\eta'_s - \eta_r)} \right) - \max(\xi, \xi') \right\} \\ &\quad + \sum_{n,m} \bar{C}^{rs}_{nm} e^{|n|\xi_r + |m|\xi'_s} e^{in\eta_r} e^{im\eta'_s}. \end{aligned}$$

Integral Formulas for $\bar{C}^{rs}_{nm}$

$$
\begin{aligned}
\bar{C}^{rs}_{00} &= \ln|Z_r - Z_s|, \quad r \neq s, \\
\bar{C}^{rr}_{00} &= -\sum_{i\neq r} \frac{\alpha_i}{\alpha_r} \ln|Z_r - Z_i| + \frac{1}{\alpha_r}\tau_0^{(r)} \\
\bar{C}^{rs}_{n0} &= \bar{C}^{sr}_{0n} = \frac{1}{2n}\oint_{Z_r} \frac{dz}{2\pi i}\frac{1}{z - Z_s} e^{-n\zeta_r(z)}, \quad n \geq 1, \\
\bar{C}^{rs}_{nm} &= \frac{1}{2nm}\oint_{Z_r}\frac{dz}{2\pi i}\oint_{Z_s}\frac{dz'}{2\pi i}\frac{1}{(z-z')^2} e^{-n\zeta_r(z) - m\zeta'_s(z')}, \quad n, m \geq 1.
\end{aligned}
$$

Reality conditions of the Green's function

$$
\bar{C}^{rs}_{nm} = \bar{C}^{rs*}_{-n-m}, \quad \bar{C}^{rs}_{-nm} = 0, \quad n, m \geq 1.
$$

Scattering Amplitude of Three Closed Strings

Scattering amplitude

$$
\begin{aligned}
\mathcal{W} &= \int DX \exp\left(i \sum_{r=1}^{M} \int P_r(\sigma) \cdot X(\tau_r, \sigma) d\sigma - \int d\tau d\sigma \mathcal{L} \right) \\
&= [\det \Delta]^{-d/2} \exp\left\{ \frac{1}{4} \left\{ \sum_r \xi_r P_0^2 - \sum_r \sum_{n=1} \frac{1}{n} P_n^{(r)} \cdot P_{-n}^{(r)} \right.\right. \\
&\quad \left.\left. + \sum_{n,m} \bar{C}^{rs}_{nm} e^{|n|\xi_r + |m|\xi'_s} P_{-n}^{(r)} \cdot P_{-m}^{(s)} \right\} \right. \\
&= \langle \mathbf{P} | \exp\left(\sum_r \xi_r L_0^{(r)} \right) | V[N] \rangle .
\end{aligned}
$$

Neumann Functions of Three-Closed-String Vertex

Neumann functions of three-closed string vertex and those of three-open string vertex:

$$
\begin{aligned}
\bar{C}^{rs}_{00} &= \bar{N}^{rs}_{00} = \ln|Z_r - Z_s|, \quad r \neq s, \\
\bar{C}^{rr}_{00} &= \bar{N}^{rr}_{00} = -\sum_{i\neq r} \frac{\alpha_i}{\alpha_r} \ln|Z_r - Z_i| + \frac{1}{\alpha_r}\tau_0, \\
\bar{C}^{rs}_{n0} &= \bar{C}^{rs}_{-n0} = \frac{1}{2}\bar{N}^{rs}_{n0} = \frac{1}{2n}\oint_{Z_r} \frac{dz}{2\pi i}\frac{1}{z - Z_s} e^{-n\zeta_r(z)}, \quad n \geq 1, \\
\bar{C}^{rs}_{nm} &= \bar{C}^{rs}_{-n-m} = \frac{1}{2}\bar{N}^{rs}_{mn} \\
&= \frac{1}{2nm}\oint_{Z_r}\frac{dz}{2\pi i}\oint_{Z_s}\frac{dz'}{2\pi i}\frac{1}{(z-z')^2} e^{-n\zeta_r(z) - m\zeta'_s(z')}, \quad n, m \geq 1, \\
\bar{C}^{rs}_{n-m} &= \bar{C}^{rs}_{-nm} = 0.
\end{aligned}
$$

Factorization of Three-Closed-String Scattering Amplitude

$$
\begin{aligned}
\mathcal{A}[1,2,3] \quad = \quad & g\langle\{\boldsymbol{k}^{(r)}\}| \\
& \exp\left\{\sum_{r,s}\Big(\sum_{n,m\geq 1}\frac{1}{2}\bar{N}^{rs}_{nm}\frac{\alpha^{(r)\dagger}_n}{2}\cdot\frac{\alpha^{(r)\dagger}_m}{2}+\sum_{n\geq 1}\bar{N}^{rs}_{n0}\frac{\alpha^{(r)\dagger}_n}{2}\cdot\frac{p^{(s)}}{2}\Big)\right\} \\
& \exp\left\{\tau_0\sum_r\frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2-1\right)\right\} \\
& \exp\left\{\sum_{r,s}\Big(\sum_{n,m\geq 1}\frac{1}{2}\bar{N}^{rs}_{nm}\frac{\tilde{\alpha}^{(r)\dagger}_n}{2}\cdot\frac{\tilde{\alpha}^{(r)\dagger}_m}{2}+\sum_{n\geq 1}\bar{N}^{rs}_{n0}\frac{\tilde{\alpha}^{(r)\dagger}_n}{2}\cdot\frac{p^{(s)}}{2}\Big)\right\} \\
& \exp\left\{\tau_0\sum_r\frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2-1\right)\right\}|0\rangle.
\end{aligned}
$$

Factorization of Three-Closed-String Scattering Amplitude

Factorization of three-closed-string scattering amplitude

$$\mathcal{A}_{\mathrm{closed}}[1,2,3] = \mathcal{A}_{\mathrm{open}}[1,2,3]\,\mathcal{A}_{\mathrm{open}}[1,2,3].$$

Scattering amplitude of three closed strings can be completely factorized into those of three open strings except for the zero modes.

Question: Can we factorize general closed string scattering amplitudes into those of open string theory?

Realization of the Kawai-Lewellen-Tye (KLT) relations in the framework of the second quantized string theory .

Three-Graviton Scattering Amplitude

Decomposition of the spin-2 field into graviton, anti-symmetric tensor, and scalar field

$$h_{\mu\nu} = \left\{\frac{1}{2}\left(h_{\mu\nu}+h_{\nu\mu}\right)-\eta_{\mu\nu}\frac{1}{d}h^{\sigma}{}_{\sigma}\right\}+\left\{\frac{1}{2}\left(h_{\mu\nu}-h_{\nu\mu}\right)\right\}+\eta_{\mu\nu}\left\{\frac{1}{d}h^{\sigma}{}_{\sigma}\right\}.$$

We choose the covariant gauge condition

$$\partial^{\mu}h_{\mu\nu}=0,$$

which becomes de Donder gauge condition for the graviton

$$\partial^{\mu}h_{\mu\nu}-\frac{1}{d-2}\partial_{\nu}h^{\sigma}{}_{\sigma}=0.$$

For three-graviton scattering, we choose the external string state as

$$|\Psi_{3G}\rangle=\prod_{r=1}^{3}\left\{h_{\mu\nu}(p^{r})\alpha_{-1}^{(r)\mu}\tilde{\alpha}_{-1}^{(r)\nu}\right\}|0\rangle.$$

Three-graviton scattering amplitude

$$\begin{aligned}
\mathcal{A}_{[3-graviton]} &= \int \prod_{r=1}^{3} dp^{(r)} \delta\left(\sum_{r=1}^{3} p^{(r)}\right) \frac{2g}{3} \langle \Psi_{3G} | E^{\text{Closed}}_{[3]}[1,2,3] | 0 \rangle \\
&= \left(\frac{2g}{3}\right) e^{-2\tau_0 \sum_{r=1}^{3} \frac{1}{\alpha_r}} \int \prod_{i=1}^{3} dp^{(i)} \delta\left(\sum_{i=1}^{3} p^{(i)}\right) \\
&\quad \langle 0 | \left\{ \prod_{i=1}^{3} h_{\mu\nu}(p^{(i)}) a_1^{(i)\mu} \cdot \tilde{a}_1^{(i)\nu} \right\} \frac{1}{2^5} \left(\sum_{r,s=1}^{3} \bar{N}^{rs}_{11} a_1^{(r)\dagger} \cdot a_1^{(s)\dagger} \right) \\
&\quad \left(\sum_{t=1}^{3} \bar{N}^{t}_{1} a_1^{(t)\dagger} \cdot \boldsymbol{p} \right) \frac{1}{2^5} \left(\sum_{l,m=1}^{3} \bar{N}^{lm}_{11} \tilde{a}_1^{(l)\dagger} \cdot \tilde{a}_1^{(m)\dagger} \right) \\
&\quad \left(\sum_{n=1}^{3} \bar{N}^{n}_{1} \tilde{a}_1^{(n)\dagger} \cdot \boldsymbol{p} \right) |0\rangle .
\end{aligned}$$

Three-Graviton Scattering Amplitude

We note that $\mathcal{A}_{[3-graviton]}$ can be written also as

$$
\begin{aligned}
\mathcal{A}_{[3-graviton]} &= \left(\frac{2g}{3}\right)\frac{1}{2^8}\int\prod_{i=1}^{3}dp^{(i)}\delta\left(\sum_{i=1}^{3}p^{(i)}\right) \\
&\quad \langle 0|\left\{\prod_{i=1}^{3}h_{\mu\nu}(p^{(i)})a_1^{(i)\mu}\cdot\tilde{a}_1^{(i)\nu}\right\}E^{\text{Open}}_{[3-Gauge]}\,\tilde{E}^{\text{Open}}_{[3-Gauge]}|0\rangle.
\end{aligned}
$$

Making use of the Neumann functions of the open string

$$
\begin{aligned}
\bar{N}^{11}_{11} &= \frac{1}{2^4}, \quad \bar{N}^{22}_{11} = \frac{1}{2^4}, \quad \bar{N}^{33}_{11} = 2^2, \\
\bar{N}^{12}_{11} &= \bar{N}^{21}_{11} = \frac{1}{2^4}, \quad \bar{N}^{23}_{11} = \bar{N}^{32}_{11} = \frac{1}{2}, \quad \bar{N}^{31}_{11} = \bar{N}^{13}_{11} = \frac{1}{2}, \\
\bar{N}^{1}_{1} &= \bar{N}^{2}_{1} = \frac{1}{4}, \quad \bar{N}^{3}_{1} = -1,
\end{aligned}
$$

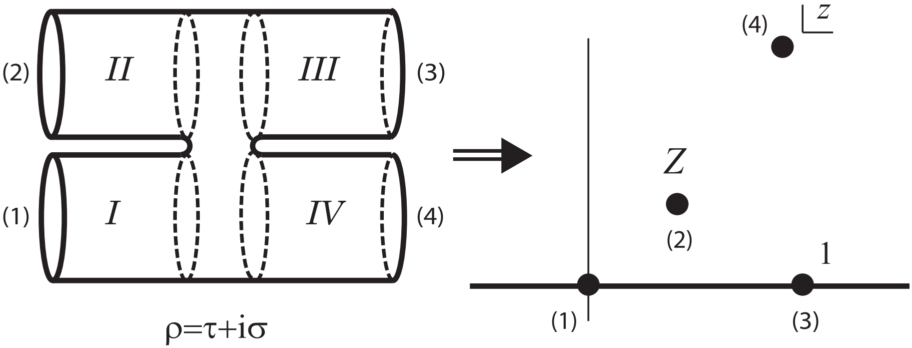
Three-Graviton Scattering Amplitude

$\mathcal{A}_{[3-graviton]}$ is precisely the three-graviton interaction term which may be obtained from the Einstein's gravity action.

$$
\begin{aligned}
\mathcal{A}_{[3-graviton]} \quad = \quad & \kappa \int \prod_{i=1}^{3} dp^{(i)} \delta\left(\sum_{i=1}^{3} p^{(i)}\right) \\
& h_{\mu_1\nu_1}(p^{(1)}) h_{\mu_2\nu_2}(p^{(2)}) h_{\mu_3\nu_3}(p^{(3)}) \\
& \left\{\eta^{\mu_1\mu_2} p^{(1)\mu_3} + \eta^{\mu_2\mu_3} p^{(2)\mu_1} + \eta^{\mu_3\mu_1} p^{(3)\mu_2}\right\} \\
& \left\{\eta^{\nu_1\nu_2} p^{(1)\nu_3} + \eta^{\nu_2\nu_3} p^{(2)\nu_1} + \eta^{\nu_3\nu_1} p^{(3)\nu_2}\right\}
\end{aligned}
$$

where $\kappa = \frac{g}{2^7 \cdot 3} = \sqrt{32\pi G_{10}}$.

Four-Closed-Scattering Amplitudes: Local coordinate patches



Conformal mapping from the string world-sheet to the complex plane

The conformal mapping from the local string patches to the complex plane may be written as follows :

$$e^{-\zeta_1} = e^{\tau_1 + i\sigma_1} \frac{(z-1)}{z(z-Z)}, \qquad \text{on the 1st string patch,}$$

$$e^{-\zeta_2} = -e^{\tau_1 + i\sigma_1} \frac{(z-1)}{z(z-Z)}, \qquad \text{on the 2nd string patch,}$$

$$e^{-\zeta_3} = e^{-\tau_2 - i\sigma_2} \frac{z(z-Z)}{z-1}, \qquad \text{on the 3rd string patch,}$$

$$e^{-\zeta_4} = -e^{-\tau_2 - i\sigma_2} \frac{z(z-Z)}{z-1}, \qquad \text{on the 4th string patch.}$$

Four-graviton scattering in the zero-slope limit

$$
\begin{aligned}
\mathcal{A}_{[4]} \quad = \quad & \frac{g^2\pi}{2^{10}} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \frac{1}{stu} \\
& \Bigg\{ \frac{ut}{2}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \frac{st}{2}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{su}{2}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \\
& -\eta^{\mu_1\mu_2}\left(tp^{(1)\mu_3}p^{(2)\mu_4} + up^{(2)\mu_3}p^{(1)\mu_4}\right) \\
& -\eta^{\mu_1\mu_3}\left(tp^{(1)\mu_2}p^{(3)\mu_4} + sp^{(3)\mu_2}p^{(1)\mu_4}\right) \\
& -\eta^{\mu_1\mu_4}\left(up^{(1)\mu_2}p^{(4)\mu_3} + sp^{(4)\mu_2}p^{(1)\mu_3}\right) \\
& -\eta^{\mu_2\mu_3}\left(up^{(2)\mu_1}p^{(3)\mu_4} + sp^{(3)\mu_1}p^{(2)\mu_4}\right) \\
& -\eta^{\mu_2\mu_4}\left(sp^{(4)\mu_1}p^{(2)\mu_3} + tp^{(2)\mu_1}p^{(4)\mu_3}\right) \\
& -\eta^{\mu_3\mu_4}\left(tp^{(3)\mu_1}p^{(4)\mu_2} + up^{(4)\mu_1}p^{(3)\mu_2}\right)\Bigg\}\Big\{\mu \Rightarrow \nu\Big\}.
\end{aligned}
$$

Four-Graviton Scattering Amplitudes

1. B. S. DeWitt, PRD **162**, 1239 (1967): Calculated the four-graviton scattering amplitude by perturbatively expanding the Einstein GR.
2. J. Schwarz, Phys. Rep. **89**, 223 (1982): Four-graviton scattering amplitude in the type-II super-string theory (by using graviton vertex operator).
3. S. Sannan, PRD **34**, 1749 (1986): Equivalence between the DeWitt's four-graviton scattering amplitude and that of string theory calculation.

What String Theory teaches us about QM Gravity

1. To describe gravity using string theory, we need only a cubic interaction term of closed strings.
2. **All local terms of higher power in Einstein action may be produced as parts of closed string diagrams generated by the cubic interaction.**
3. It may be possible to develop a perturbation theory of gravity, extending linearized gravity systematically in the framework of SFT.
4. KLT factorization works at the level of SFT.

Classical Solutions with D-Brane Sources and Black p-Branes

Closed string field theory action

$$S = \int \left\{ \frac{1}{2}\langle\Phi|\mathcal{K}\Phi\rangle + \frac{g}{3}\left(\langle\Phi|\Phi\circ\Phi\rangle + \langle\Phi\circ\Phi|\Phi\rangle + \langle\Phi|D_p\rangle\right\}.$$

Classical equation of motion with a D-brane source, J_D

$$\mathcal{K}\Phi + g\Phi\circ\Phi = J_D.$$

Perturbative solutions:

$$\Phi = \frac{1}{\mathcal{K}-i\epsilon}J_D - g\frac{1}{\mathcal{K}-i\epsilon}\left\{\frac{1}{\mathcal{K}-i\epsilon}J_D \circ \frac{1}{\mathcal{K}-i\epsilon}J_D\right\} + \dots .$$

The zero-th order by P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda, and R. Russo (1997).

Conclusions and Future Works

1. M-Loop Diagrams of YM
   $\Leftrightarrow$ M-Loop Open String Diagrams with Vertex Operators
   $\Leftrightarrow$ Tree Diagrams of Closed String with M external states
2. Perturbative Approaches to AdS/CFT
3. Perturbative QM Gravity, extending Linearized Gravity
4. BRST invariant Formulations
5. Super-symmetric Extensions