# From Navier-Stokes to Maxwell via Einstein

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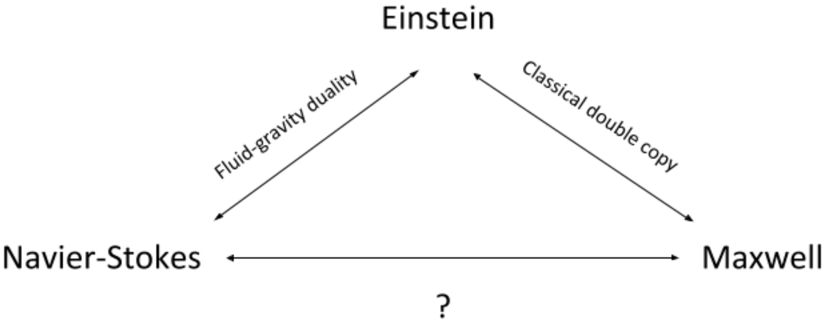

Einstein
Fluid-gravity duality
Classical double copy
Navier-Stokes
Maxwell
?

# Outline

- Fluid gravity duality (Rindler background)
- Classical double copy
- Petrov type $\Leftrightarrow$ restricted fluid solutions
- Type D double copy
- Type N double copy

# Navier-Stokes $\leftrightarrow$ Einstein

**Fluid gravity duality:** Spacetime fluctuations near a horizon described by Navier-Stokes[1]

Rindler: $ds_0^2 = -r d\tau^2 + 2 d\tau dr + \delta_{ij} dx^i dx^j$,

at $r = r_c$, perturb, identify $\kappa_{ab} - \gamma_{ab}\kappa \propto T_{ab}^{NS}$, demand reg. & infalling BCs at $r = 0$ generate

$$\begin{aligned} ds^2 = ds_0^2 &- 2(1 - r/r_c) v_i dx^i d\tau - 2(v_i/r_c) dx^i dr \\ &+ (1 - r/r_c)[(v^2 + 2P) d\tau^2 + (v_i v_j/r_c) dx^i dx^j] + (v^2 + 2P)/r_c d\tau dr \\ &- (r^2 - r_c^2)/r_c \partial^2 v_i dx^i d\tau + O(\epsilon^3), \end{aligned}$$

$$\boxed{\begin{aligned} G_{\tau\tau} &= 0 + O(\epsilon^4) \Rightarrow \partial_i v_i = 0, \\ G_{\tau i} &= 0 + O(\epsilon^4) \Rightarrow \partial_\tau v_i - \eta \partial^2 v_i + \partial_i P + (v \cdot \partial) v_i = 0 \end{aligned}}$$

[1] 1101.2451

# Hydrodynamic limit

-Derivatives and fields must satisfy

$$\partial_i \sim \epsilon, \quad \partial_\tau \sim \epsilon^2, \quad v \sim \epsilon, \quad P \sim \epsilon^2,$$

for $\epsilon$ being order in hydrodynamic expansion

If $(v_i(x^i,\tau), P(x^i,\tau))$ is a solution,

$$v_i^\epsilon(x^i,\tau) = \epsilon v_i(\epsilon x^i, \epsilon^2\tau)$$
$$P^\epsilon(x^i,\tau) = \epsilon^2 P(\epsilon x^i, \epsilon^2\tau)$$

is a solution for $\epsilon^2 \to 0$.

- Equivalent to near horizon limit via coordinate rescaling

- All double copy results are valid only up to some order of $\epsilon$

# BCJ double copy

- Color factors $c_k$ satisfy Jacobi $c_i + c_j + c_k = 0$, grav. amp. obtained from YM amp. via[2]

$$\mathcal{A}^{\text{YM}} \sim \sum_k \frac{n_k c_k}{\text{props}} \xrightarrow{\quad c_k \to \tilde{n}_k \quad} \sum_k \frac{n_k \tilde{n}_k}{\text{props}} \sim \mathcal{M}^{\text{grav}}$$

where kinematic numerators satisfy $\tilde{n}_i + \tilde{n}_j + \tilde{n}_k = 0$

- Zeroth copy for bi-adjoint scalars $\phi^{aa'}$

$$\mathcal{A}^{\text{YM}} \sim \sum_k \frac{n_k c_k}{\text{props}} \xrightarrow{\quad n_k \to \tilde{c}_k \quad} \sum_k \frac{c_k \tilde{c}_k}{\text{props}} \sim \mathcal{A}^{\text{scalar}}$$

[2] 1909.01358

# Classical double copy: Kerr-Schild

- Kerr-Schild coords[3] $g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu$, w/ $g^{\mu\nu} k_\mu k_\nu = \eta^{\mu\nu} k_\mu k_\nu = 0$

**if $g_{\mu\nu}$ satisfies Einstein, then**

$$\boxed{A^a_\mu = \phi c^a k_\mu}$$

**satisfies U(1) YM over $\eta_{\mu\nu}$.** Like BCJ s.t.

$$A^a_\nu = \phi c^a k_\nu \quad \xrightarrow{c^a \to k_\mu} \quad \phi k_\mu k_\nu = h_{\mu\nu}.$$

- Zeroth copy here is

$$A^a_\nu = \phi c^a k_\nu \quad \xrightarrow{k_\nu \to c^{a'}} \quad \phi c^a c^{a'} = \phi^{aa'}$$

[3] 1410.0239

# Classical double copy: Weyl

- Spinor formalism: field strength $f_{AB}$ and Weyl spinor $C_{ABCD}$,

$$f_{AB} = \frac{1}{2} F_{\mu\nu} \sigma^{\mu\nu}_{AB}, \quad C_{ABCD} = \frac{1}{4} W_{\mu\nu\alpha\beta} \sigma^{\mu\nu}_{AB} \sigma^{\alpha\beta}_{CD},$$

w/ $\sigma^{\mu\nu}_{AB} = \sigma^{[\mu}_{A\dot{C}} \bar{\sigma}^{\nu]\dot{C}}{}_{B}$, and $\sigma^{\mu}_{A\dot{A}} = e^{\mu}{}_{a} \sigma^{a}_{A\dot{A}}$ for usual $\sigma^a = (1, \vec{\sigma})$

- **If $C_{ABCD}$ built from Petrov type D or (some) type N vacuum solution, then**

$$\boxed{C_{ABCD} = \frac{1}{S} f_{(AB} f_{CD)}}$$

**is s.t. $f_{AB}$ satisfies Maxwell.** Zeroth $S$ satisfies $\Box^{(0)} S = 0$ over flat background

# Petrov type $\Rightarrow$ restricted fluid solutions

- Weyl scalars $\Psi_I$, spinor basis $\{o_A, \iota_B\}$

$$C_{ABCD} = \Psi_0 \iota_A \iota_B \iota_C \iota_D - 4\Psi_1 o_{(A} \iota_B \iota_C \iota_{D)} + 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)} - 4\Psi_3 o_{(A} o_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D$$

* Restrictions amount to Type D $\sim \Psi_{I\neq 2} = 0$, type N $\sim \Psi_{I\neq 4} = 0$

- For fluids metric,

$$\Psi_0 = \Psi_1 = \Psi_3 = 0 + O(\epsilon^3)$$

$$\Psi_2 = -i\frac{\epsilon^2}{4r_c}\partial_{[x} v_{y]} + O(\epsilon^3)$$

$$\Psi_4 = -\frac{\epsilon^2}{2r}(\partial_x v_x - \partial_y v_y + i\partial_{\{x} v_{y\}}) + O(\epsilon^3)$$

# Type D double copy: constant vorticity

- Demand $\Psi_4 = 0$, N-S solution

$$v_x = -\omega y, \quad v_y = \omega x, \quad P = \frac{\omega^2}{2}(x^2+y^2) \quad \Rightarrow \quad \Psi_2 = -i\epsilon^2 \frac{\omega}{2r_c} + O(\epsilon^3)$$

- Weyl double copy $C_{ABCD} = \frac{1}{S} f_{(AB} f_{CD)}$ gives

$$S = \frac{i\omega r_c}{3} e^{2i\theta}, \qquad f_{AB} = e^{i\theta}\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $\Box^{(0)} S = 0$ trivially. Invert $f_{AB}$ to get $F_{\mu\nu}$, E's and B's

$$E_\nu = F_{\nu\mu}\xi^\mu = \omega\cos\theta\delta^r_\nu, \qquad B_\nu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\xi^\mu = -\omega\sin\theta\delta^r_\nu$$

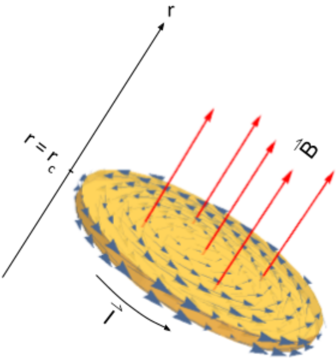
w/ timelike Killing $\xi = \partial_\tau$

# Type D double copy: constant vorticity

- Set global phase $\theta = -\pi/2$, left w/

$$\boxed{\vec{B} = \omega \hat{r}}$$

- $\vec{B} = \nabla \times \vec{A}$ for $\vec{A} \propto \vec{v}$ (up to gauge), 'current' $I \sim \omega/n$



## Type N double copy: potential flows

- Demand $\Psi_2 \propto \partial_{[x} v_{y]} = 0 \Rightarrow v_i = \partial_i \phi$ for holomorphic + antiholomorphic

$$\phi(z, \bar{z}) = f(z) + \bar{f}(\bar{z}), \qquad z = x + iy, \qquad \bar{z} = z^*$$

- $\Psi_4 = -\frac{2}{r} \partial_{\bar{z}}^2 \bar{f}(\bar{z})$ in terms of $\phi(z, \bar{z})$

- Weyl double copy $C_{ABCD} = \frac{1}{S} f_{(AB} f_{CD)}$

$$S = -\frac{e^{2i\theta}}{2\partial_{\bar{z}}^2 \bar{f}(\bar{z})}, \qquad f_{AB} = \frac{e^{i\theta}}{\sqrt{r}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- $\Box^{(0)} S = 0$ non-trivially satisfied for *arbitrary* $\phi$

# Type N double copy: potential flows

| Solution | Potential $\phi(z, \bar{z})$ | $\Psi_4$ |
|---|---|---|
| Source/Sink | $\alpha \ln(z\bar{z})$ | $2\ \alpha\, r^{-1}\bar{z}^{-2}$ |
| Source to Sink (dipole) | $\frac{\alpha\delta}{2}\frac{z+\bar{z}}{z\bar{z}}$ | $2\, r^{-1}\alpha\delta\bar{z}^{-3}$ |
| Line Vortex | $\frac{\alpha}{2i}\ln\left(\frac{z}{\bar{z}}\right)$ | $i\alpha\, r^{-1}\bar{z}^{-2}$ |
| Extensional flow | $-\frac{\alpha}{4}(z^2+\bar{z}^2)$ | $\frac{\alpha}{r}$ |

Table: Some examples of standard fluid solutions and the corresponding non-vanishing scalar $\Psi_4$ for type N solutions. For the dipole flow, $\delta$ refers to the distance between the source and the sink.

# Type N double copy: potential flows

- Invert $f_{AB}$ to get $F_{\mu\nu}$, E's and B's

$$\vec{E} = (0, \sin\theta, -\cos\theta), \qquad \vec{B} = (0, 0, \cos\theta, -\sin\theta)$$

- Choosing $\theta = -\pi/2$, get non-zero Poynting

$$\vec{S} = -\hat{r},$$

dissipation: grav. dual carries energy from $r = r_c$ slice towards null horizon, same flow seen in EM fields

## Summary and Results

- Start w/ Rindler, perturb near horizon at $r = r_c$, generate fluid metric demanding Einstein eqns hold

- Compute Weyl scalars, restrict velocity fields $\Leftrightarrow$ Petrov type D or N, evaluate Weyl double copy

- Can think of Helmholtz decomp. $v_i = \partial_i \phi + \varepsilon_{ijk} \partial_j V_k$ for $\vec{V} = |V|(\hat{x} \times \hat{y})$, contains both cases:

Type D: $\phi = 0, \quad \vec{V} \propto \vec{A},$

constant vorticity $\Leftrightarrow$ solenoid w/ $I \propto \omega / n$,

type N: $\vec{V} = 0, \quad \phi = f(z) + \bar{f}(\bar{z}),$

potential flows $\Leftrightarrow$ constant, ortho E and B, dissipation

# Outlook

- Any fluid-dual metric map to single copy gauge field and zeroth copy scalar, a la Helmholtz decomp.?

- Horizons story: interplay between single copy $\vec{E}$ and $\vec{B}$ and horizon locations[4]; insight from fluid-grav.?

- AdS-CFT perspective: where/how does double copy fit in?

[4] D. Easson, C. Keeler, T.M. 2007.16186