Superconformal index of M2-branes and giant graviton expansion

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Superconformal index

$$I = \mathrm{Tr}\Big[(-1)^F \prod_i q_i^{C_i}\Big] = \mathrm{Tr}\Big[(-1)^F e^{-\beta\{Q,Q^\dagger\}} \prod_i q_i^{C_i}\Big] = \mathrm{Tr}_{\{Q,Q^\dagger\}=0}\Big[(-1)^F \prod_i q_i^{C_i}\Big]$$

$$([Q, C_i] = 0)$$

Only BPS ground states contributes

Finite number at each order in q_i

$$I = 1 + \bigcirc q_1 + \bigcirc q_1^2 q_2 + \cdots$$

Independent of continuous parameters

➡ calculable for strong coupling ➡ useful for check of dualities, holography, symmetry enhancement, ...

General structure in holographic theories - 1

Theories on *N* D3-branes $\longleftrightarrow$ $\mathrm{AdS}_5 \times Y_5$

" M2-branes $\longleftrightarrow$ $\mathrm{AdS}_4 \times Y_7$

" M5-branes $\longleftrightarrow$ $\mathrm{AdS}_7 \times Y_4$

Large *N* limit exists:

$$I_N = 1 + a_1^{(N)} q + a_2^{(N)} q^2 + \cdots$$

➡ $a_n^{(N)}$ for all finite *n* saturate to finite values $a_n^{(\infty)}$ as $N \to \infty$

I_∞ = superconformal index of graviton multiplet in $\mathrm{AdS}_m \times Y_n$

[Kinney,Maldacena,Minwalla,Raju,'05][Bhattacharya,Bhattacharyya,Minwalla,Raju,'08]

General structure in holographic theories - 2

$$I_{N=1} = 1 + a_1^{(\infty)} q + a_2^{(1)} q^2 + a_3^{(1)} q^3 + a_4^{(1)} q^4 + \cdots$$
$$I_{N=2} = 1 + a_1^{(\infty)} q + a_2^{(\infty)} q^2 + a_3^{(2)} q^3 + a_4^{(2)} q^4 + \cdots$$
$$I_{N=3} = 1 + a_1^{(\infty)} q + a_2^{(\infty)} q^2 + a_3^{(\infty)} q^3 + a_4^{(3)} q^4 + \cdots$$
$$\vdots$$

$$\boxed{\frac{I_N}{I_\infty} = 1 + \mathcal{O}(q^{N+1})}$$

Deviation at finite *N* is due to "overcounting" of gauge invariant operators

c.f. trace constraint on $\mathrm{tr} X^n$

$$\mathrm{tr} X^{N+1} = f(\mathrm{tr} X, \mathrm{tr} X^2, \cdots \mathrm{tr} X^N)$$

Gravity interpretation

Finite N ➡ finite $S^n \subset \mathrm{AdS}_m \times S^n$ ➡ "giant gravitons" with bounded size

[McGreevy,Susskind,Toumbas,'00]

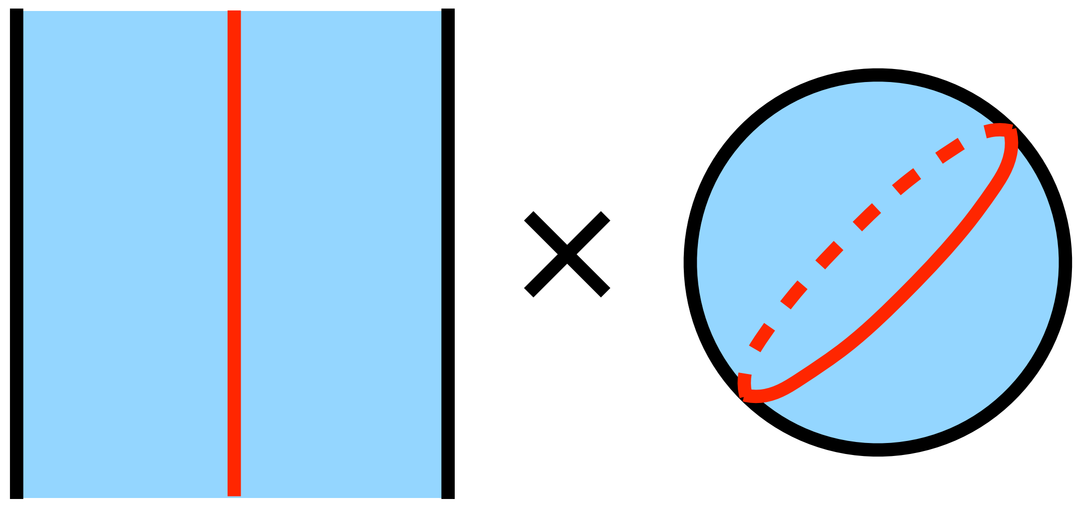
Can also be interpreted as wrapped branes [Mikhailov,'00]

$\mathrm{AdS}_5 \times S^5$ ➡ D3 on $\mathbb{R} \times S^3$

$\mathrm{AdS}_4 \times S^7$ ➡ M5 on $\mathbb{R} \times S^5$

$\mathrm{AdS}_7 \times S^4$ ➡ M2 on $\mathbb{R} \times S^2$

energy $\sim (R_{S^n})^{p+1} \sim N$



$$\frac{I_N}{I_\infty} = 1 + c_1(q)q^N + c_2(q)q^{2N} + \cdots$$

: "giant graviton expansion"

fluctuation modes around wrapped brane

fluctuation modes around 2 wrapped branes

Self-similarity structure

Fluctuation modes = superconformal index on wrapped branes

$$\frac{I_{N\,\mathrm{D3}}(q)}{I_{\infty\,\mathrm{D3}}(q)} = 1 + \sum_{m=1}^{\infty} q^{mN} I_{m\,\mathrm{D3}}(q')$$

$$\frac{I_{N\,\mathrm{M2}}(q)}{I_{\infty\,\mathrm{M2}}}(q) = 1 + \sum_{m=1}^{\infty} q^{mN} I_{m\,\mathrm{M5}}(q')$$

$$\frac{I_{N\,\mathrm{M5}}(q)}{I_{\infty\,\mathrm{M5}}}(q) = 1 + \sum_{m=1}^{\infty} q^{mN} I_{m\,\mathrm{M2}}(q')$$

Q. Can we see these relations analytically?

→ difficult...
- Determine all giant graviton coefficients
- Obtain $I_N(q)$ in closed form for general *N*

Solution: simplified indices

Simplified indices preserving more SUSY can be solved exactly

4d $\mathcal{N}=4$ SYM in Schur limit (= 1/4 BPS)

➡ $I_N(q)$ can be calculated exactly by Fermi gas formalism [Bourdier,Drukker,Felix,'15][Hatsuda,Okazaki,'22]

⬇

Detailed analysis of giant graviton expansion

- Different expansions & analytic continuations [Arai,Imamura,19][Gaiotto,Lee,'21][Imamura,'22]...
- Multi-D3 giant from gravity side [Eleftheriou,Murthy,Rossello,'23][Deddo,Liu,Pando Zayas,Saskowski,'24]...
- Other gauge groups [Fujiwara,Imamura,Mori,Murayama,Yokoyama,'23][Du,Huang,Wang,'23]...
- With Wilson lines [Imamura,'24][Beccaria,'24]

⋮

This talk: Same analysis can be done for M2 indices in Coulomb limit

Plan of talk

1. 3d U(*N*) ADHM theory

2. Coulomb limit and giant graviton expansion

3. M5-interpretation ($m = 1, \infty$)

4. M5-interpretation (general *m*)

Superconformal index of 3d $\mathcal{N}=4$

- spacetime: $p_\mu,\ j_{\mu\nu},\ h,\ k_\mu$
- super(conformal): $\mathcal{Q}^I_\alpha,\ \mathcal{S}^I_\alpha$
- R-symmetry: $SO(4) = SU(2)_H \times SU(2)_C$

Choose one $\mathcal{Q}^I_\alpha$

	h	j_{12}	H	C	f_C	f_H
$\mathcal{Q}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	1	0	0

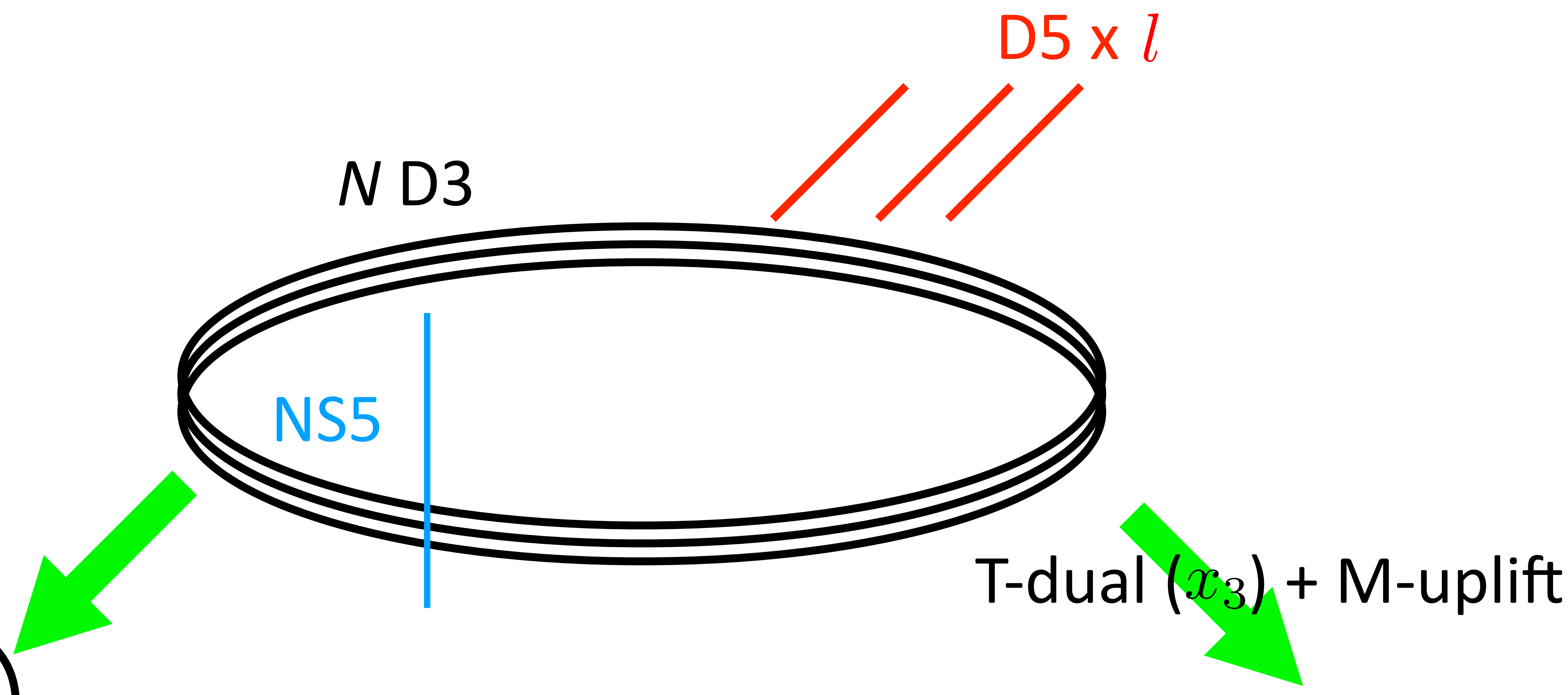
$$\left(\{\mathcal{Q},\mathcal{Q}^\dagger\} = h - j_{12} - \frac{C+H}{2}\right)$$

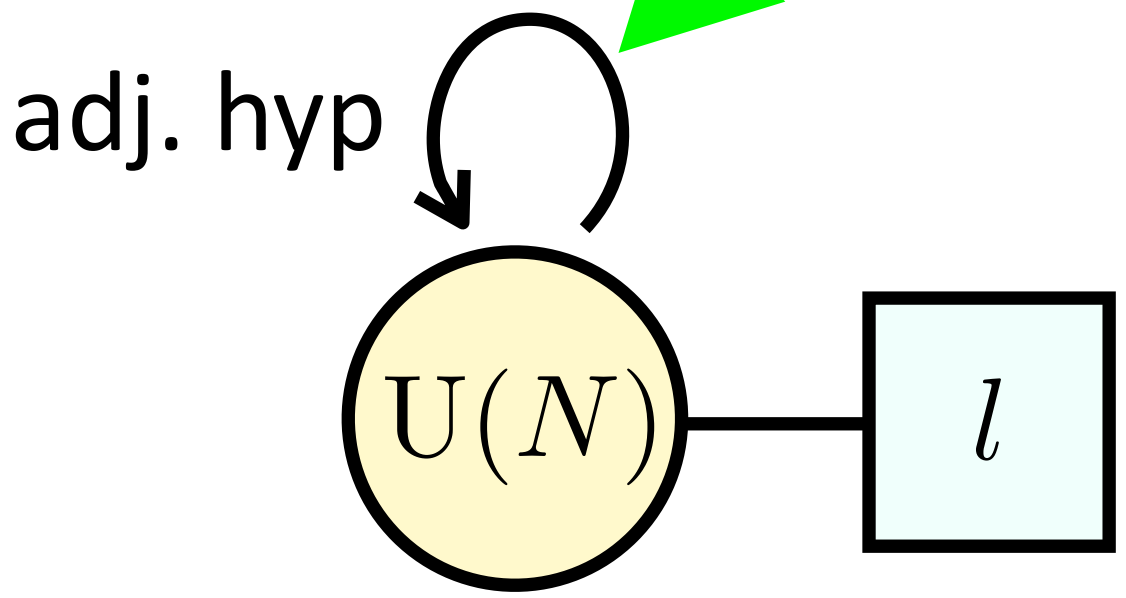
$$I = \mathrm{Tr}[(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} q^{j_{12}+\frac{H+C}{4}} t^{H-C} z^{f_C} x^{f_H}]$$

$[\cdot, \mathcal{Q}] = 0$ ⇒ β-indep.

M2 theories in IIB brane construction

IIB	0	1	2	3(S^1)	4	5	6	7	8	9
D3	-	-	-	-						
NS5	-	-	-		-	-	-			
D5	-	-	-					-	-	-





$=$ N M2 on $\mathbb{C}^2/\mathbb{Z}_l \times \mathbb{C}^2$

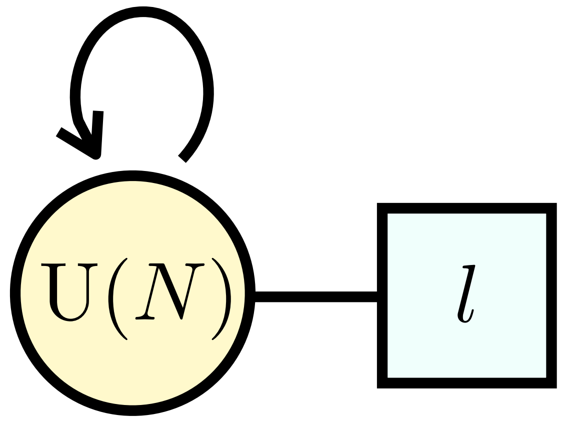
[de Boer,Hori,Ooguri,Oz,'96][Benini,Closset,Cremonesi,'09]

This theory is called "ADHM model"

Superconformal index of ADHM model

Superconformal index of gauge theory is constructed by

single particle index ➡ multi-particle uplift $\left(\sum_n a_n x^n \overset{\text{PE}}{\to} \prod_n \frac{1}{(1-x^n)^{a_n}}\right)$ ➡ gauge singlet



$$I_N = \frac{1}{N!}\sum_{m_i\in\mathbb{Z}}\int \frac{ds_i}{2\pi i s_i}\prod_{i\neq j}\left(1-q^{\frac{|m_i-m_j|}{2}}\frac{s_i}{s_j}\right)\prod_{i,j}\frac{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}}t^2\frac{s_i}{s_j};q)_\infty}{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}}t^{-2}\frac{s_i}{s_j};q)_\infty}\prod_{i,j}\prod_{\pm}\frac{(q^{\frac{3}{4}+\frac{|m_i-m_j|}{2}}t^{-1}x^{\pm1}\frac{s_i}{s_j};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_j|}{2}}tx^{\pm1}\frac{s_i}{s_j};q)_\infty}$$

$$\times\prod_i\prod_{\alpha=1}^{l}\prod_{\pm}\frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}}t^{-1}(s_iy_\alpha)^{\pm1};q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}}t(s_iy_\alpha)^{\pm1};q)_\infty} z^{l\sum_i m_i} q^{\frac{l}{4}\sum_i|m_i|} t^{-l\sum_i|m_i|}$$

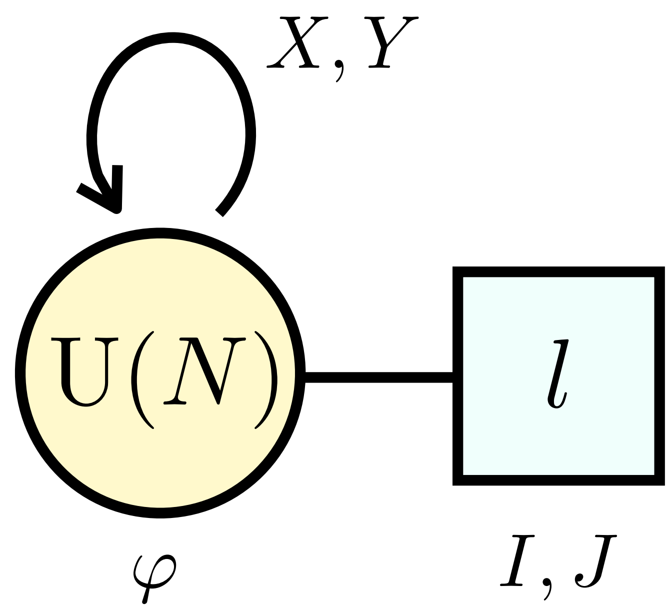
m_i : monopole flux on S^2 [Kim,'09]

$$\left(\frac{1}{(x;q)_\infty} = \prod_{n=0}^{\infty}\frac{1}{1-xq^n} = \mathrm{PE}[x+\underset{\uparrow\,\partial}{qx}+\underset{\uparrow\,\partial^2}{q^2x}+\cdots]\right)$$

How to calculate in small *q* expansion:

truncate $\sum_{m_i}\left(q^\nu \to \text{at most } |m_i|\leq\frac{4\nu}{l}\right)$ ➡ expand in *q* ➡ keep only $s_1^0 s_2^0\cdots s_N^0$

Examples



$$I^{\mathrm{U(1)ADHM\text{-}[1]}} = 1 + \left[(\underbrace{x}_{X} + \underbrace{x^{-1}}_{Y})t + (\underbrace{z}_{v^1} + \underbrace{z^{-1}}_{v^{-1}})t^{-1}\right]q^{1/4}$$

$$+ \left[\underbrace{xz}_{v^1X} + \underbrace{x^{-1}z^{-1}}_{v^{-1}Y} + \underbrace{xz^{-1}}_{v^{-1}X} + \underbrace{x^{-1}z}_{v^1Y} + (\underbrace{1}_{XY} + \underbrace{x^2}_{X^2} + \underbrace{x^{-2}}_{Y^2})t^2 + (\underbrace{1}_{\varphi} + \underbrace{z^2}_{v^2} + \underbrace{z^{-2}}_{v^{-2}})t^{-2}\right]q^{1/2}$$

$$+ \Big[(\underbrace{x^2z}_{v^1X^2} + \underbrace{x^{-2}z^{-1}}_{v^{-1}Y^2} + \underbrace{x^{-2}z}_{v^1Y^2} + \underbrace{x^2z^{-1}}_{v^{-1}X^2})t + (\underbrace{xz^2}_{v^2X} + \underbrace{x^{-1}z^{-2}}_{v^{-2}Y} + \underbrace{xz^{-2}}_{v^{-2}X} + \underbrace{x^{-1}z^2}_{v^2Y})t^{-1}$$

$$+ (\underbrace{x}_{X^2Y} + \underbrace{x^{-1}}_{XY^2} + \underbrace{x^3}_{X^3} + \underbrace{x^{-3}}_{Y^3})t^3 + (\underbrace{z}_{v^1\varphi} + \underbrace{z^{-1}}_{v^{-1}\varphi} + \underbrace{z^3}_{v^3} + \underbrace{z^{-3}}_{v^{-3}})t^{-3}\Big]q^{3/4} + \cdots$$

$$I^{\mathrm{U(2)ADHM\text{-}[1]}} = 1 + \left[(\underbrace{x}_{\mathrm{Tr}X} + \underbrace{x^{-1}}_{\mathrm{Tr}Y})t + (\underbrace{z}_{v^{1,0}} + \underbrace{z^{-1}}_{v^{-1,0}})t^{-1}\right]q^{1/4} + \Big[\underbrace{2xz}_{\substack{v^{1,0}X^{(1)},\\ v^{1,0}X^{(2)}}} + \underbrace{2x^{-1}z^{-1}}_{\substack{v^{-1,0}Y^{(1)},\\ v^{-1,0}Y^{(2)}}} + \underbrace{2xz^{-1}}_{\substack{v^{-1,0}X^{(1)},\\ v^{-1,0}X^{(2)}}} + \underbrace{2x^{-1}z}_{\substack{v^{1,0}Y^{(1)},\\ v^{1,0}Y^{(2)}}}$$

$$+ (\underbrace{2}_{\substack{\mathrm{Tr}XY,\\ \mathrm{Tr}X\mathrm{Tr}Y}} + \underbrace{2x^2}_{\substack{\mathrm{Tr}X^2,\\ (\mathrm{Tr}X)^2}} + \underbrace{2x^{-2}}_{\substack{\mathrm{Tr}Y^2\\ (\mathrm{Tr}Y)^2}})t^2 + (\underbrace{2}_{\substack{\mathrm{Tr}\varphi,\\ v^{1,-1}}} + \underbrace{2z^2}_{\substack{v^{2,0},\\ v^{1,1}}} + \underbrace{2z^{-2}}_{\substack{v^{-2,0},\\ v^{-1,-1}}})t^{-2}\Big]q^{1/2} + \cdots$$

[Hayashi,TN,Okazaki,'22]

Increasingly difficult for larger *N* (practically $N \sim$ 5 or 6) & higher order in *q*

Plan of talk

✓ 1. 3d U(*N*) ADHM theory

2. Coulomb limit and giant graviton expansion

3. M5-interpretation ($m = 1, \infty$)

4. M5-interpretation (general *m*)

Coulomb limit $q, t \to 0, \quad \mathfrak{t} = q^{\frac{1}{4}} t^{-1}$

q,t-dependence of each term is $q^{\frac{n}{4}} t^{-n\sim n}$ ➡ keep only extreme operators $q^{\frac{n}{4}} t^{-n}$

$$I_N = \frac{1}{N!} \sum_{m_i \in \mathbb{Z}} \int \frac{ds_i}{2\pi i s_i} \prod_{i\neq j} \boxed{\left(1 - q^{\frac{|m_i - m_j|}{2}} \frac{s_i}{s_j}\right)} \prod_{i,j} \frac{\cancel{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}} t^2 \frac{s_i}{s_j}; q)_\infty}}{\boxed{(q^{\frac{1}{2}+\frac{|m_i-m_j|}{2}} t^{-2} \frac{s_i}{s_j}; q)_\infty}} \cancel{\prod_{i,j} \prod_{\pm} \frac{(q^{\frac{3}{4}+\frac{|m_i-m_j|}{2}} t^{-1} x^{\pm 1} \frac{s_i}{s_j}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i-m_j|}{2}} t x^{\pm 1} \frac{s_i}{s_j}; q)_\infty}}$$
$$\times \cancel{\prod_i \prod_{\alpha=1}^{l} \prod_{\pm} \frac{(q^{\frac{3}{4}+\frac{|m_i|}{2}} t^{-1} (s_i y_\alpha)^{\pm 1}; q)_\infty}{(q^{\frac{1}{4}+\frac{|m_i|}{2}} t (s_i y_\alpha)^{\pm 1}; q)_\infty}} z^{l \sum_i m_i} q^{\frac{l}{4}\sum_i |m_i|} t^{-l \sum_i |m_i|}$$

$$\rightarrow \begin{cases} 1 - (\mathfrak{t}^2) \frac{s_i}{s_j} & (m_i = m_j) \\ 1 & (\text{otherwise}) \end{cases}$$ ➡ $\int d^N s$ factorize in pieces

It is useful to label monopole charges $\{m_i\}$ by $\nu_m = \#\{i \,|\, m_i = m\}$

➡ $$I_N \to \mathcal{I}_N^{(C)} = \sum_{\substack{\nu_m \geq 0 \\ (\sum_m \nu_m = N)}} \left(\prod_{m=-\infty}^{\infty} \frac{1}{\nu_m!} \int d^{\nu_m} \sigma (\cdots) \right)$$

$$\left(\text{example: } m_i = (1, 0, -3, 2, 1) \to \nu_m = \begin{cases} 1 & (m = -3, 0, 2) \\ 2 & (m = 1) \\ 0 & (otherwise) \end{cases} \right)$$

Coulomb limit as Fermi gas

Constrained sum can be simplified by considering grand canonical index

$$\Xi(\kappa)=\sum_{N=0}^{\infty}\mathcal{I}_N^{(C)}=\prod_{m=-\infty}^{\infty}\tilde{\Xi}(\mathfrak{t}^{l|m|}z^{lm}\kappa)\qquad \tilde{\Xi}(\kappa)=\sum_{\nu=0}^{\infty}\kappa^{\nu}\frac{1}{\nu!}\int\frac{d\sigma_i}{2\pi i\sigma_i}\frac{\prod_{i\neq j}^{\nu}(1-\frac{\sigma_i}{\sigma_j})}{\prod_{i,j=1}^{\nu}(1-\mathfrak{t}^2\frac{\sigma_i}{\sigma_j})}$$

$$\tilde{\Xi}(\kappa)=\sum_{\nu=0}^{\infty}\kappa^{\nu}\frac{\mathfrak{t}^{-\nu(\nu-1)}}{\nu!}\int\frac{d\sigma_i}{2\pi i\sigma_i}\det_{i,j}\left[\frac{1}{\frac{\sigma_i}{\sigma_j}-\mathfrak{t}^2}\right]$$

→ ν-particle free Fermions on S^1_α $\qquad \sigma=e^{2\pi i\alpha}$

$\langle\alpha_i|\rho(\hat{p})|\alpha_j\rangle$

$\mathrm{tr}\hat{\rho}^n$ can be calculated in momentum basis as

$$\mathrm{tr}\hat{\rho}^n=\frac{1}{1-\mathfrak{t}^{2n}}$$

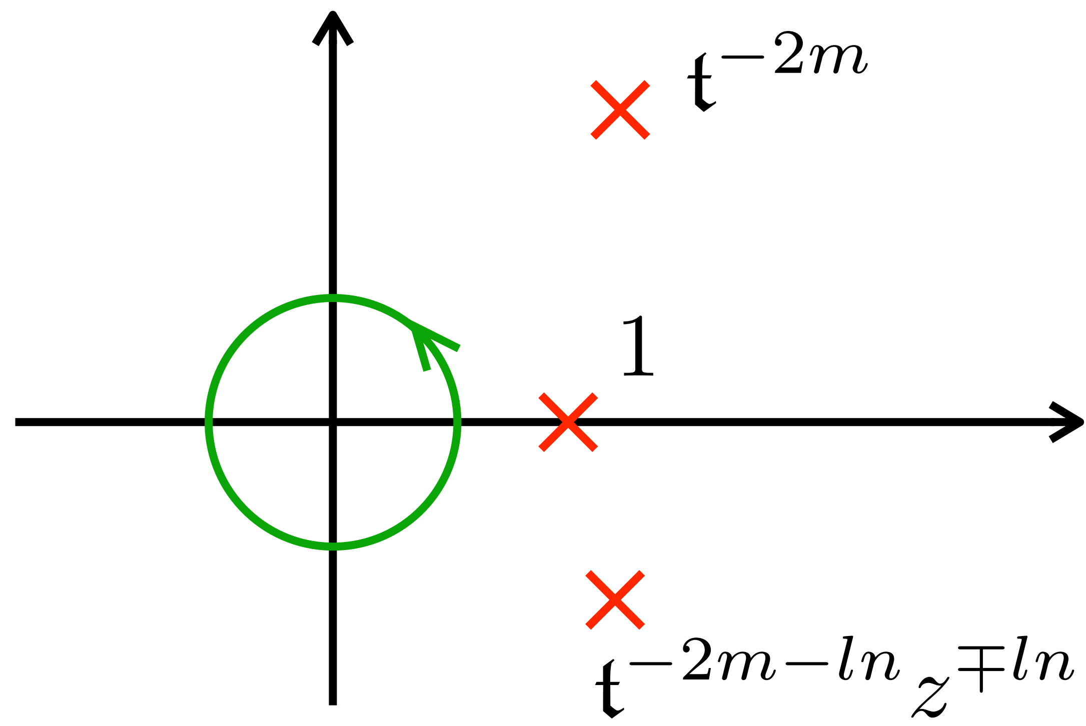
→

$$\Xi(\kappa)=\prod_{m=0}^{\infty}\frac{1}{1-\mathfrak{t}^{2m}\kappa}\prod_{m=0}^{\infty}\prod_{n=1}^{\infty}\prod_{\pm}\frac{1}{1-\mathfrak{t}^{2m+ln}z^{\pm ln}\kappa}$$

A "giant graviton" expansion

$$\mathcal{I}_N^{(C)} = \oint_{|\kappa|=\epsilon} \frac{d\kappa}{2\pi i\kappa}\kappa^{-N}\Xi(\kappa) = -\sum_{w\neq 0}\mathrm{Res}\left[\frac{\kappa^{-N}}{\kappa}\Xi(\kappa), \kappa\to w\right]$$

$\kappa = 1 \to \mathcal{I}_\infty^{(C)}$ $\qquad$ $\kappa = \mathfrak{t}^{-\bigcirc} \to \mathfrak{t}^{\bigcirc N}$



$$\frac{\mathcal{I}_N^{(C)}}{\mathcal{I}_\infty^{(C)}} = 1 + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left[f_{m,n}^{(l)}(z^{lm}\mathfrak{t}^{2n+lm})^N + g_{m,n}^{(l)}(z^{-lm}\mathfrak{t}^{2n+lm})^N\right] + \sum_{m=1}^{\infty} h_m^{(l)}\mathfrak{t}^{2mN}$$

[Gaiotto,Lee,'21][Hayashi,TN,Okazaki,'22]

For $l = 1$

$$\frac{\mathcal{I}_N^{l=1(C)}}{\mathcal{I}_\infty^{l=1(C)}} = \sum_{m,n\geq 0} f_{m,n}^{(l=1)}((z\mathfrak{t})^m(z^{-1}\mathfrak{t})^n)^N$$

$$f_{m,n}^{(l=1)} = \prod_{a=1}^{m}\prod_{b=0}^{\infty}\frac{1}{1-(z\mathfrak{t})^{-a}(z^{-1}\mathfrak{t})^b}\prod_{a=1}^{n}\prod_{b=0}^{\infty}\frac{1}{1-(z^{-1}\mathfrak{t})^{-a}(z\mathfrak{t})^b}\prod_{a=1}^{m}\prod_{b=1}^{n}\frac{1}{1-(z\mathfrak{t})^{-a}(z^{-1}\mathfrak{t})^{-b}}$$

Single-sum expansion

Original expansion $(|z|=1, |\mathfrak{t}|<1)$:

$$\frac{\mathcal{I}_N^{l=1(C)}(x_1,x_2)}{\mathcal{I}_\infty^{l=1(C)}(x_1,x_2)} = \sum_{m,n\geq 0} f_{m,n}^{(l=1)}(x_1,x_2) x_1^{mN} x_2^{nN} \qquad \begin{aligned} x_1 &= z\mathfrak{t} \\ x_2 &= z^{-1}\mathfrak{t} \end{aligned}$$

$$f_{m,n}^{(l=1)}(x_1,x_2) = \prod_{a=1}^{m}\prod_{b=0}^{\infty}\frac{1}{1-x_1^{-a}x_2^{b}} \prod_{a=1}^{n}\prod_{b=0}^{\infty}\frac{1}{1-x_2^{-a}x_1^{b}} \prod_{a=1}^{m}\prod_{b=1}^{n}\frac{1}{1-x_1^{-a}x_2^{-b}}$$

We focus on single-sum expansion

$$\frac{\mathcal{I}_N^{l=1(C)}(x_1,x_2)}{\mathcal{I}_\infty^{l=1(C)}(x_1,x_2)} \text{``}=\text{''} \sum_{m=0}^{\infty} f_{m,0}^{(l=1)}(x_1,x_2) x_1^{mN}$$

"=" is justified if we expand in x_2 and truncate at finite (but arbitrarily large) order x_2^{ν}

$$\left(\because f_{m,n(n>0)}^{(l=1)} \propto \prod^{\infty} x_2^{positive} = x_2^{\infty}\right)$$

Inverse single-sum expansion

Single-sum expansion for general l flavors:

$$\frac{\mathcal{I}_N^{l(C)}(x_1,x_2)}{\mathcal{I}_\infty^{l(C)}(x_1,x_2)} \text{ “$=$” } \sum_{m=0}^{\infty} f_{m,0}^{(l)}(x_1,x_2)x_1^{lmN} \qquad \left(f_{m,0}^{(l)}(x_1,x_2)=\prod_{a=1}^{m}\prod_{b=0}^{\infty}\prod_{c=0}^{l-1}\frac{1}{1-x_1^{-la+c}x_2^{lb+c}}\right)$$

If we define $F_m^{(l)}=f_{m,0}^{(l)}(x_1^{-1},x_1x_2)$ ➡

$$\frac{F_N^{(l)}(x_1,x_2)}{F_\infty^{(l)}(x_1,x_2)} \text{ “$=$” } \sum_{m=0}^{\infty} x_1^{lmN}\mathcal{I}_m^{l(C)}(x_1^{-1},x_1x_2)$$

[Hayashi,TN,Okazaki, *to appear*]

Expected interpretation $(l=1)$:

$f_{m,0}$ = Contribution of *m* M5-giants in $\mathrm{AdS}_4\times S^7$ to index of *N* M2

F_N = Index on *N* M5

$\mathcal{I}_m^{(C)}(x_1^{-1},x_1x_2)$ = Contribution of *m* M2-giants in $\mathrm{AdS}_7\times S^4$ to index of *N* M5

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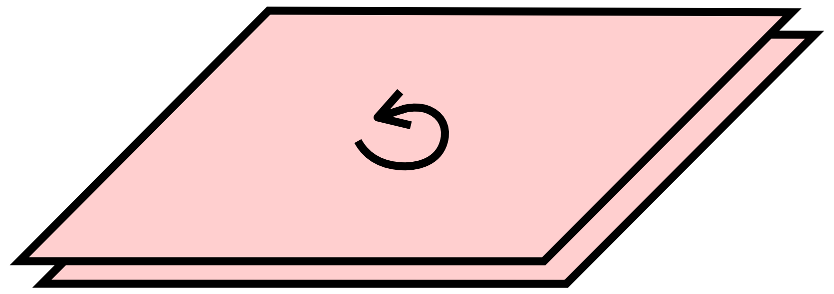
Comparison with M5 indices

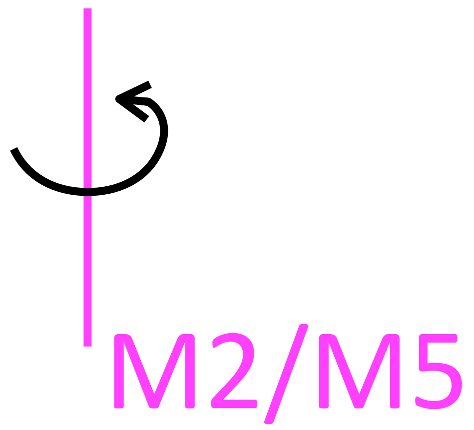
Gravity picture suggests $f_{m,0}(x_1, x_2) = I^{\text{6d}}_{m\text{M5}}$ under some non-trivial change of variables

Rough idea:

$j^{\text{3d}}, J^{\text{6d}}$ = rotation along M2/M5-branes



$r^{\text{3d}}, R^{\text{6d}}$ = rotation around M2/M5-branes



$$\begin{array}{ccc} \text{AdS}_4 & \times & S^7 \\ \cup & & \cup \\ \mathbb{R} & \times & S^5 \end{array} \Rightarrow \begin{pmatrix} h^{\text{3d}} \\ j^{\text{3d}} \\ r^{\text{3d}} \end{pmatrix} \sim \begin{pmatrix} H^{\text{6d}} \\ R^{\text{6d}} \\ J^{\text{6d}} \end{pmatrix}$$

(M5: $\mathbb{R} \times S^5$)

Convention for M2/M5 indices

M2 index

- spacetime: $p_\mu,\ j_{\mu\nu},\ h,\ k_\mu$
- super(conformal): $\mathcal{Q}^I_\alpha,\ \mathcal{S}^I_\alpha$ ⟶ Choose one $\mathcal{Q}^I_\alpha$
- R-symmetry: $SO(8)_r$

	h	j_{12}	r_{12}	r_{34}	r_{56}	r_{78}
$\mathcal{Q}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$$I_{\rm M2} = {\rm Tr}[(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} q^{\frac{h+j_{12}}{2}} u_1^{r_{12}} u_2^{r_{12}} u_3^{r_{12}} u_4^{r_{12}}]$$

$$(u_1 = zt^{-1},\ u_2 = xt,\ u_3 = z^{-1}t^{-1},\ u_4 = x^{-1}t)$$

M5 index

- spacetime: $P_\mu, J_{\mu\nu}, H, K_\mu$
- super(conformal): Q^I_α, S^I_α ⟶ Choose one Q^I_α
- R-symmetry: $SO(5)_R$

	H	J_{12}	J_{34}	J_{56}	R_{12}	R_{34}
Q	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$$I_{\rm M5} = {\rm Tr}[(-1)^F e^{-\beta\{Q,Q^\dagger\}} p^{H+\frac{1}{3}(J_{12}+J_{34}+J_{56})} y_1^{J_{12}} y_2^{J_{34}} y_3^{J_{56}} u^{R_{12}-R_{34}}] \qquad (y_1y_2y_3 = 1)$$

Parameter identification - step 1

$$\frac{I_{N\mathrm{M2}}}{I_{\infty\mathrm{M2}}} = 1 + f_{1,0}(q^{\frac{1}{4}} t^{-1} z)^N + \cdots$$

ground state contribution of M5 wrapped on $S^5_{z_1=0} \subset S^7$ $\left(S^7 : \sum_{a=1}^{4} |z_a|^2 = 1\right)$

: preserves $Q^I_{\alpha\,(h-r_{12}=0)}$ ➡ subalgebra $\supset$ $so(3)_{j_{ij}} \times so(6)_{r_{ab}} \times u(1)_{h-\frac{1}{2}r_{12}} \times u(1)_{h-r_{12}}$, $a,b=3,\cdots,8$ ①

Taking into account of inverse expansion, consider also the contribution in I_{M5} from wrapped M2 on $S^2_{Z_1=0} \subset S^4$ $\left(S^4 : \sum_{a=1}^{2} |Z_a|^2 = 1\right)$

: preserves $Q^I_{\alpha\,(H-R_{12}=0)}$ ➡ subalgebra $\supset$ $so(6)_{J_{ij}} \times so(3)_{R_{ab}} \times u(1)_{H-2R_{12}} \times u(1)_{H-R_{12}}$, $a,b=3,4,5$ ②

Match ① with ② ➡ $h = \frac{H}{2} - \frac{3R_{12}}{2},\ j_{12} = R_{34},\ r_{12} = -R_{12},\ r_{34} = J_{12},\ r_{56} = J_{34},\ r_{78} = J_{56}$

Parameter identification - step 2

Write I_{M2} in terms of Cartans (H, J_{ij}, R_{ab}) of M5 SCFT

$$I_{\mathrm{M2}} = \mathrm{Tr}[(-1)^F (q^{\frac{3}{16}} t^{\frac{1}{4}} z^{-\frac{1}{4}})^{H+\frac{J_{12}+J_{34}+J_{56}}{3}} (t^{\frac{2}{3}} z^{\frac{1}{3}} x)^{J_{12}} (t^{-\frac{4}{3}} z^{-\frac{2}{3}})^{J_{34}} (t^{\frac{2}{3}} z^{\frac{1}{3}} x^{-1})^{J_{56}} (q^{-\frac{5}{8}} t^{\frac{1}{2}} z^{-\frac{1}{2}})^{R_{12}-R_{34}}]$$

Hence we conclude

$$\frac{I_{N\mathrm{M2}}}{I_{\infty\mathrm{M2}}} = 1 + \sum_{m=1}^{\infty} (q^{\frac{1}{4}} t^{-1} z)^{mN} I_{m\mathrm{M5}}\big|_{p=q^{\frac{3}{16}} t^{\frac{1}{4}} z^{-\frac{1}{4}}, y_1=t^{\frac{2}{3}} z^{\frac{1}{3}} x, y_2=t^{-\frac{4}{3}} z^{-\frac{2}{3}}, y_3=t^{\frac{2}{3}} z^{\frac{1}{3}} x^{-1}, u=q^{-\frac{5}{8}} t^{\frac{1}{2}} z^{-\frac{1}{2}}} + \cdots$$

[Arai,Fujiwara,Imamura,Mori,Yokoyama,'20]

In Coulomb limit

$$\frac{\mathcal{I}^{(C)}_{N\mathrm{M2}}}{\mathcal{I}^{(C)}_{\infty\mathrm{M2}}} = 1 + \sum_{m=1}^{\infty} f_{m,0}(x_1, x_2) x_1^{mN}$$

$$x_1 = z\mathfrak{t}$$
$$x_2 = z^{-1}\mathfrak{t}$$

$$f_{m,0}(x_1, x_2) = \lim_{\substack{q,t\to 0,\\ q^{\frac{1}{4}} t^{-1} = \mathfrak{t}:\text{fixed}}} I_{m\mathrm{M5}} = I_{m\mathrm{M5}}(p^2 u = x_1^{-1}, p^{\frac{4}{3}} y_2 = x_2; p^2 u^{-1} = p^{\frac{4}{3}} y_1 = p^{\frac{4}{3}} y_3 = 0)$$

Check for *m*=1 and *m*=∞

$$I_{\mathrm{M5}} = \mathrm{Tr}[(-1)^F e^{-\beta\{Q,Q^\dagger\}} p^{H+\frac{1}{3}(J_{12}+J_{34}+J_{56})} y_1^{J_{12}} y_2^{J_{34}} y_3^{J_{56}} u^{R_{12}-R_{34}}]$$

$$(2\{Q,Q^\dagger\} = H - (J_{12}+J_{34}+J_{56}) - 2(R_{12}+R_{34}))$$

	H	$J_{12}+J_{34}+J_{56}$	R_{12}	R_{34}
$\Phi^{(1,0)}$	2	0	1	0
$\Phi^{(0,1)}$	2	0	0	1
$\Psi^{++}_{++-}, \Psi^{++}_{+-+}, \Psi^{++}_{-++}$	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\partial_1^+, \partial_2^+, \partial_3^+$	1	1	0	0
$(\partial\Psi)^{++}_{+++} = 0$	$\frac{7}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

① Single M5 ➡ free tensor multiplet

$$I_{\mathrm{singleM5}} = \mathrm{PE}\Big[\frac{p^2(u+u^{-1}) - p^{\frac{8}{3}}(y_1y_2+y_2y_3+y_3y_1) + p^4}{(1-p^{\frac{4}{3}}y_1)(1-p^{\frac{4}{3}}y_2)(1-p^{\frac{4}{3}}y_3)}\Big] \overset{(C)}{\to} \mathrm{PE}\Big[\frac{x_1^{-1}}{1-x_2}\Big] = f_{1,0}(x_1,x_2)$$

② ∞ M5 ➡ graviton index

$$I_{\infty\mathrm{M5}} = \mathrm{PE}\Big[\frac{p^2(1-p^4)(u+u^{-1}) - p^{\frac{8}{3}}(y_1y_2+y_2y_3+y_3y_1) + p^{\frac{16}{3}}(y_1+y_2+y_3)}{(1-p^{\frac{4}{3}}y_1)(1-p^{\frac{4}{3}}y_2)(1-p^{\frac{4}{3}}y_3)(1-p^2u)(1-p^2u^{-1})}\Big] \overset{(C)}{\to} \mathrm{PE}\Big[\frac{x_1^{-1}}{(1-x_1^{-1})(1-x_2)}\Big] = f_{\infty,0}(x_1,x_2)$$

[Bhattacharya,Bhattacharyya,Minwalla,Raju,'08]

Plan of talk

✓ 1. 3d U(*N*) ADHM theory

✓ 2. Coulomb limit and giant graviton expansion

✓ 3. M5-interpretation ($m = 1, \infty$)

4. M5-interpretation (general *m*)

Superconformal index of multiple M5

Can be calculated by using the proposal *m* M5 on S^1 = 5d U(*m*) $\mathcal{N}=2$ SYM

KK modes ⟷ instanton particles

$$I_{\text{M5}} = Z^{\text{M5}}_{S^5 \times_t S^1} \sim Z^{5\text{d}\mathcal{N}=1^*\text{SYM}}_{S^5_b}$$

[Douglas,'10][Lambert,Papageorgakis,Schmidt-Sommerfeld,'10]

But it is still difficult to obtain $I_{m\text{M5}}$ in closed form (⟷ sum over all instantons)

Exception: **Unrefined limit**

$$I^{\text{unrefined}}_{m\text{M5}}(p) = I_{m\text{M5}}(u = p^{-\frac{2}{3}}, y_1 = y_2 = y_3 = 1) = \prod_{a=1}^{m} \prod_{b=0}^{\infty} \frac{1}{1 - p^{\frac{4(a+b)}{3}}}$$

[Kim,Kim,Kim,'12][Kim,Kim,Kim,Lee,'13]

Q. Can we compare this with giant graviton coefficients $f_{m,0}(x_1, x_2)$ in **Coulomb limit**?

Coulomb limit vs 6d unrefined limit

$$I_{\mathrm{M5}}|_{\mathrm{M2parameters}} = I_{\mathrm{M5}}(p = t\mathfrak{t}^{\frac{3}{4}} z^{-\frac{1}{4}}, y_1 = t^{\frac{2}{3}} z^{\frac{1}{3}} x, y_2 = t^{-\frac{4}{3}} z^{-\frac{2}{3}}, y_3 = t^{\frac{2}{3}} z^{\frac{1}{3}} x^{-1}, u = t^{-2}\mathfrak{t}^{-\frac{5}{2}} z^{-\frac{1}{2}})$$

$$= \mathrm{Tr}[(-1)^F t^{\Delta'} \mathfrak{t}^{\frac{3H}{4} + \frac{J_{12}+J_{34}+J_{56}}{4} - \frac{5(R_{12}-R_{34})}{2}} z^{-\frac{H}{4} + \frac{J_{12}}{4} - \frac{3J_{34}}{4} + \frac{J_{56}}{4} - \frac{R_{12}-R_{34}}{2}} x^{J_{12}-J_{56}}]$$

(Coulomb limit: $t \to 0$)

$$\Delta' = H + J_{12} - J_{34} + J_{56} - 2R_{12} + 2R_{34} = 2\{Q', (Q')^\dagger\}$$

	H	J_{12}	J_{34}	J_{56}	R_{12}	R_{34}	$\frac{3H}{4} + \frac{J_{12}+J_{34}+J_{56}}{4} - \frac{5(R_{12}-R_{34})}{2}$	$-\frac{H}{4} + \frac{J_{12}}{4} - \frac{3J_{34}}{4} + \frac{J_{56}}{4} - \frac{R_{12}-R_{34}}{2}$
Q	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
Q'	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-2	0

When $\mathfrak{t} = 1$, $I_{\mathrm{M5}}|_{\mathrm{M2parameters}}$ is independent of *t*

[Beem,Rastelli,van Rees,'14]

$$I_{m\mathrm{M5}}|_{\mathrm{M2parameters}}(\mathfrak{t} = 1, t = 0, x = 1) = I_{m\mathrm{M5}}|_{\mathrm{M2parameters}}(\mathfrak{t} = 1, t = z^{-\frac{1}{2}}, x = 1)$$

$$\| \qquad\qquad \|$$

$$f_{m,0}(x_1 = z, x_2 = z^{-1}) \stackrel{?}{=} I_{m\mathrm{M5}}^{\mathrm{unrefined}}(p = z^{-\frac{3}{4}})$$

Coulomb limit vs 6d unrefined limit

$$I_{\mathrm{M5}}|_{\mathrm{M2parameters}} = I_{\mathrm{M5}}(p = t\mathfrak{t}^{\frac{3}{4}} z^{-\frac{1}{4}}, y_1 = t^{\frac{2}{3}} z^{\frac{1}{3}} x, y_2 = t^{-\frac{4}{3}} z^{-\frac{2}{3}}, y_3 = t^{\frac{2}{3}} z^{\frac{1}{3}} x^{-1}, u = t^{-2}\mathfrak{t}^{-\frac{5}{2}} z^{-\frac{1}{2}})$$

$$= \mathrm{Tr}[(-1)^F t^{\Delta'} \mathfrak{t}^{\frac{3H}{4} + \frac{J_{12}+J_{34}+J_{56}}{4} - \frac{5(R_{12}-R_{34})}{2}} z^{-\frac{H}{4} + \frac{J_{12}}{4} - \frac{3J_{34}}{4} + \frac{J_{56}}{4} - \frac{R_{12}-R_{34}}{2}} x^{J_{12}-J_{56}}]$$

(Coulomb limit: $t \to 0$)

$$\Delta' = H + J_{12} - J_{34} + J_{56} - 2R_{12} + 2R_{34} = 2\{Q', (Q')^\dagger\}$$

	H	J_{12}	J_{34}	J_{56}	R_{12}	R_{34}	$\frac{3H}{4} + \frac{J_{12}+J_{34}+J_{56}}{4} - \frac{5(R_{12}-R_{34})}{2}$	$-\frac{H}{4} + \frac{J_{12}}{4} - \frac{3J_{34}}{4} + \frac{J_{56}}{4} - \frac{R_{12}-R_{34}}{2}$
Q	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
Q'	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-2	0

When $\mathfrak{t} = 1$, $I_{\mathrm{M5}}|_{\mathrm{M2parameters}}$ is independent of *t*

[Beem,Rastelli,van Rees,'14]

$$I_{m\mathrm{M5}}|_{\mathrm{M2parameters}}(\mathfrak{t} = 1, t = 0, x = 1) = I_{m\mathrm{M5}}|_{\mathrm{M2parameters}}(\mathfrak{t} = 1, t = z^{-\frac{1}{2}}, x = 1)$$

$$||\qquad\qquad||$$

$$f_{m,0}(x_1 = z, x_2 = z^{-1}) \overset{\checkmark}{=} I^{\mathrm{unrefined}}_{m\mathrm{M5}}(p = z^{-\frac{3}{4}})$$

Summary

- M2-M5 giant graviton expansion works!

$$\frac{\mathcal{I}^{(C)}_{N\mathrm{M2}}(x_1,x_2)}{\mathcal{I}^{(C)}_{\infty\mathrm{M2}}(x_1,x_2)}=\sum_{m=0}^{\infty}x_1^{mN}I_{m\mathrm{M5}}(x_1^{-1},x_1x_2),\qquad \frac{I_{N\mathrm{M5}}(x_1,x_2)}{I_{\infty\mathrm{M5}}(x_1,x_2)}=\sum_{m=0}^{\infty}x_1^{mN}\mathcal{I}^{(C)}_{m\mathrm{M2}}(x_1^{-1},x_1x_2)$$

- Propose 2-parameter refinement of M5-index which might be solvable

Future works

- Derivation of giant graviton coefficients from gravity side?

[Eleftherious,Murthy,Rossello,'23] (D3, 1/2 BPS)

- Higgs limit? $\mathcal{I}^{\mathrm{U}(N)\mathrm{ADHM}\text{-}[l](H)}=\frac{1}{\mathfrak{t}^{2lN}}Z^{\mathrm{5dSU}(l)\mathrm{pureYM}}_{\#\mathrm{inst}=N}$

$I_{\mathrm{M5}}=Z^{\mathrm{M5}}_{S^5\times S^1}\sim Z^{\mathrm{5dSYM}}_{S^5}$

new "5d $\longleftrightarrow$ 5d correspondence"???

- Giant graviton expansion of Wilson loops (M2 wrappend on $\mathrm{AdS}_2\times S^1$?)

[Hayashi,TN,Okazaki,'24]

- Chern-Simons matter theories? [Imamura,Kimura,'08]