Towards elucidation of quantum criticality of Ising model on 2d dynamical triangulations

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(IAR, Nagoya U)

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YS and T. Tanaka, Phys. Rev. D98 (2018)

J. Ambjørn, YS and T. Tanaka, Phys. Rev. D101(2020)

@ the 4th international conference on holography, string theory and discrete approaches

Ising model on 2d dynamical triangulations (DT) [Kazakov, 1986] [Boulatov-Kazakov, 1987]

(1) Continuous phase transition at non-zero temperature T_c.

(2) Physics around the critical point is described by

2d gravity coupled to fermion

Reconsider criticality of Ising model on 2d DT [YS-Tanaka, 2017]

(1) Introduce a “loop-counting” parameter θ.

(2) Tuning θ, one can reduce $T_c(\theta)$ to absolute zero.

(3) Continuum theories around absolute zero are NOT

2d gravity coupled to fermion → Continuum two-MM

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(2) Physics around the critical point is described by

2d gravity coupled to fermion

There exists one parameter connecting two criticalities
[Ambjørn-YS-Tanaka, 2020]

Reconsider criticality of Ising model on 2d DT [YS-Tanaka, 2017]

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Outline

(1) Ising model on 2d DT

(2) zero-temperature vs. finite temperature

(3) Summary & Discussion

Ising model on 2d DT

Ising model on a honeycomb lattice (G):

$$Z_G(\beta) = \sum_{\sigma} \prod_{<i,j>} e^{\beta \sigma_i \sigma_j}$$

Ising spin:

$$\sigma_i = \pm 1$$

Exactly solved in the thermodynamic limit:

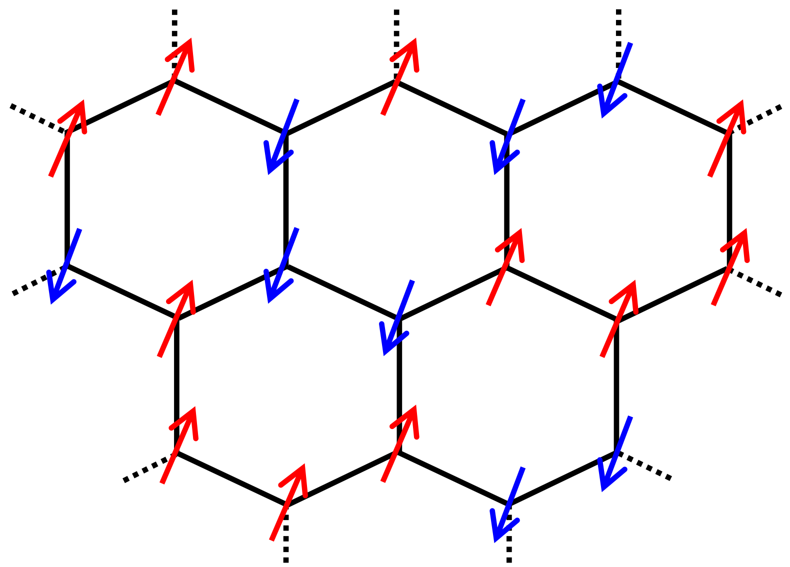
[Weiner, 1950; Houtappel, 1950]

(1) 2nd order phase transition at

$$\beta = \beta_c \neq \infty$$

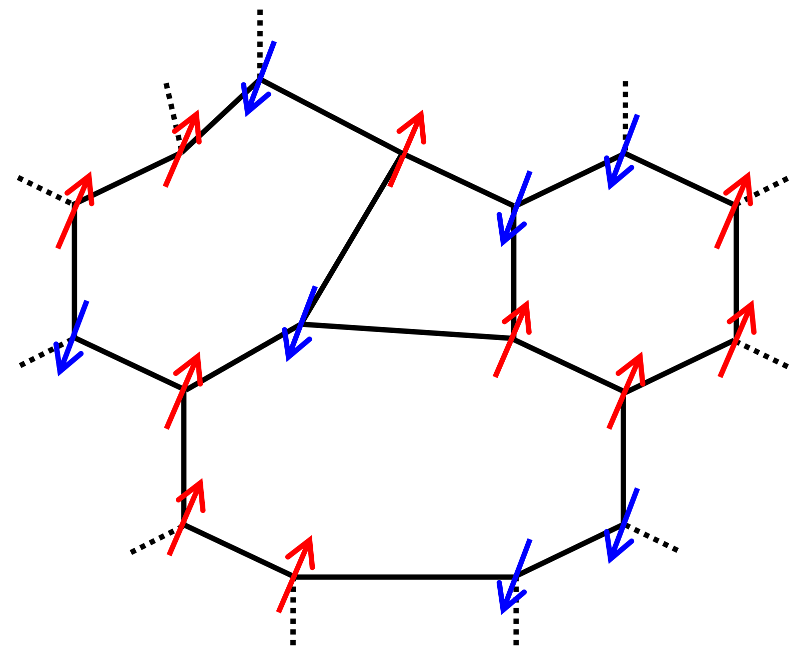
(2) Physics around β_c described by

2d fermion



Ising model on a planar graph (w/ coordination number = 3) (G'):

$$Z_{G'}(\beta) = \sum_{\sigma} \prod_{<i,j>} e^{\beta \sigma_i \sigma_j}$$



Summing all planar graphs (w/ coordination number = 3),

$$\{G, G', G'', \cdots\}$$

one can construct a solvable model (Ising model on 2d DT).

[Kazakov, 1986]

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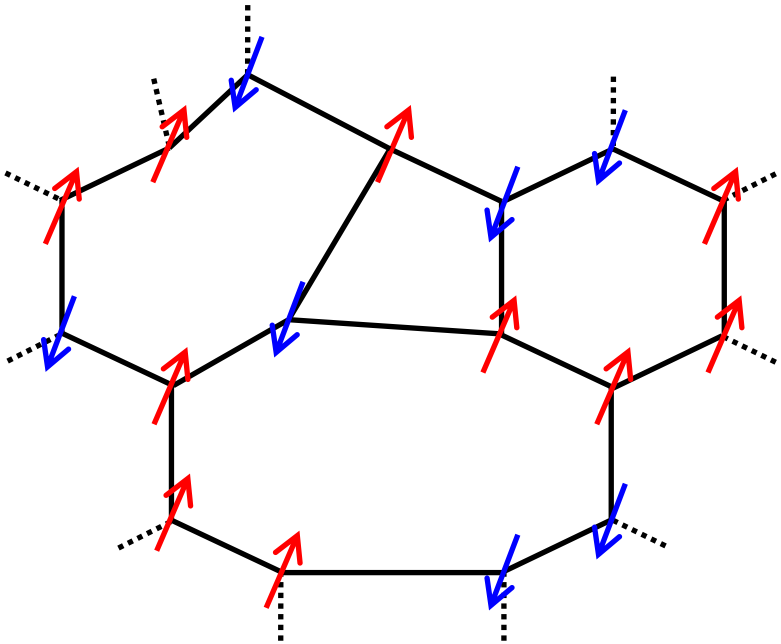

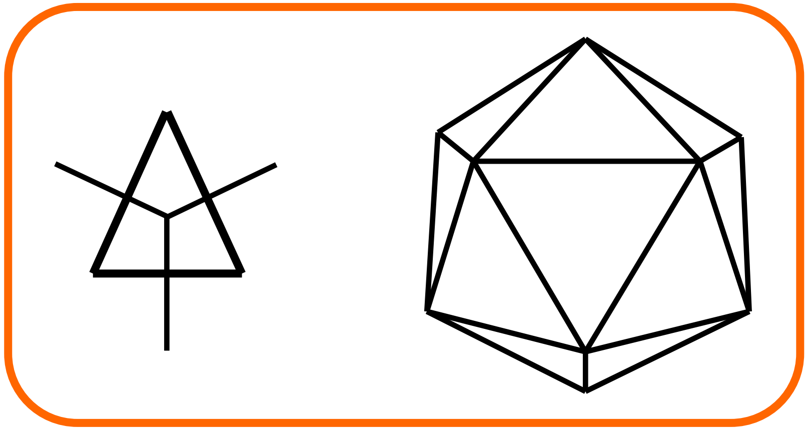

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[Kazakov, 1986]

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

$$Z(\beta, g) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} g^{n(G)} \boxed{Z_G(\beta)}$$

Ising model on G

g : weight of a vertex

n(G) := #(vertices in G)

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Ising model on G

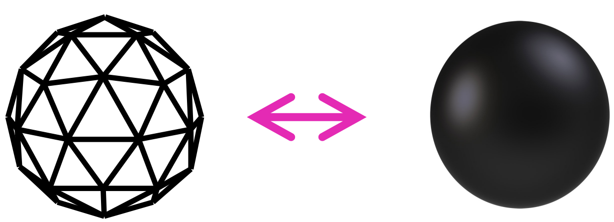
g : weight of a vertex

n(G) := #(vertices in G)

g is related to the cosmological constant λ:

$$g = e^{-\lambda}$$

$$g^n = e^{-\lambda n} \leftrightarrow e^{-\Lambda_0 \int d^2x \sqrt{\det(g_{ab})}}$$



Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

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$$=: \sum_n g^n \boxed{Z_n(\beta)}$$

Partition function (of Ising model dressed by quantum gravity)

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

$$Z(\beta, g) = \sum_G \frac{1}{|\mathrm{Aut}(G)|} g^{n(G)} Z_G(\beta)$$

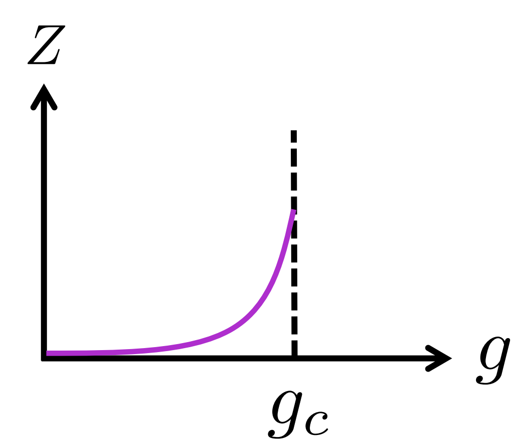
$$=: \sum_n g^n \boxed{Z_n(\beta)}$$

Since for n >>1

$$\boxed{Z_n(\beta)} \propto n^{\gamma-3}(1/g_c)^n(1+\mathcal{O}(1/n))$$

there exists a finite radius of convergence

$$g_c = \lim_{n\to\infty} \frac{Z_n}{Z_{n+1}}$$



The sum diverges for g > g_c

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

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Since for n >>1

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the free energy per vertex is

$$\boxed{f(\beta) = -\frac{1}{\beta} \lim_{n\to\infty} \frac{1}{n} \log \boxed{Z_n(\beta)} = \frac{1}{\beta} \log[g_c(\beta)]}$$

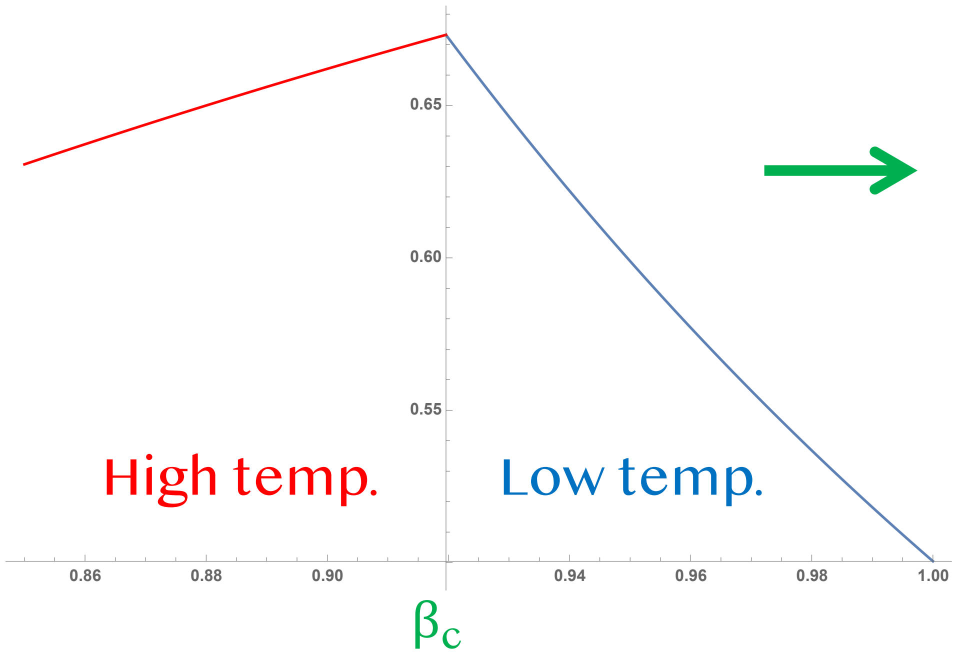
$f(\beta)$ becomes singular at $\beta_c = \log[1+2\sqrt{7}]/2$.

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

The specific heat

$$C = -\beta^2 \frac{d^2}{d\beta^2}(\beta f(\beta)) \qquad\qquad f(\beta) = \frac{1}{\beta}\log[g_c(\beta)]$$

has a cusp at $\beta=\beta_c$



→ 3rd-order phase transition

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

Around g_c

String susceptibility

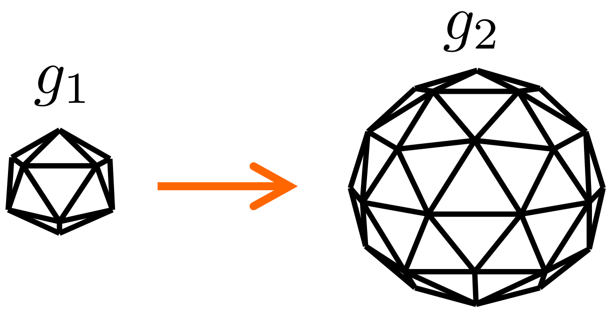
$$Z(\beta, g) \sim (g_c(\beta) - g(\beta))^{\boxed{2-\gamma}}$$

$\gamma = -1/2$ @ $\beta \neq \beta_c$

$\gamma = -1/3$ @ $\beta = \beta_c$

the average number of vertices blows up

$$\langle n \rangle = g \frac{\partial}{\partial g} \log Z(\beta, g) \Big|_{sing} \sim \frac{1}{g_c - g}$$



$0 < g_1 < g_2 < g_c$

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Large volume behavior:

$$Z(\beta,g) = \sum_n \boxed{g^n Z_n(\beta)} \xrightarrow{n \gg 1} \boxed{(g/g_c)^n n^{\gamma-3}}$$

Here

$$(g/g_c)^n = e^{-(\lambda-\lambda_c)n} = e^{-\frac{(\lambda-\lambda_c)}{\varepsilon^2} n\varepsilon^2} =: e^{-\Lambda A}$$

$$A := n\varepsilon^2$$

$$\Lambda := \frac{\lambda - \lambda_c}{\varepsilon^2}$$

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$$\langle n \rangle = g \frac{\partial}{\partial g} \log Z(\beta,g)\Big|_{sing} \sim \frac{1}{g_c - g}$$

Continuum limit:

$$Z(\beta,g) = \sum_n \boxed{g^n Z_n(\beta)} \xrightarrow[g \to g_c \quad \varepsilon \to 0]{\text{Continuum limit}} \propto \boxed{e^{-\Lambda A} A^{\gamma-3}}$$

A kept fixed

Pure Liouville gravity @ $\beta \neq \beta_c$

Liouville gravity coupled to fermion @ $\beta = \beta_c$

Definition via Hermitian NxN two-matrix model: [Kazakov, 1986]

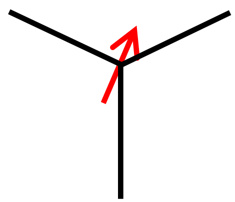
$$I_N(\beta, g) = \int D\varphi_+ D\varphi_- e^{-N\mathrm{tr}U(\varphi_+,\varphi_-)}$$

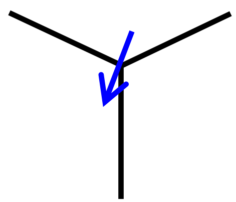
where

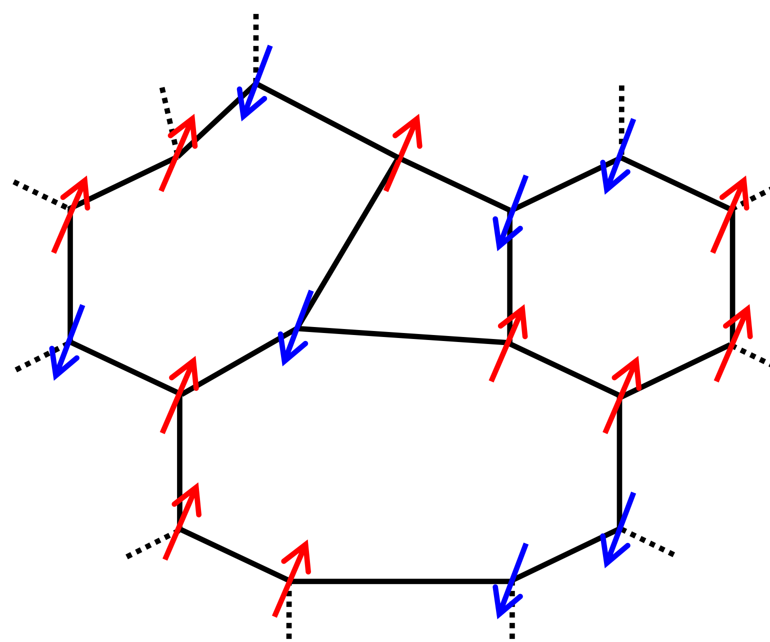
weight of vertex

$$U(\varphi_+, \varphi_-) = \frac{e^{\beta}}{\sinh(2\beta)}(\varphi_+^2 + \varphi_-^2 - 2e^{-2\beta}\varphi_+\varphi_-) - \frac{g}{3}(\varphi_+^3 + \varphi_-^3)$$

Vertices:

$$\mathrm{tr}\left(\varphi_+^3\right) \sim$$ 

$$\mathrm{tr}\left(\varphi_-^3\right) \sim$$ 



Definition via Hermitian NxN two-matrix model: [Kazakov, 1986]

$$I_N(\beta, g) = \int D\varphi_+ D\varphi_- e^{-N\mathrm{tr}U(\varphi_+,\varphi_-)}$$

where nearest-neighbor interactions

$$U(\varphi_+, \varphi_-) = \boxed{\frac{e^{\beta}}{\sinh(2\beta)}(\varphi_+^2 + \varphi_-^2 - 2e^{-2\beta}\varphi_+\varphi_-)} - \frac{g}{3}(\varphi_+^3 + \varphi_-^3)$$

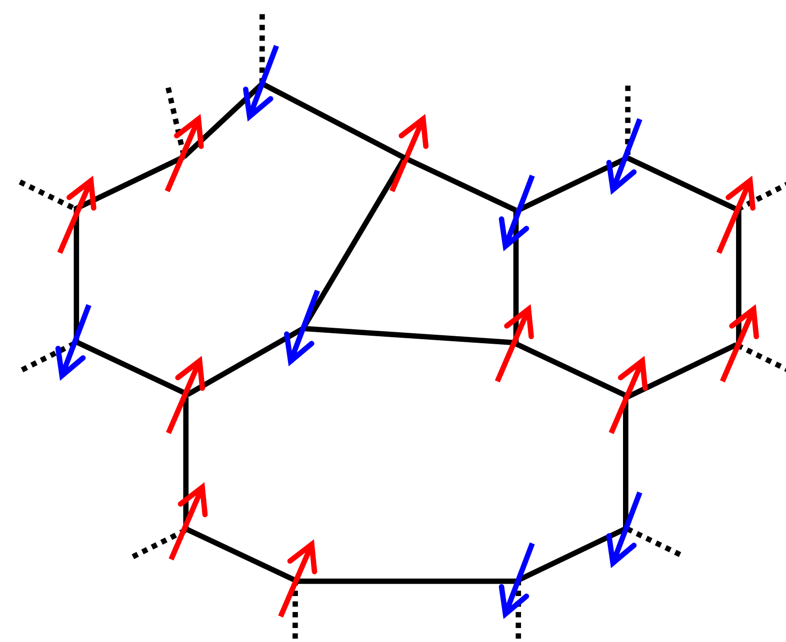
Propagators:

$$\langle \varphi_+ \varphi_+ \rangle = \frac{1}{N} e^{\beta\sigma_+\sigma_+} \sim$$

$$\langle \varphi_- \varphi_+ \rangle = \frac{1}{N} e^{\beta\sigma_-\sigma_+} \sim$$

$$\langle \varphi_+ \varphi_- \rangle = \frac{1}{N} e^{\beta\sigma_+\sigma_-} \sim$$

$$\langle \varphi_- \varphi_- \rangle = \frac{1}{N} e^{\beta\sigma_-\sigma_-} \sim$$



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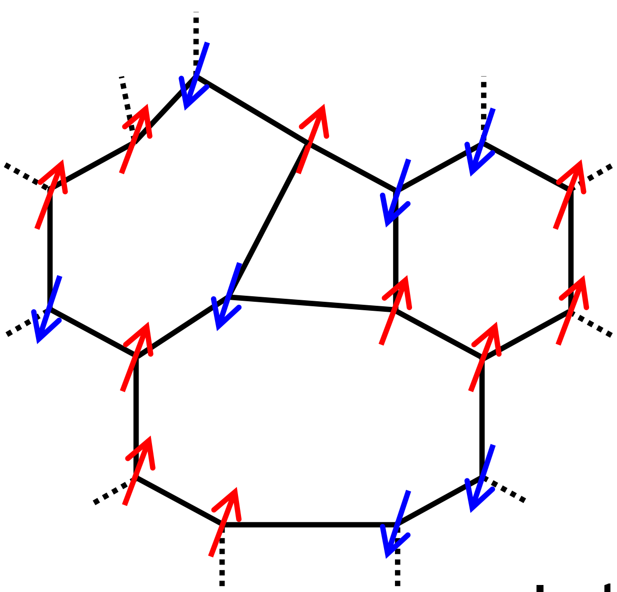

$$\sim N^{2-2h}$$

h=0 for planar
h>0 for non-planar

In the large-N limit, planar graphs survive.

$$N = e^{1/G_0}$$

Definition via Hermitian NxN two-matrix model: [Kazakov, 1986]

$$I_N(\beta, g) = \int D\varphi_+ D\varphi_- e^{-N\mathrm{tr}U(\varphi_+,\varphi_-)}$$

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Define Ising model on DT via the matrix model:

$$\boxed{Z(\beta, g) = \lim_{N\to\infty} \frac{1}{N^2} \log\left(\frac{I_N(\beta, g)}{I_N(\beta, 0)}\right)}$$

$$= \sum_G \frac{1}{|\mathrm{Aut}(G)|} g^{n(G)} Z_G(\beta)$$

(G: a connected planar graph)

Zero-temperature

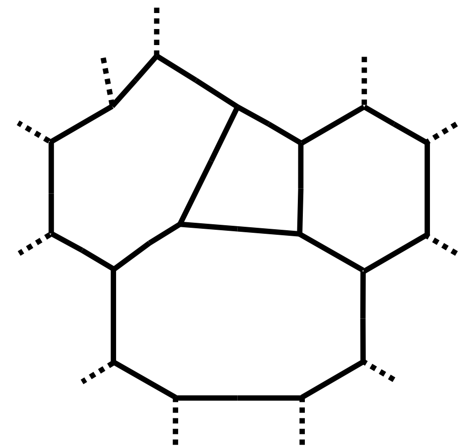
v.s

Finite temperature

Boulatov-Kazakov potential: [Kazakov, 1986] [Boulatov-Kazakov, 1987]

$$U^{(2)}(\psi_+,\psi_-) = \frac{1}{2}\left(\psi_+^2 + \psi_-^2 - 2c_{BK}\psi_+\psi_-\right) - \frac{g_{BK}}{3}\left(\psi_+^3 + \psi_-^3\right)$$

where $c_{BK} = e^{-2\beta_{BK}}$

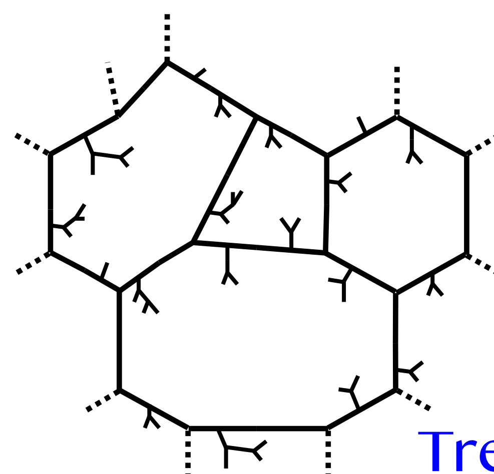


Skeleton graph

Our potential: [YS-Tanaka, 2017] [Fuji-YS-Watabiki, 2011]

$$U^{(0)}(\varphi_+,\varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2 + \varphi_-^2 - 2c\varphi_+\varphi_-\right) - \underline{g(\varphi_+ + \varphi_-)} - \frac{g}{3}\left(\varphi_+^3 + \varphi_-^3\right)\right)$$

where $c = e^{-2\beta}$



Trees are attached!

θ and linear term were introduced in a one-matrix model.
[Ambjørn-Loll-Watabiki-Westra-Zohren, 2008]

Boulatov-Kazakov potential: [Kazakov, 1986] [Boulatov-Kazakov, 1987]

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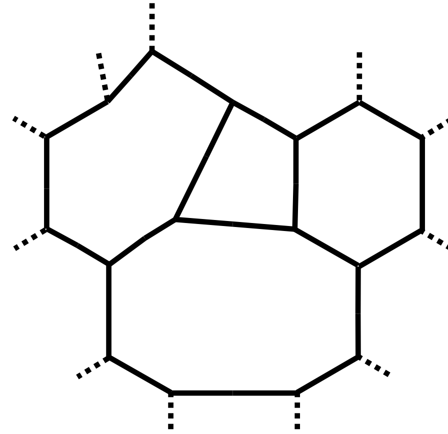

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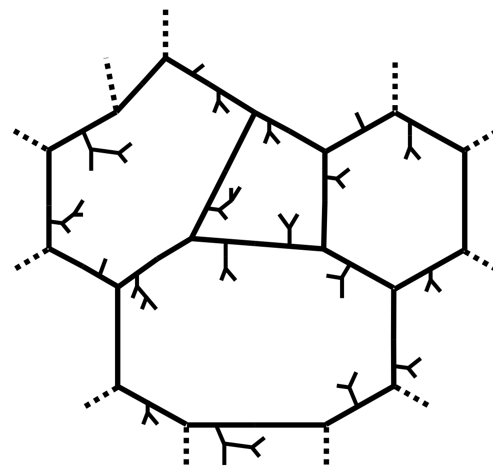
$$U^{(0)}(\varphi_+,\varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2 + \varphi_-^2 - 2c\varphi_+\varphi_-\right) - g(\varphi_+ + \varphi_-) - \frac{g}{3}\left(\varphi_+^3 + \varphi_-^3\right)\right)$$

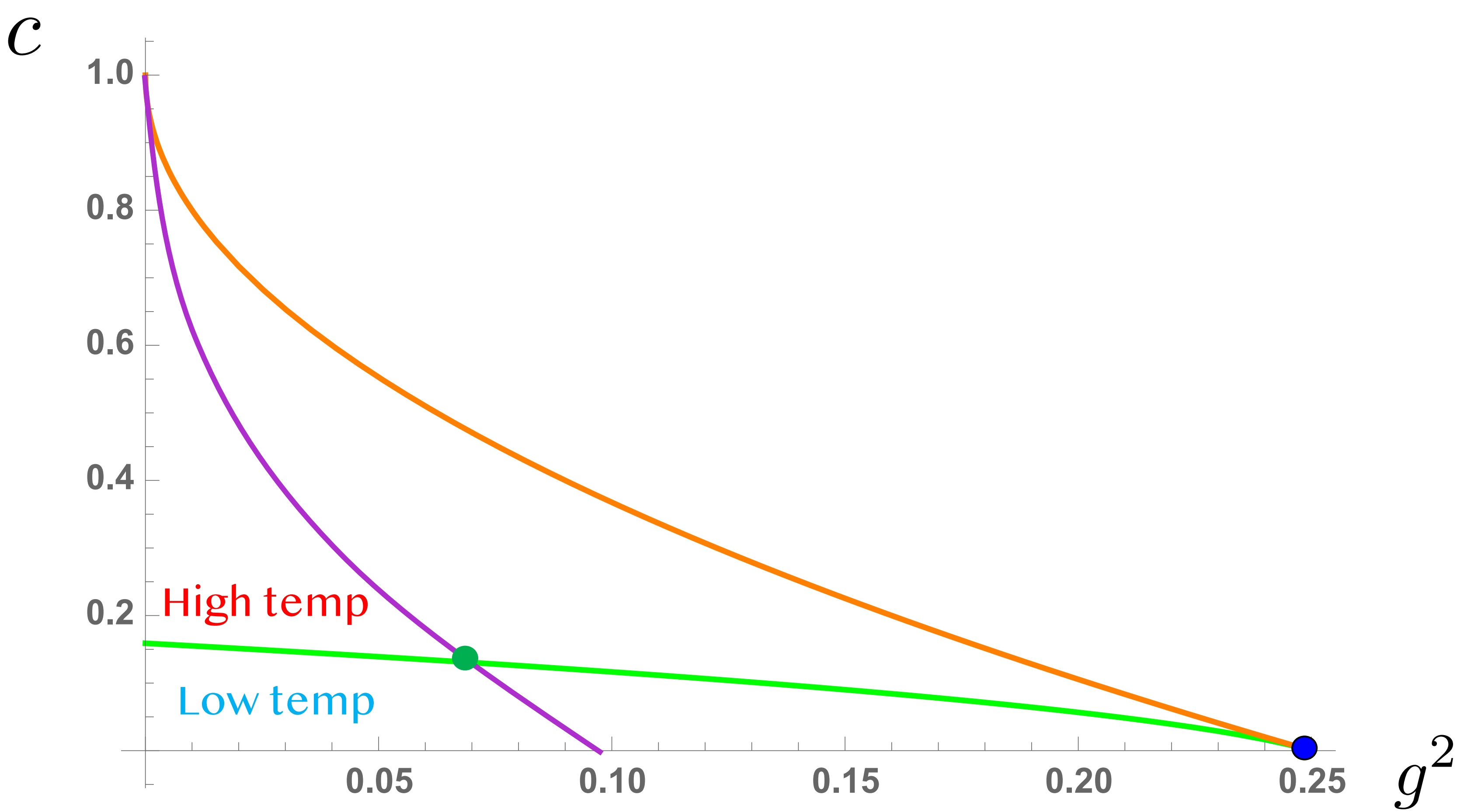
where $c = e^{-2\beta}$

When θ<<1,
trees become dominant



$\sim \theta^{\#(loops)-2}$

Phase diagram [YS-Tanaka, 2017] [Ambjørn-YS-Tanaka, 2020]



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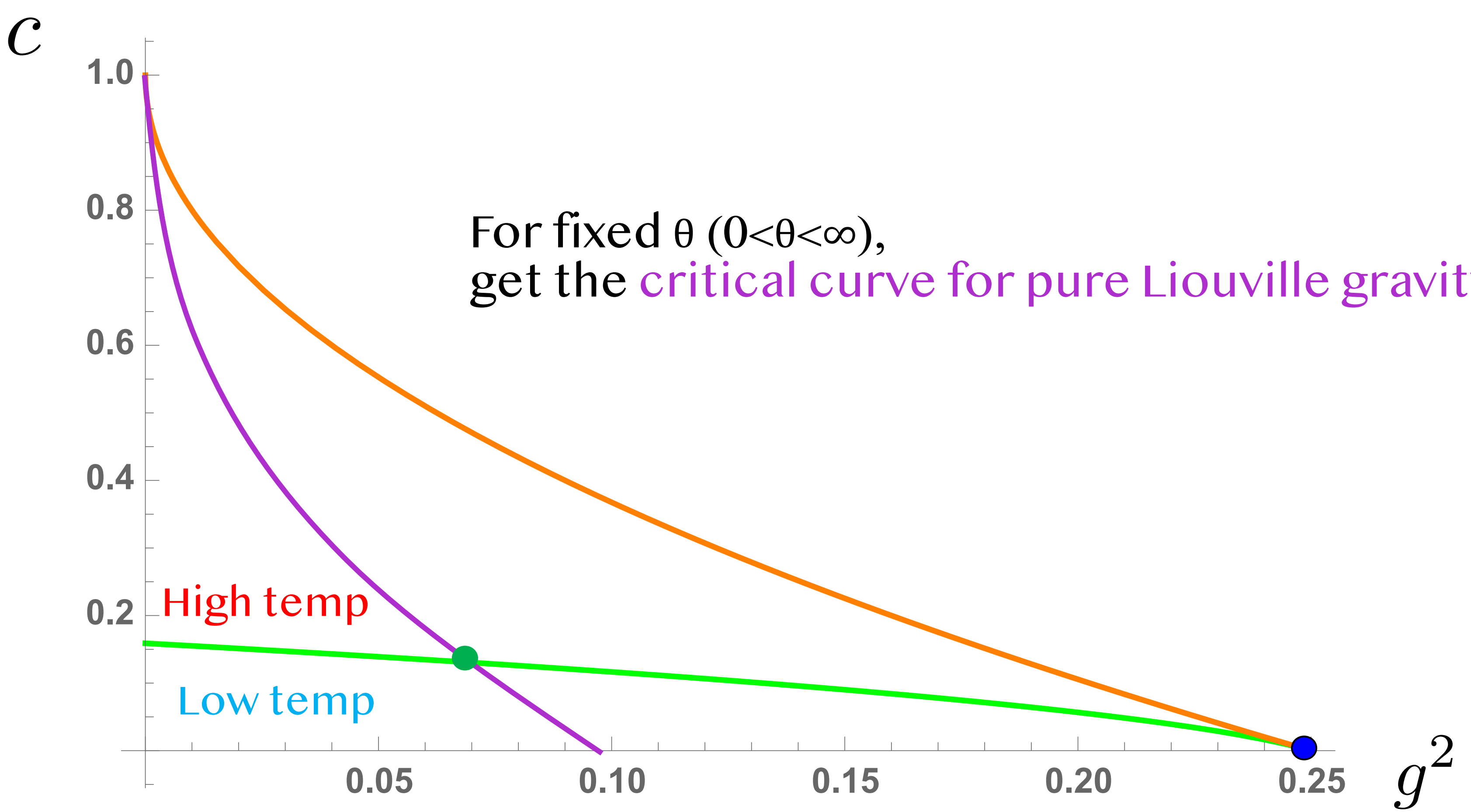
3 parameters in our model:

g (weight for vertices), **c** (temperature), **θ** (loop-counting parameter)

Phase diagram [YS-Tanaka, 2017] [Ambjørn-YS-Tanaka, 2020]



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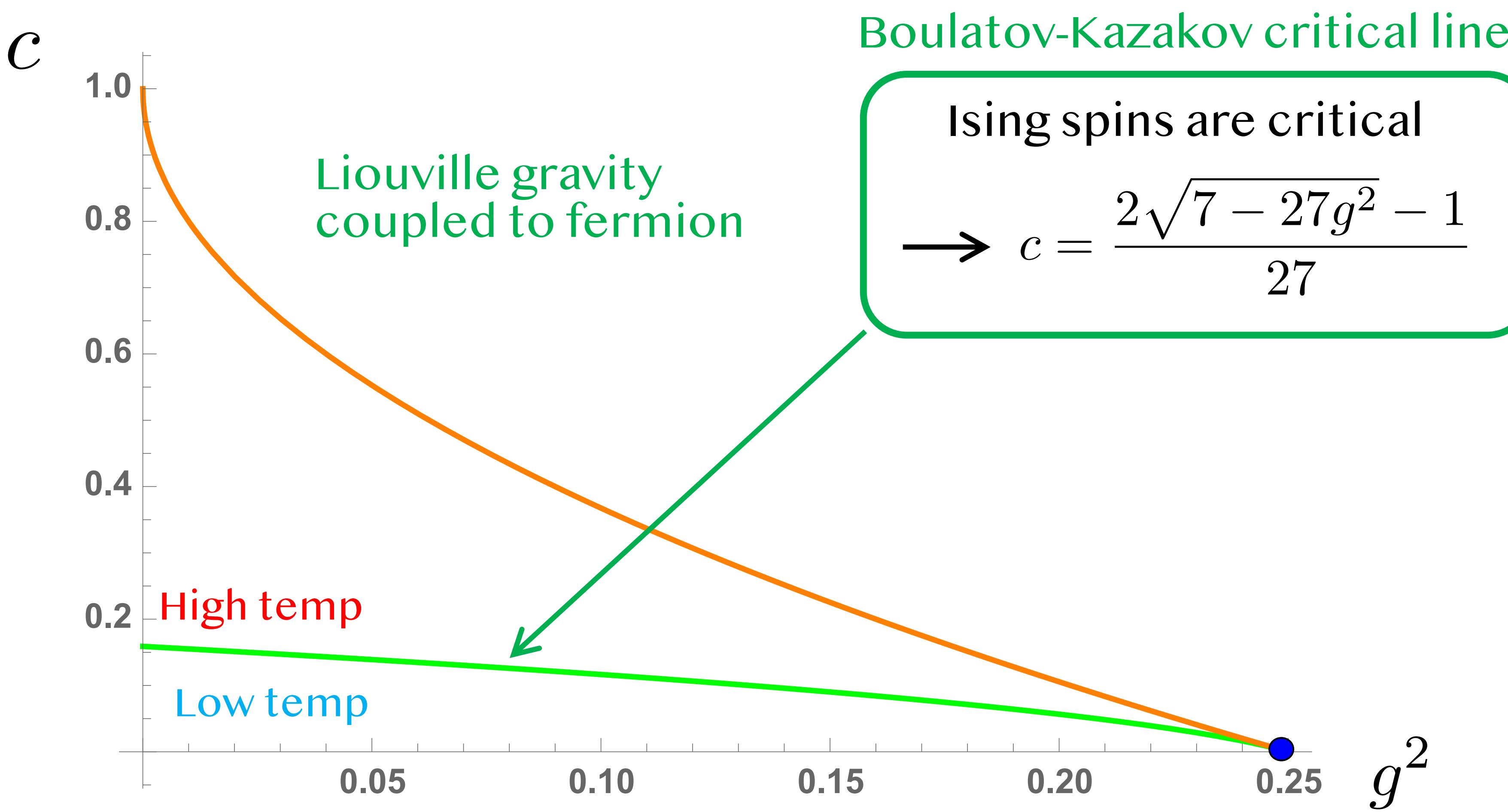

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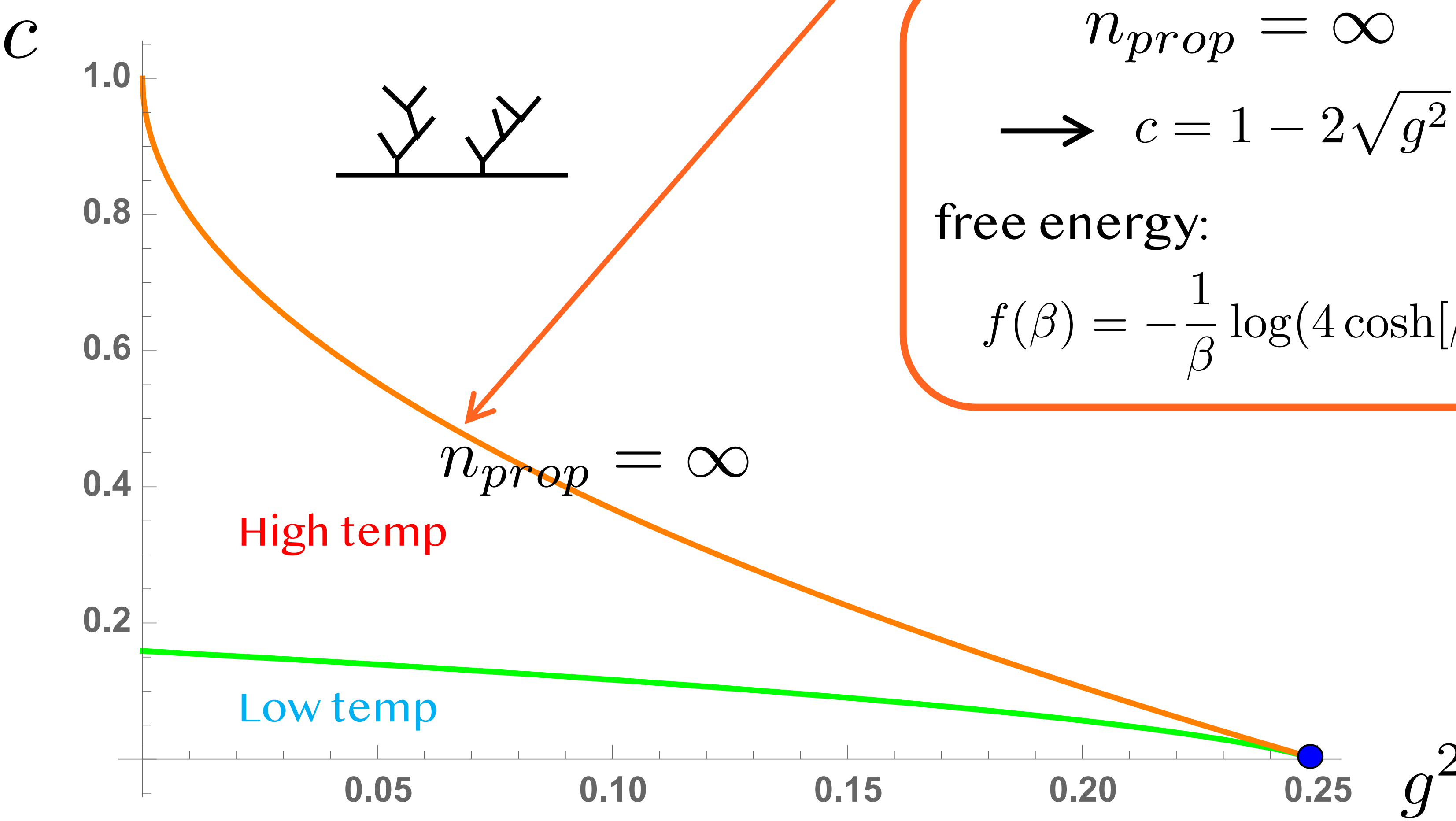
Critical line for dominant trees



$$n_{prop} = \infty$$

$$\longrightarrow \quad c = 1 - 2\sqrt{g^2}$$

free energy:

$$f(\beta) = -\frac{1}{\beta}\log(4\cosh[\beta])$$

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3 parameters in our model:

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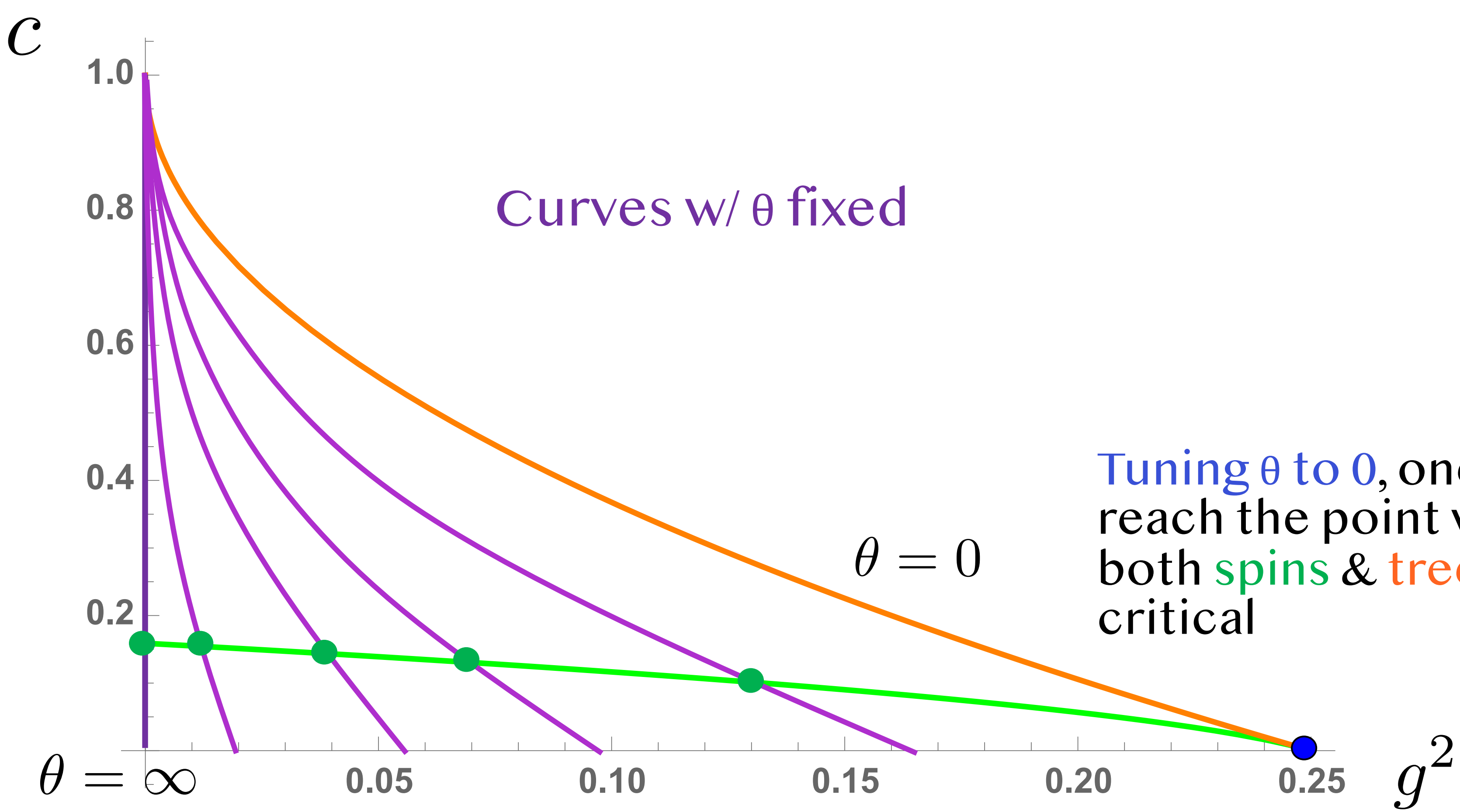
Phase diagram [YS-Tanaka, 2017] [Ambjørn-YS-Tanaka, 2020]



Tuning θ to 0, one can reach the point where both spins & trees are critical

---

3 parameters in our model:

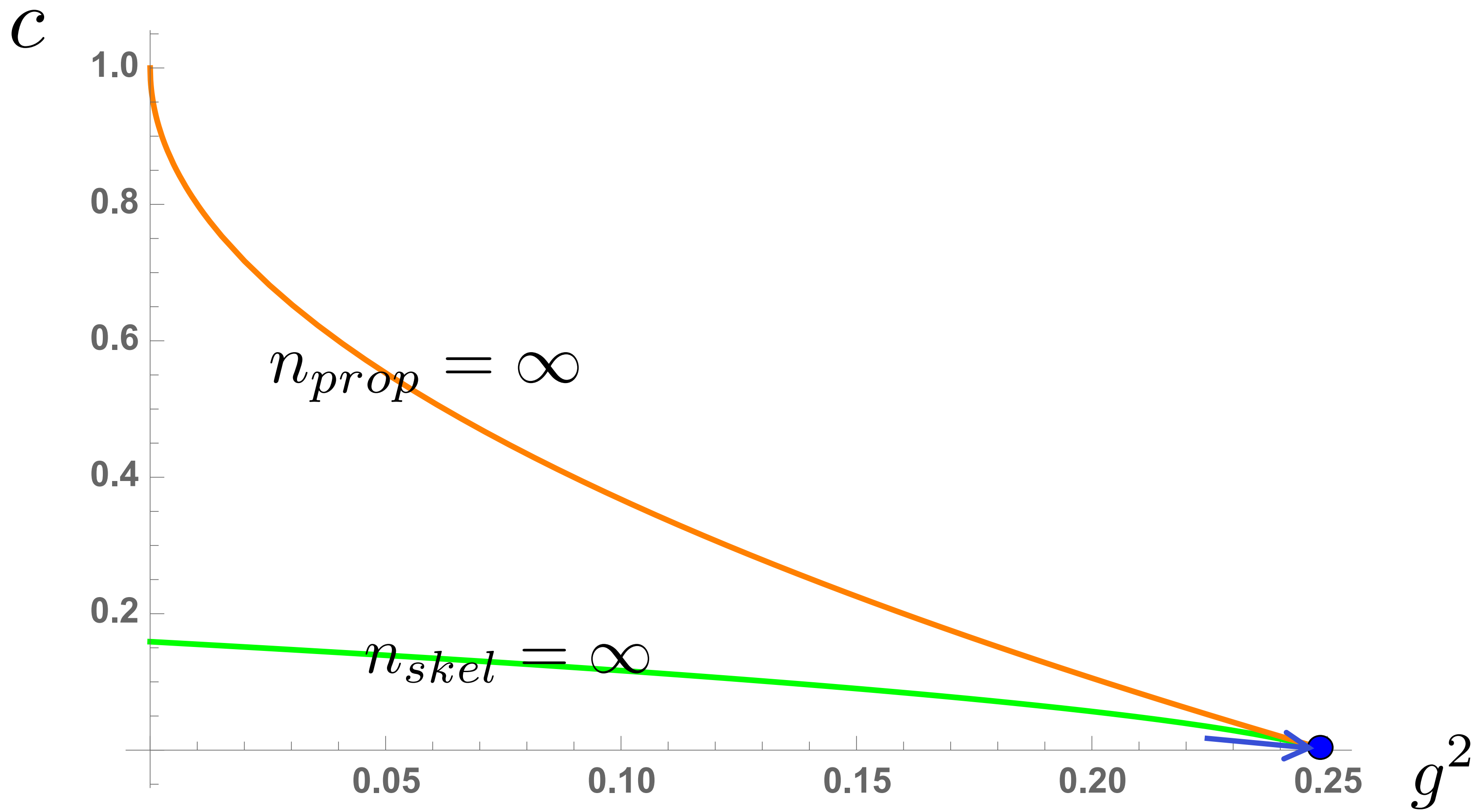
g (weight for vertices), **c** (temperature), **θ** (loop-counting parameter)

Continuum limit around the zero temperature

Scaling of parameters:

$$\boxed{g = g_c(\theta)e^{-\Lambda\varepsilon^2} \qquad c = c_c(\theta) \qquad \theta = \Theta_a \varepsilon^a}$$

$(0 < a \leq 3)$



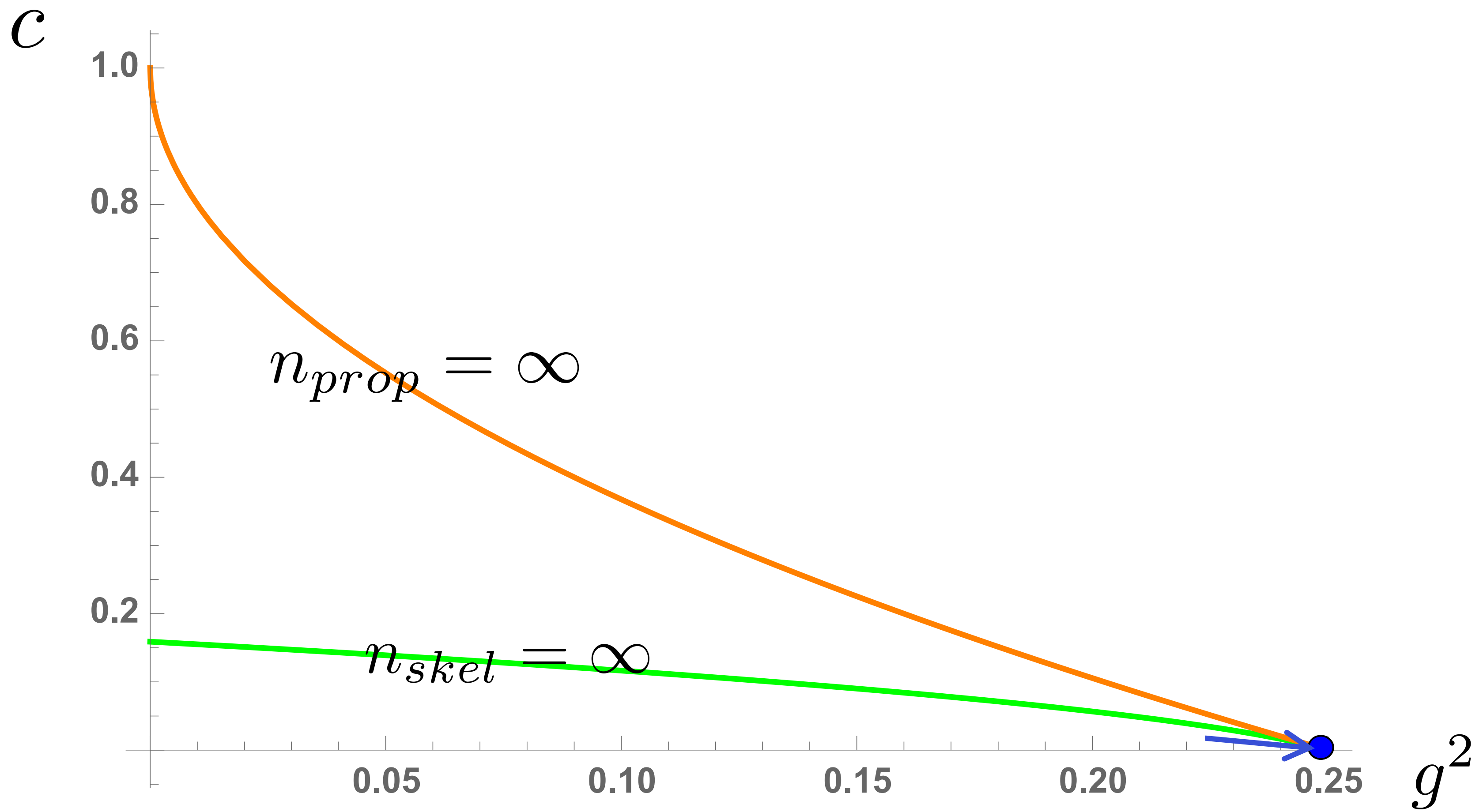
Continuum limit around the zero temperature

Scaling of parameters:

$$\boxed{g = g_c(\theta) e^{-\Lambda \varepsilon^2} \qquad c = c_c(\theta) \qquad \theta = \Theta_a \varepsilon^a}$$

$$(g/g_c)^n = e^{-\Lambda n \varepsilon^2} = e^{-\Lambda A}$$

$$(0 < a \leq 3)$$



Continuum limit around the zero temperature

Scaling of parameters:

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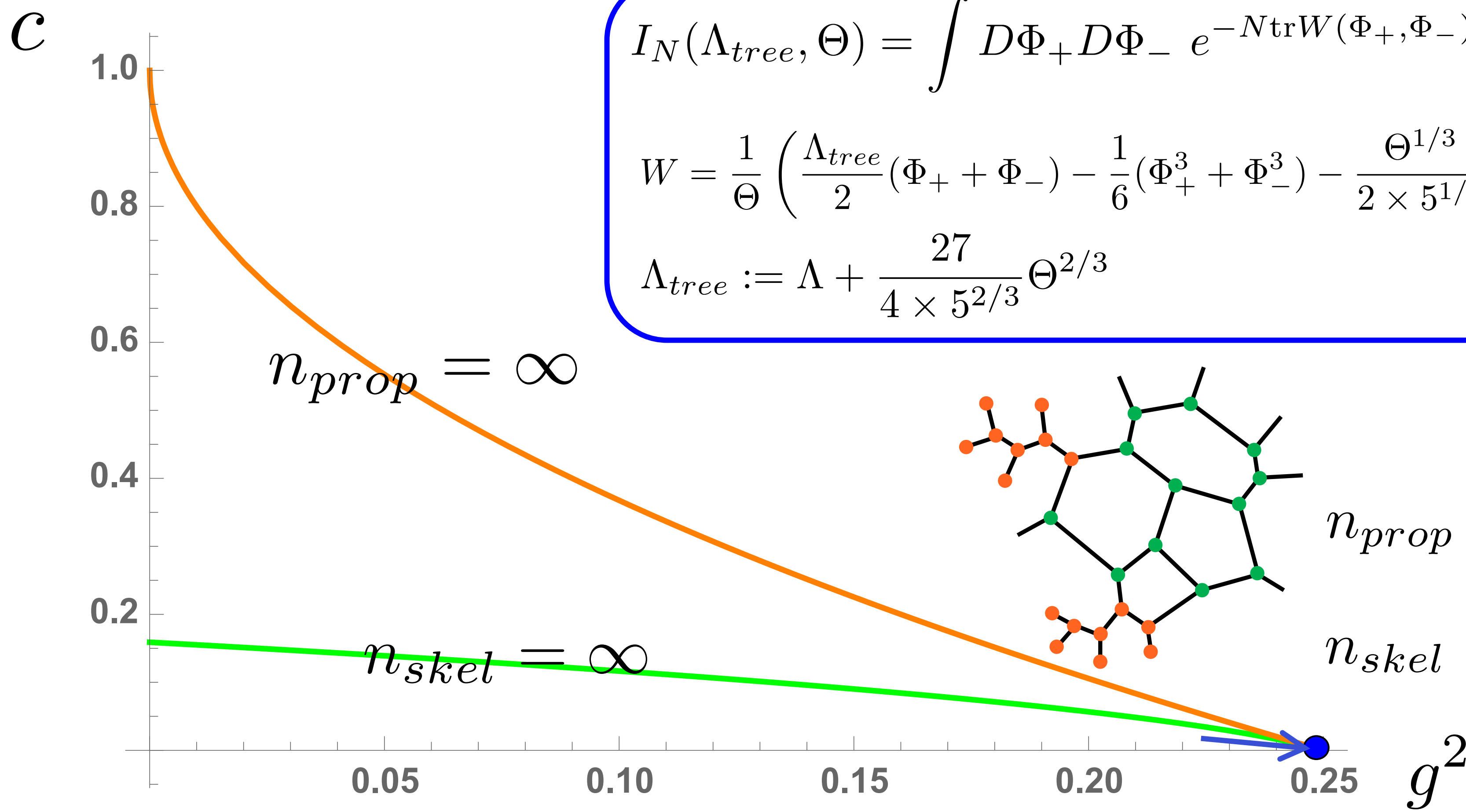
Continuum two-matrix model:

$$I_N(\Lambda_{tree},\Theta) = \int D\Phi_+ D\Phi_- \ e^{-N\mathrm{tr}W(\Phi_+,\Phi_-)}$$

$$W = \frac{1}{\Theta}\left(\frac{\Lambda_{tree}}{2}(\Phi_+ + \Phi_-) - \frac{1}{6}(\Phi_+^3 + \Phi_-^3) - \frac{\Theta^{1/3}}{2\times 5^{1/3}}\Phi_+\Phi_-\right)$$

$$\Lambda_{tree} := \Lambda + \frac{27}{4\times 5^{2/3}}\Theta^{2/3}$$



$$n_{prop} \sim \frac{1}{\varepsilon^2}$$

$$n_{skel} \sim \mathcal{O}(1)$$

Continuum limit around the zero temperature

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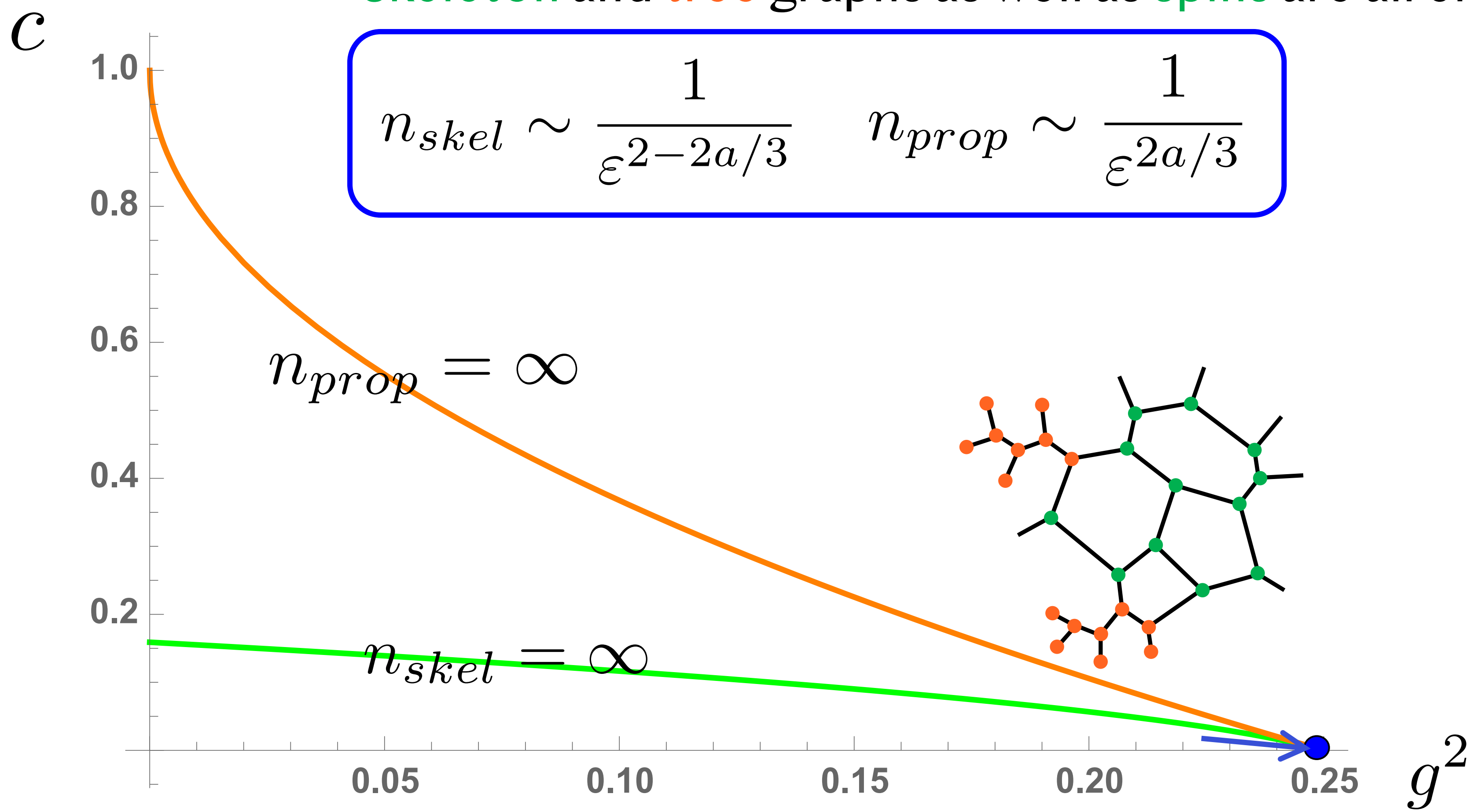
$$(0 < a < 3)$$

Skeleton and tree graphs as well as spins are all critical

$$\boxed{n_{skel} \sim \frac{1}{\varepsilon^{2-2a/3}} \qquad n_{prop} \sim \frac{1}{\varepsilon^{2a/3}}}$$



Continuum limit around the zero temperature

Scaling of parameters:

$$\boxed{g = g_c(\theta) e^{-\Lambda \varepsilon^2} \qquad c = c_c(\theta) \qquad \theta = \Theta_a \varepsilon^a}$$

$$(0 < a < 3)$$

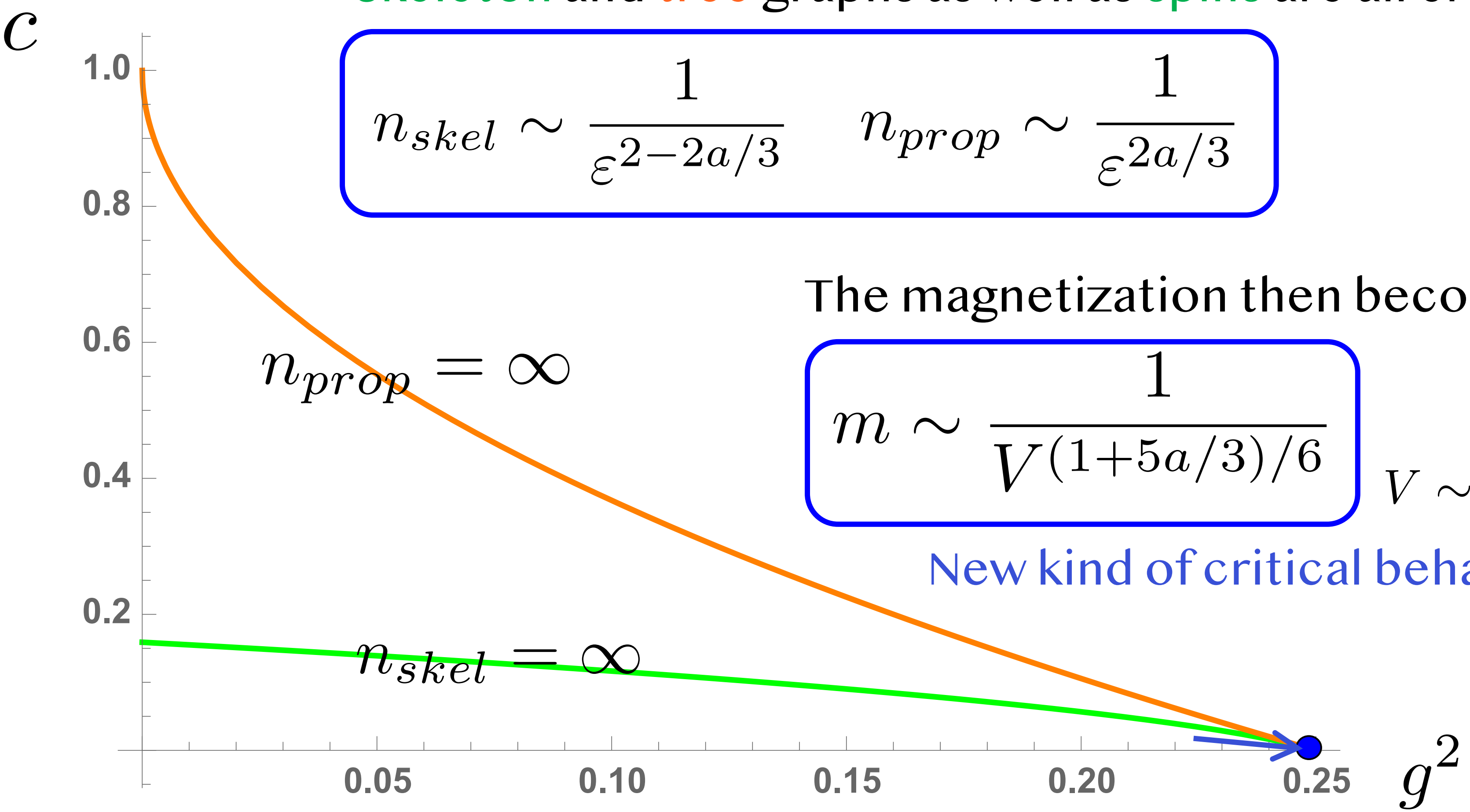
Skeleton and tree graphs as well as spins are all critical

$$\boxed{n_{skel} \sim \frac{1}{\varepsilon^{2-2a/3}} \qquad n_{prop} \sim \frac{1}{\varepsilon^{2a/3}}}$$

The magnetization then becomes

$$\boxed{m \sim \frac{1}{V^{(1+5a/3)/6}}} \qquad V \sim n \sim \xi^d$$

New kind of critical behavior



Summary & Discussion

Summary

We have shown:

[1] $T_c(\theta)$ can reach the zero temperature by tuning θ to 0.

[2] If tuning θ as $\theta = \Theta\varepsilon^3$, the criticality is governed by trees.

The continuum limit is given by the continuum two-matrix model.

[3] There exists a parameter that characterizes cool down "speed".

In particular, if $\theta = \Theta_a \varepsilon^a \ \ (0 < a < 3)$

tree graphs, skeleton graphs and spins are all important;

the magnetization behaves as $m \sim 1/V^{(1+5a/3)/6}$

[4] On the critical line governed by trees, the free energy becomes

$f(\beta) = -\frac{1}{\beta}\log(4\cosh[\beta])$ ⟷ Free energy of 1d spin chain

Discussion

[A] What are the continuum theories at $T_c = 0$?

[B] What is the physical meaning of a in the continuum theory?

[C] The hyper-scaling relation may be violated: $\alpha \neq 2 - \nu d$

Beyond our model...

[D] Can we extend our argument to the O(n) model?

[E] Can we couple quantum matter to random lattices?

[F] It might be better to change the lattice structure in a homogeneous manner to reduce T_c.

Discussion

Turning off spin DOF by c=0, one can essentially consider the one MM

[Ambjørn-Loll-Watabiki-Westra-Zohren, 2008]

$$V = \frac{1}{\theta}\left(\frac{1}{2}\phi^2 - g\phi - \frac{g}{3}\phi^3\right)$$

The continuum theory w/ the scaling $g = g_c(\theta)e^{-\Lambda\varepsilon^2}$ $\theta = \Theta\varepsilon^3$

→ Generalized CDT
(characterized by finite number of baby universes)

If further setting $\Theta = 0$

→ Continuum limit of CDT
= 2d projectable Horava-Lifshitz gravity

[Ambjørn-Glaser-YS-Watabiki, 2013]