Semi-classical saddles of three-dimensional gravity via holography

Yasuaki Hikida (YITP, Kyoto U.)

In corroboration with

Heng-Yu Chen (NTU), Yusuke Taki (YITP), Takahiro Uetoko (Kagawa)

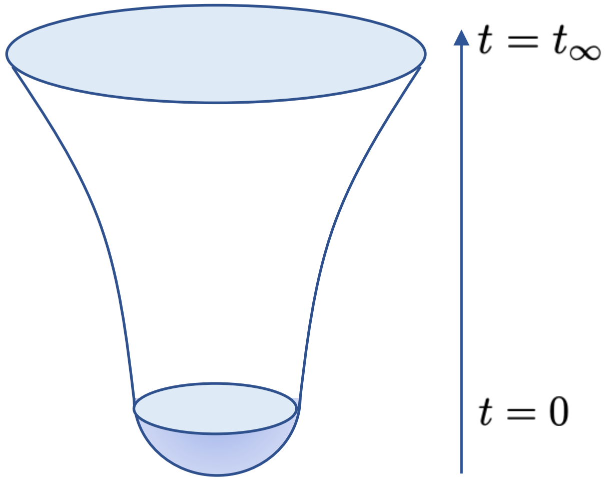
Refs. PRD107(2023)L101902; PRD108(2023)066005; PRD110(2024) 2026018; JHEP07(2024)283

cf. YH-Nishioka-Takayanagi-Taki, PRL'22; JHEP'22

August 23, 2024@7th International Conference on Holography and String Theory in Da Nang

Quantum gravity

- Path integral formulation
  - We may define quantum gravity in the path integral formulation
  - a crucial problem is which geometry should be integrated over or which saddles should be summed
- Complex geometry
  - Like usual quantum field theory, we could make the path integral convergent by working with complex geometry
  - A canonical example is no-boundary proposal by Hartle-Hawking, where the universe starts from hemi-sphere and approaches to dS space



Allowable complex geometry

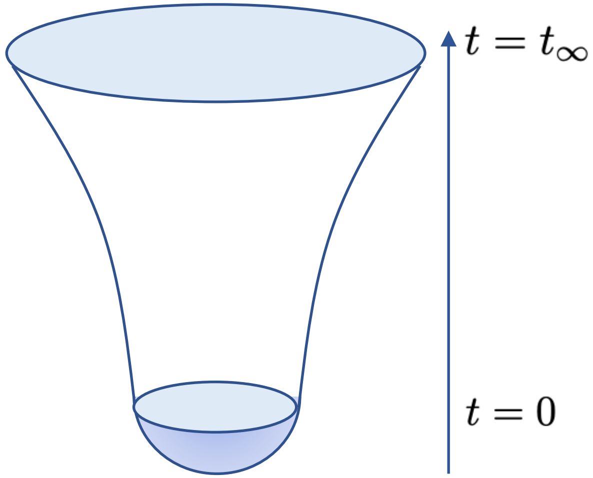
[Louko-Sorkin ’97;Kontsevich-Segal ’21;Witten ’21]

- A complexified metric of S^{d+1}

$$ds^2 = \ell_{\mathrm{dS}}^2(\theta'(u)^2 du^2 + \cos^2\theta(u) d\Omega_d^2)$$

  - Let us assume that the universe starts from nothing at $u=0$ and approaches to dS_{d+1} for $u \to \infty$
  - There is a family of complex geometry labeled by *n*

$$\cos\theta(u=0) = 0 \longrightarrow \theta(u=0) = (n+1/2)\pi \ (n \in \mathbb{Z})$$



- A criteria of *D*-dim. allowable geometry is

$$\mathrm{Re}\left(\sqrt{\det g} g^{i_1 j_1} \dots g^{i_q j_q} F_{i_1 \dots i_q} F_{j_1 \dots j_q}\right) > 0, \ 0 \le q \le D$$

⟹ Only geometry with *n*=-1,0 are allowable, which reproduces the geometry of Hartle-Hawking

Semi-classical saddles via holography

- Holographic method
  - We determine the saddles in gravity path integral via holography and provide their geometrical interpretation
  - As a concrete example, we analyze 3d pure gravity with positive/negative cosmological constant from dual 2d CFT described by Liouville field theory
- Our results
  - For positive cosmological constant, we reproduce Witten’s result as desired
  - For negative cosmological constant, we find that the saddles in the gravity path integral correspond to geometries with three time-like directions
  - The same results can be obtained from mini-superspace approach to gravity

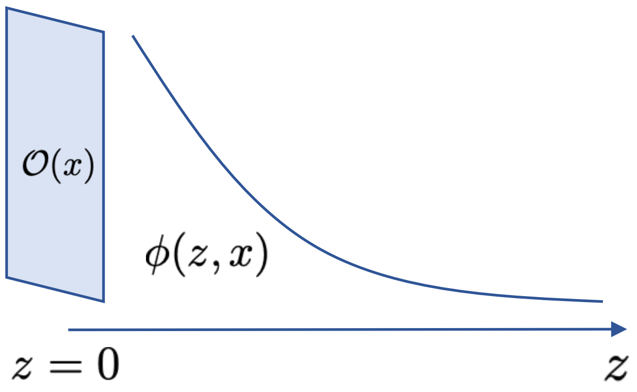
The plan of this talk

- Introduction
- Holographic duality
- Dual CFT description
- Mini-superspace approach
- Conclusion

Holographic duality

AdS/CFT correspondence

- Poincare coordinates (boundary at $z = 0$)



$$ds^2 = \frac{\ell_{\mathrm{AdS}}^2}{z^2}\left(dz^2 - dt^2 + \sum_{j=1}^{d-1}(dz^j)^2\right)$$

  - Map between AdS bulk fields and CFT operators

AdS bulk fields $\phi(z,x)$ ⟺ CFT operators $\mathcal{O}(x)$

- GKP-Witten relation '98
  - Gravity scattering amplitudes ⬄ CFT correlation functions

$$\left\langle \prod_{i=1}^{n} \mathcal{O}(x_i) \right\rangle = \prod_{i=1}^{n} \frac{\delta}{\delta\phi_0(x_i)} \left\langle \exp\left(\int d^d x \phi_0(x)\mathcal{O}(x)\right)\right\rangle \Bigg|_{\phi_0=0}$$

$$\mathcal{Z}_{\mathrm{AdS}}\left[\phi(z=0,x)=\phi_0\right] = \left\langle e^{\int d^d x \phi_0(x)\mathcal{O}(x)} \right\rangle$$

dS/CFT correspondence

[Maldacena '03]

- A way to describe gravity theory on dS space is utilizing wave functional of universe

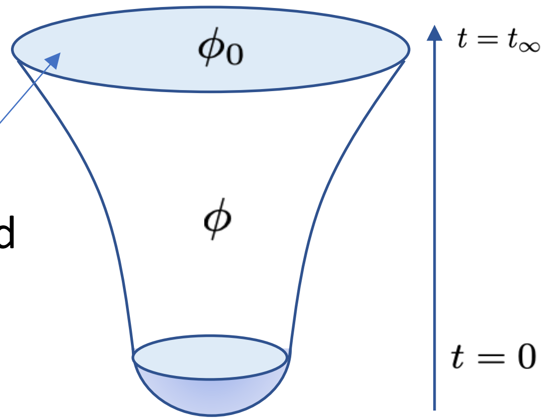
$$\Psi_{\mathrm{dS}}[h, \phi_0] = \int \mathcal{D}g\mathcal{D}\phi \exp iS[g, \phi]$$

$$\text{with } g = h, \phi = \phi_0 \text{ at } t = t_\infty$$

- The wave functional is identified as generating functional of correlation functions in dual CFT

$$\Psi_{\mathrm{dS}}[\phi_0] = \left\langle \exp\left(\int d^d x \phi_0(x) \mathcal{O}(x)\right)\right\rangle$$

Correlators are computed by dual Euclidean CFT



Gaberdiel-Gopakumar duality for AdS_3

[Castro-Gopakumar-Gutperle-Raeymaekers '12; Gaberdiel-Gopakumar '12]
(see [Gaberdiel-Gopakumar '11] for original proposal)

- A version of Gaberdiel-Gopakumar duality

Higher-spin AdS_3 gravity ($\simeq$SL(N) CS theory) with matters at the classical limit ⟺ 2d coset model with large central charge $\frac{\mathrm{SU}(N)_k \times \mathrm{SU}(N)_1}{\mathrm{SU}(N)_{k+1}}$

Spins of gauge fields $s = 2, 3, \ldots, N$

- The simplest case with N=2

Einstein gravity on AdS_3 with matters at the classical limit ⟺ 2d coset model with large central charge $\frac{\mathrm{SU}(2)_k \times \mathrm{SU}(2)_1}{\mathrm{SU}(2)_{k+1}}$

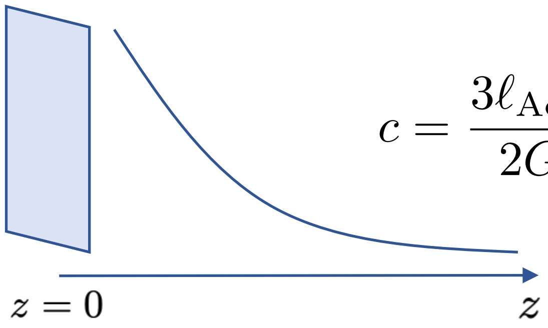
The coset describes analytic continuation of Virasoro-minimal model, which was shown to reduce to Liouville theory [Creutzig-YH '21]

Central charge and the level of coset model

- A version of Gaberdiel-Gopakumar duality

Einstein gravity on AdS_3 with matters at the classical limit ⟷ 2d coset model with large central charge $\frac{\mathrm{SU}(2)_k \times \mathrm{SU}(2)_1}{\mathrm{SU}(2)_{k+1}}$

- Comparison of symmetry algebra
  - Near the boundary of AdS_3 there appears Virasoro symmetry with central charge [Brown-Henneaux '86]



$$c = \frac{3\ell_{\mathrm{AdS}}}{2G} \to \infty$$

  - The central charge of the coset is

$$c = 1 - \frac{6}{(k+2)(k+3)}$$

  - To have large central charge, we have to set

$$k \to -3 - \frac{6}{c} + \mathcal{O}(c^{-2})$$

Analytic continuation from AdS_3 to dS_3

[YH-Nishioka-Takayanagi-Taki '22; '22]

- Formally we can move from AdS_3 to dS_3 by replacing $\ell_{\rm AdS} \to -i\ell_{\rm dS}$
- Gaberdiel-Gopakumar duality becomes

Einstein gravity on dS_3 with matters at the classical limit	⟺	2d coset model with imaginary central charge $\frac{\mathrm{SU}(2)_k \times \mathrm{SU}(2)_1}{\mathrm{SU}(2)_{k+1}}$

- Comparison of central charge [Strominger '01]

$$c = 1 - \frac{6}{(k+2)(k+3)} = -ic^{(g)}, \quad c^{(g)} = \frac{3\ell_{\rm dS}}{2G} \to \infty \quad \Longleftrightarrow \quad k \to -3 + i\frac{6}{c^{(g)}} + \mathcal{O}(c^{(g)-2})$$

- Comparison of partition functions
  - We compute the partition functions of dual CFT at the large central charge limit and find agreement with gravity counterparts

Dual CFT description

Liouville field theory

- The action of Liouville field theory is

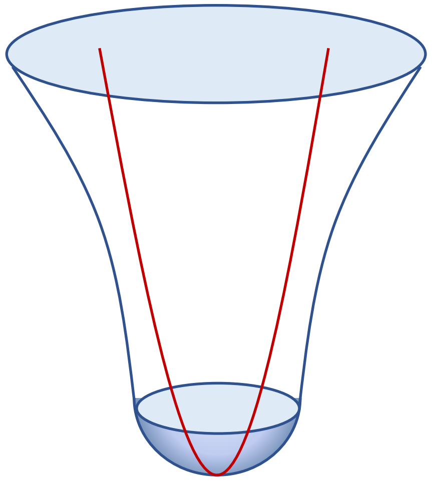
$$S_{\rm L}=\frac{1}{2\pi}\int d^2z\sqrt{g}\left[\partial\phi\bar{\partial}\phi+\frac{Q}{4}\mathcal{R}\phi+\pi\mu e^{2b\phi}\right],\ \ Q=b+1/b$$

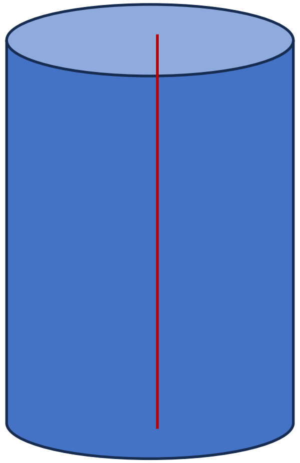
- The large central charge limit is realized by

$$c=1+6(b+b^{-1})^2 \Longrightarrow b^{-2}=\frac{c}{6}-\frac{13}{6}+\cdots$$

- The wave functional or partition function is related to 2-pt. function
  - Here $\eta=\alpha b$ kept finite (and set $\eta\to 0$ for simplicity)

$$\langle V_\alpha(z_1)V_\alpha(z_2)\rangle=\int\mathcal{D}\phi e^{-S_L}e^{2\alpha(\phi(z_1)+\phi(z_2))}$$





Semi-classical saddles for CFT 2-pt. function

[Harlow-Maltz-Witten '11]

- Once $\phi_c^{(0)}$ is a solution to the EOM $\partial\bar{\partial}\phi_c = 2\pi\mu b^2 e^{\phi_c}$, then the same is true for $\phi_c^{(n)} = \phi_c^{(0)} + 2\pi i n$ $(\phi_c = 2b\phi)$

  ⟹ The integer *n* labels different complex saddles of Liouville field theory

- Semi-classical expression of 2-pt. function can be read off from its exact result as

$$\lim_{\eta=\alpha b\to 0} \langle V_\alpha(z_1)V_\alpha(z_2)\rangle \propto \begin{cases} e^{\pi i/b^2} - e^{-\pi i/b^2} = \sum_{n=-1,0} (-1)^n e^{(2n+1)\pi i/b^2} & \text{for Re } b^{-2} < 0 \\ \dfrac{1}{e^{-\pi i/b^2} - e^{\pi i/b^2}} = \sum_{n=0,1,\ldots\infty} e^{(2n+1)\pi i/b^2} & \text{for Re } b^{-2} > 0 \end{cases}$$

Semi-classical saddles of dS gravity

- The wave functional for dS_3 can be described by the limit of 2-pt. function

$$\Psi_{\rm dS} = \lim_{\eta=\alpha b\to 0} \langle V_\alpha(z_1) V_\alpha(z_2)\rangle$$

- The parameter *b* can be written in terms of gravity parameters as

$$b^{-2} = -i\frac{c^{(g)}}{6} - \frac{13}{6} + \cdots = -i\frac{\ell_{\rm dS}}{4G} - \frac{13}{6} + \cdots \quad \Longrightarrow \quad {\rm Re}\, b^{-2} < 0$$

- The wave functional for for dS_3 can be decomposed as

$$\Psi_{\rm dS} \sim \sum_{n=-1,0} (-1)^n e^{S^{(n)}_{\rm GH}/2 + i\mathcal{I}}\,, \qquad S^{(n)}_{\rm GH} = \frac{(2n+1)\pi\ell_{\rm dS}}{2G}$$

$\Longrightarrow$ We should pick up saddle points of dS gravity with *n*=-1,0
and the result reproduces the allowable geometry of Witten

[Chen-YH-Taki-Uetoko ’23;’23]

Semi-classical saddles of AdS gravity

- The partition function for AdS_3 can be described by the limit of 2-pt. function

$$\mathcal{Z}_{\text{AdS}} = \lim_{\eta=\alpha b\to 0} \langle V_\alpha(z_1) V_\alpha(z_2) \rangle$$

- The parameter *b* can be written in terms of gravity parameters as

$$b^{-2} = \frac{c}{6} - \frac{13}{6} + \cdots = \frac{\ell_{\text{AdS}}}{4G} - \frac{13}{6} + \cdots \quad \Longrightarrow \quad \text{Re}\, b^{-2} > 0$$

- The partition function for AdS_3 can be decomposed as

$$\mathcal{Z}_{\text{AdS}} \sim \sum_{n=0,1,2,\ldots} \Theta_n \mathcal{Z}_0\,, \quad \Theta_n = e^{\frac{\ell_{\text{AdS}}}{2G} n\pi i}\,,$$

→ Which geometry corresponds to the saddle point labeled by *n* and why the sum is taken over *n*=0,1,…?

Geometry corresponding to saddle

- Ansatz for the geometry

$$ds^2 = \ell_{\mathrm{AdS}}^2(\theta'(u)^2 du^2 + \sinh^2\theta(u) d\Omega^2)$$

  - We assume that the manifold truncates at $u = 0$ and approaches to Euclidean AdS_3 for $u \to \infty$
  - There is a family of complex geometry labeled by *n*

$$\sinh\theta(u=0) = 0 \quad \Longrightarrow \quad \theta(u=0) = n\pi i \ (n \in \mathbb{Z})$$

- Geometrical interpretation

$$\theta = n\pi i(1-u) \ (0 \leq u \leq 1), \ \ \theta = u - 1 \ (1 < u)$$

  - Euclidean AdS_3 for $1 < u$ and 3-sphere with 3 time directions for $0 \leq u \leq 1$
  - The 3-sphere can be generated by a large gauge transformation in Chern-Simons formulation of gravity and the phase factor can be reproduced

Mini-superspace approach

Mini-superspace approach

cf. [Feldbrugge-Lehners-Turok'17;Di Tucci-Heller-Lehners'20]

- We want to compute path integral for AdS_3 partition function with $I[g]$ as Einstein-Hilbert action

$$\mathcal{Z}_{\text{AdS}}[h] = \int \mathcal{D}g e^{-I[g]}$$

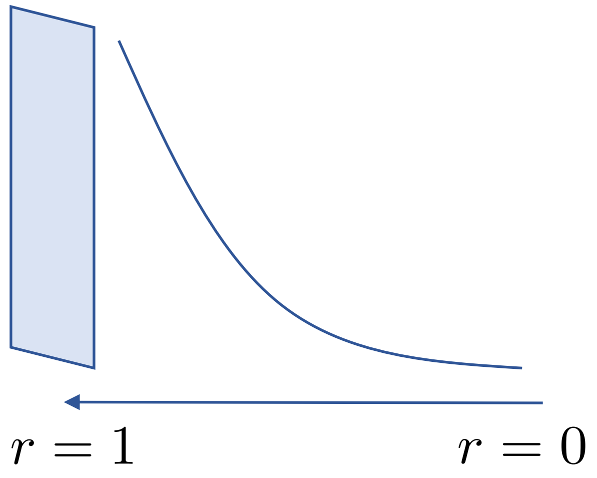
- We consider a reduced model with the following ansatz of metric

$$ds^2 = \ell_{\text{AdS}}^2 \left[N(r)^2 dr^2 + a(r)^2 d\Omega^2\right] \quad (0 \le r \le 1)$$

- The path integral reduce to

$$\mathcal{Z} = \int_{\mathcal{C}} dN \int \mathcal{D}a(r) \exp\left[\frac{\ell_{\text{AdS}}}{2G}\int_0^1 dr\, N\left(\frac{1}{N^2}\frac{d^2 a}{dr^2} + a^2 + 1\right)\right]$$

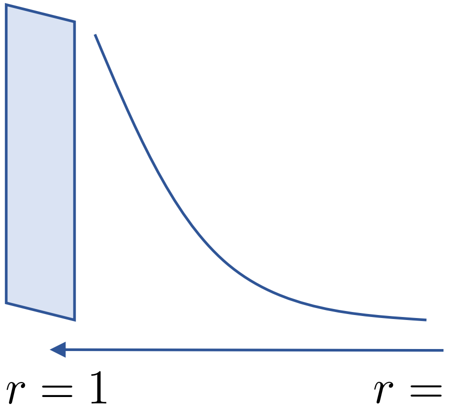
  - We set $N(r) = N$ by fixing a gauge and integrate over N along a contour $\mathcal{C}$



Reduce to one-parameter integration

- The EOM for *a*(*r*) is $d^2a/d\tau^2 - N^2a = 0$
- A solution subject to boundary conditions $a(0) = 0, a(1) = a_1$ is

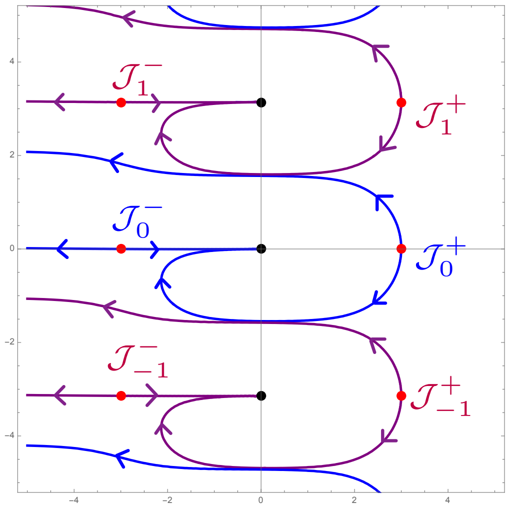
$$\bar{a}^{(N)}(r) = \frac{a_1}{\sinh N}\sinh(Nr)$$



- The path integral is approximated by

$$\mathcal{Z} = \int_{\mathcal{C}} dN e^{-I[N]},\ I[N] = -\frac{\ell_{\rm dS}}{2G}(N + a_1^2 \coth N)$$

- The contour for *N* is given by a set of Lefschetz thimbles

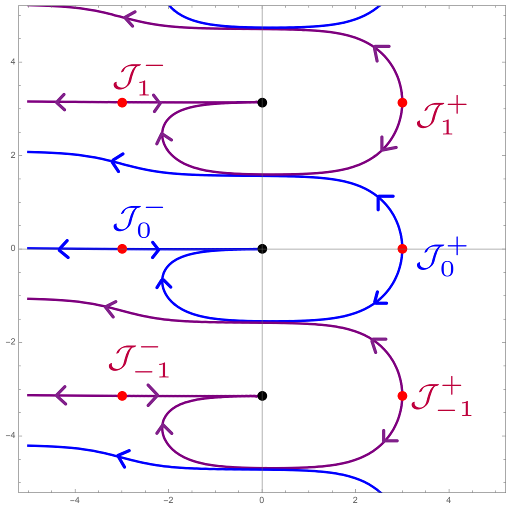


Lefschetz thimbles

- How to determine Lefschetz thimbles
  1. Compute the saddle points by solving $\partial I[N]/\partial N = 0$

$$N_n^+ = n\pi i + \ln\left(a_1 + \sqrt{a_1^2+1}\right), \qquad N_n^- = n\pi i - \ln\left(a_1 + \sqrt{a_1^2+1}\right)$$

  2. Find out steepest descents from the saddle point satisfying $\mathrm{Im}\, I[N] = 0$ as denoted by $\mathcal{J}_n^\pm$

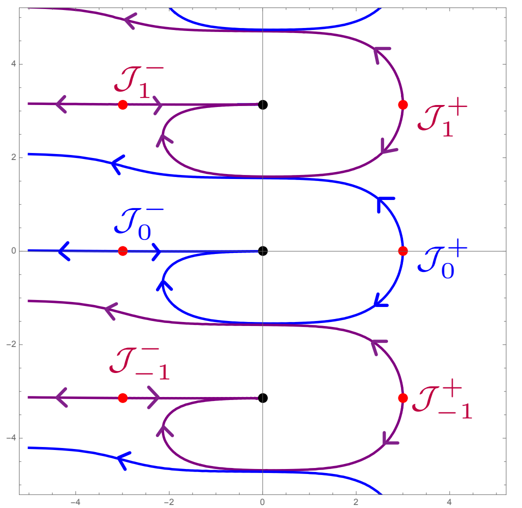


- How to find the contour
  1. Start from a natural contour, i.e., along the positive real axis.
  2. Deform the contour such as to be given by the sum of Lefschetz thimbles

$$\mathcal{C} = \sum_{n=0}^{\infty} \mathcal{J}_n^+ - \sum_{n=1}^{\infty} \mathcal{J}_n^-$$

Evaluation of path integral

[Chen-YH-Taki-Uetoko '24;'24]



- Each contribution from the saddle point is

$$\mathcal{Z}_n^{\pm} \sim e^{\frac{n\pi i \ell_{\mathrm{AdS}}}{2G}} (2a_1)^{\pm \frac{\ell_{\mathrm{AdS}}}{2G}}$$

  - For large a_1, a series of contributions $\mathcal{Z}_n^-$ vanishes
- The path integral is given by the sum as

$$\mathcal{Z} \sim \sum_{n=0}^{\infty} \mathcal{Z}_n{}^{+} \sim \sum_{n=0}^{\infty} e^{\frac{n\pi i \ell_{\mathrm{AdS}}}{2G}}$$

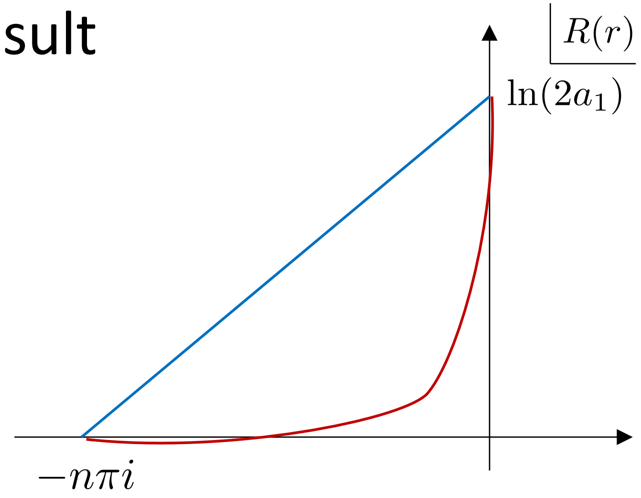
⟹ Reproduce the previous result

- We can change the radial coordinate as

$$Nr \to R(r) = -n\pi i (1-r)^q + \ln(2a_1) r^q$$

cf. [Lehners'21]

  - Reproduce the previous radius coordinate for $q = 1$
  - Reproduce the geometry from ansatz for $q \to \infty$



Conclusion

Summary & Future problems

- We determined the saddles in gravity path integral via holography and examined their geometrical interpretation
- For positive cosmological constant, we reproduce the allowable complex geometry of Witten
- For negative cosmological constant, the saddle points in gravity path integral correspond to geometries with three time-like directions
- The same results can be obtained from mini-superspace approach
- The feature may be specific to 3d pure gravity and it is important to examine other cases

Thank you for your attention