Construction of Superconducting dome
and
Quantum Critical Region

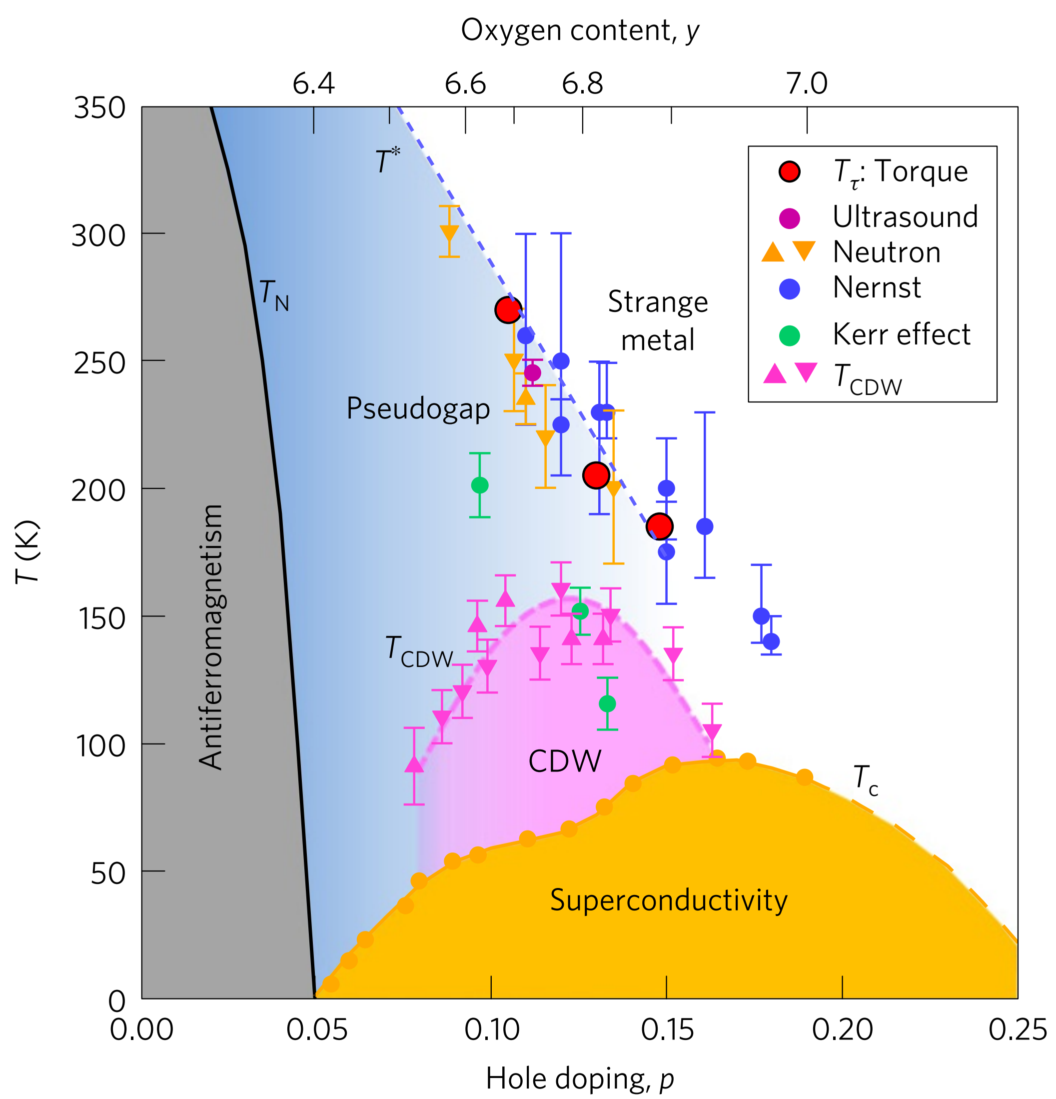
Yunseok Seo(Kookmin Univ.)

August 24, 2024

Based on 2312.06321 and on-going work
with Kyung Kiu Kim and Sejin Kim,

Introduction

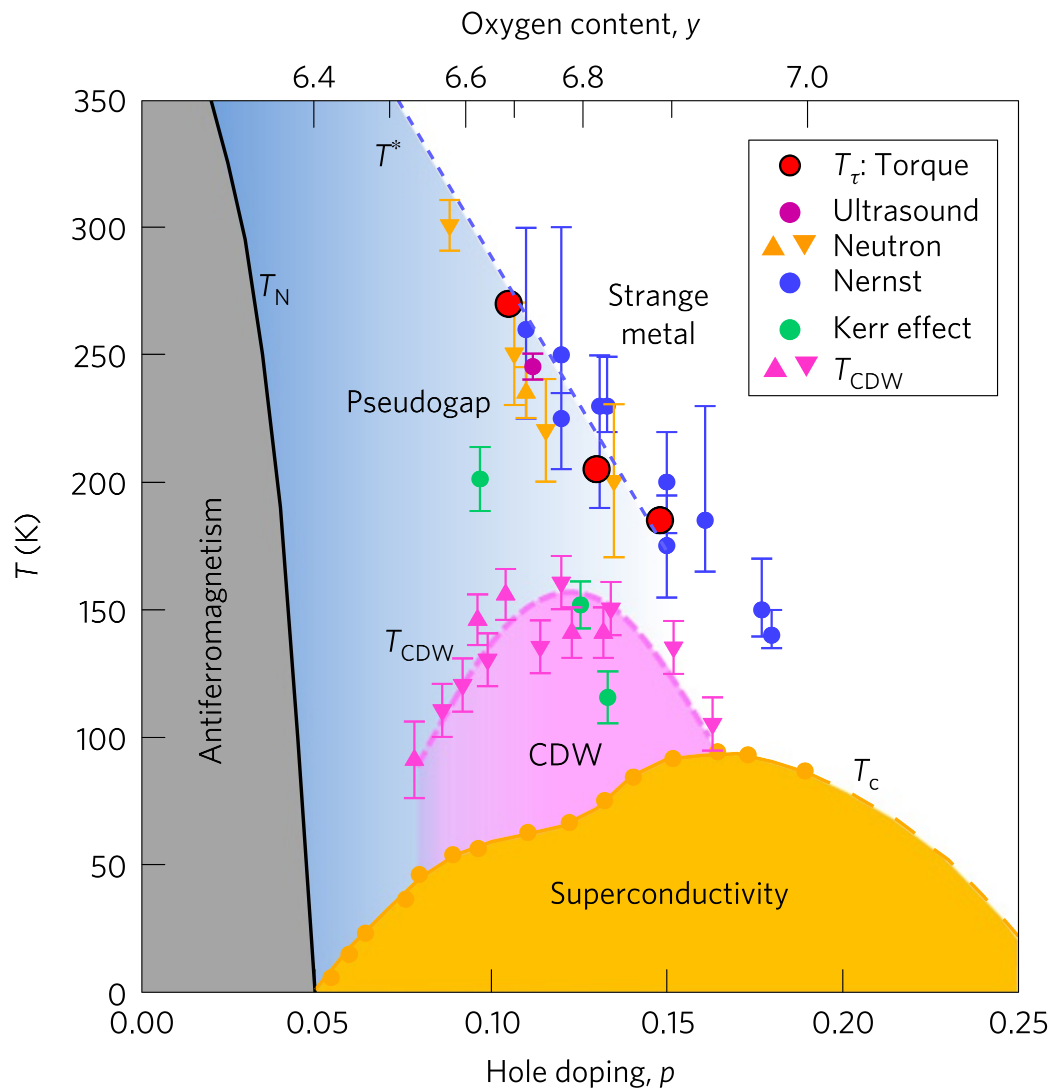
■ Strongly correlated system



YBCO: Nature Physics(2017)

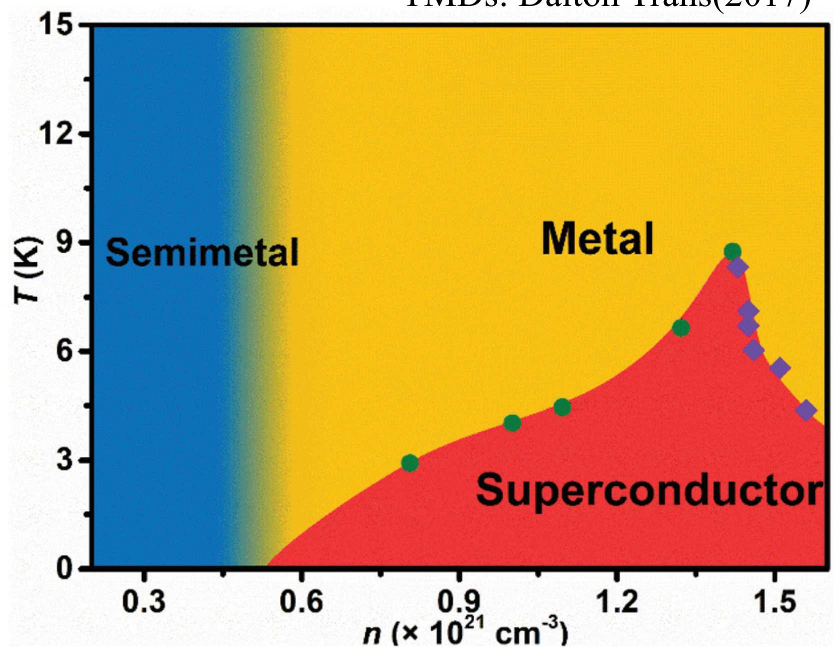
Introduction

■ Strongly correlated system

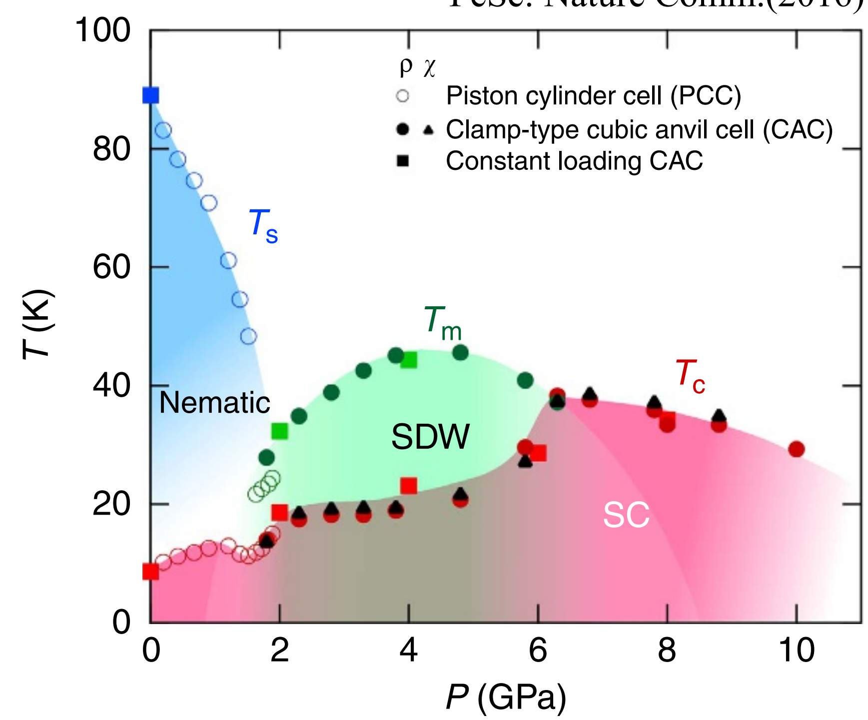


YBCO: Nature Physics(2017)

TMDs: Dalton Trans(2017)



FeSe: Nature Comm.(2016)



Background Geometry

■ Action in 3+1 dim.

Building a Holographic Superconductor #3
Sean A. Hartnoll (Santa Barbara, KITP), Christopher P. Herzog (Princeton U.), Gary T. Horowitz (UC, Santa Barbara)
(Mar, 2008)
Published in: *Phys.Rev.Lett.* 101 (2008) 031601 • e-Print: 0803.3295 [hep-th]
pdf DOI cite claim reference search 1,535 citations

$$S_B = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} - \frac{1}{2}\sum_{i=1}^{2}(\partial\chi^i) - \frac{1}{4}F^2 - |D\phi|^2 - m^2|\phi|^2\right) + S_{NED}\,,$$

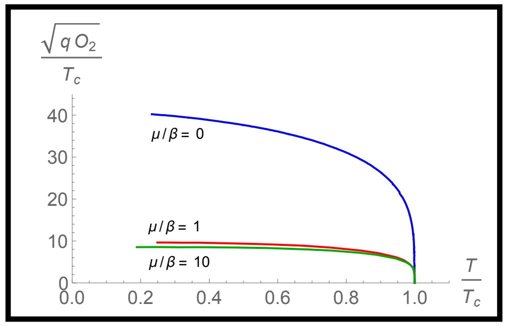
Background Geometry

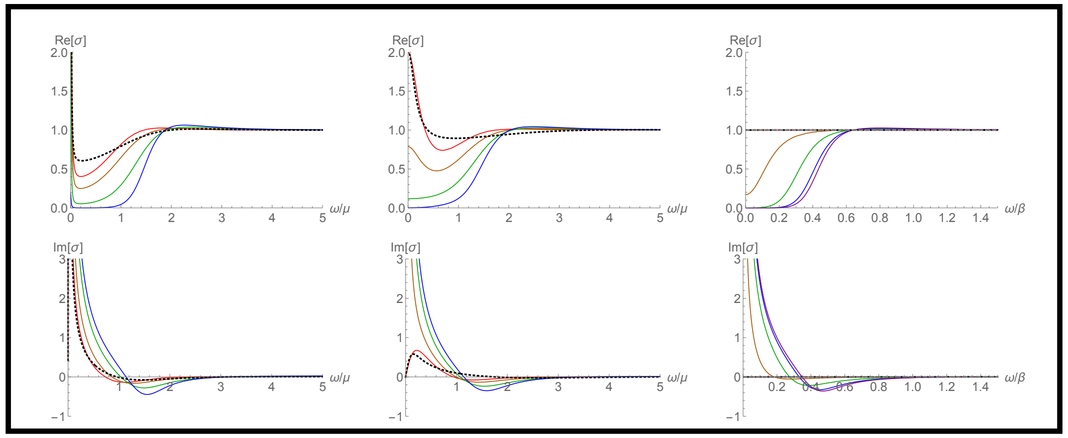
■ Action in 3+1 dim.

Building a Holographic Superconductor #3
Sean A. Hartnoll (Santa Barbara, KITP), Christopher P. Herzog (Princeton U.), Gary T. Horowitz (UC, Santa Barbara) (Mar, 2008)
Published in: *Phys.Rev.Lett.* 101 (2008) 031601 • e-Print: 0803.3295 [hep-th]
pdf DOI cite claim reference search 1,535 citations

$$S_B = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} - \frac{1}{2}\sum_{i=1}^{2}(\partial\chi^i) - \frac{1}{4}F^2 - |D\phi|^2 - m^2|\phi|^2\right) + S_{\cancel{NED}}\,;$$

A Simple Holographic Superconductor with Momentum Relaxation #58
Keun-Young Kim (GIST, Gwangju), Kyung Kiu Kim (GIST, Gwangju), Miok Park (Korea Inst. Advanced Study, Seoul) (Jan 2, 2015)
Published in: *JHEP* 04 (2015) 152 • e-Print: 1501.00446 [hep-th]
pdf DOI cite claim reference search 50 citations





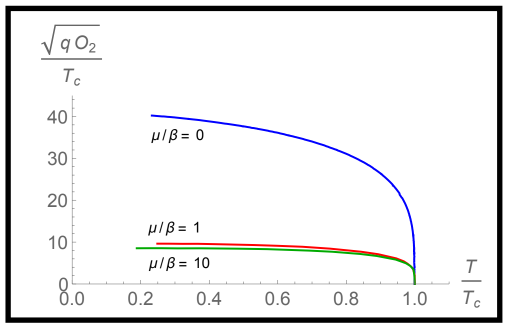
Background Geometry

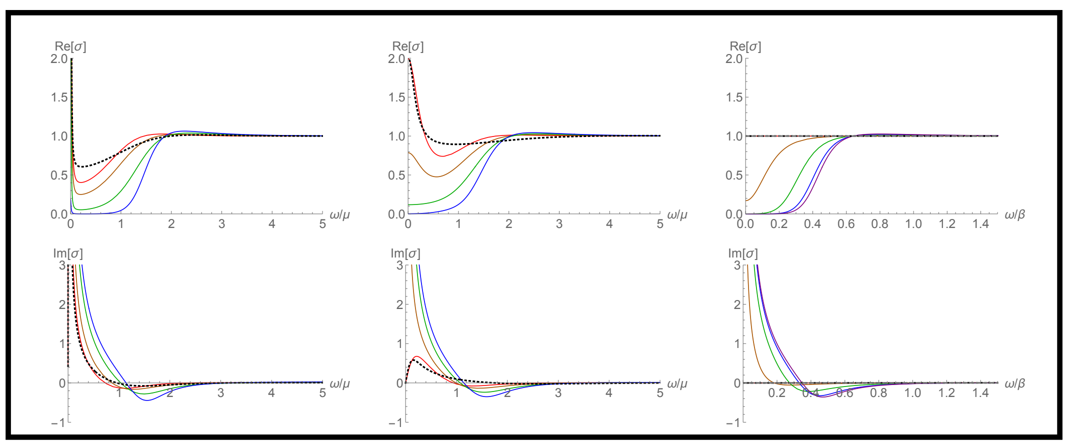
- Action in 3+1 dim.

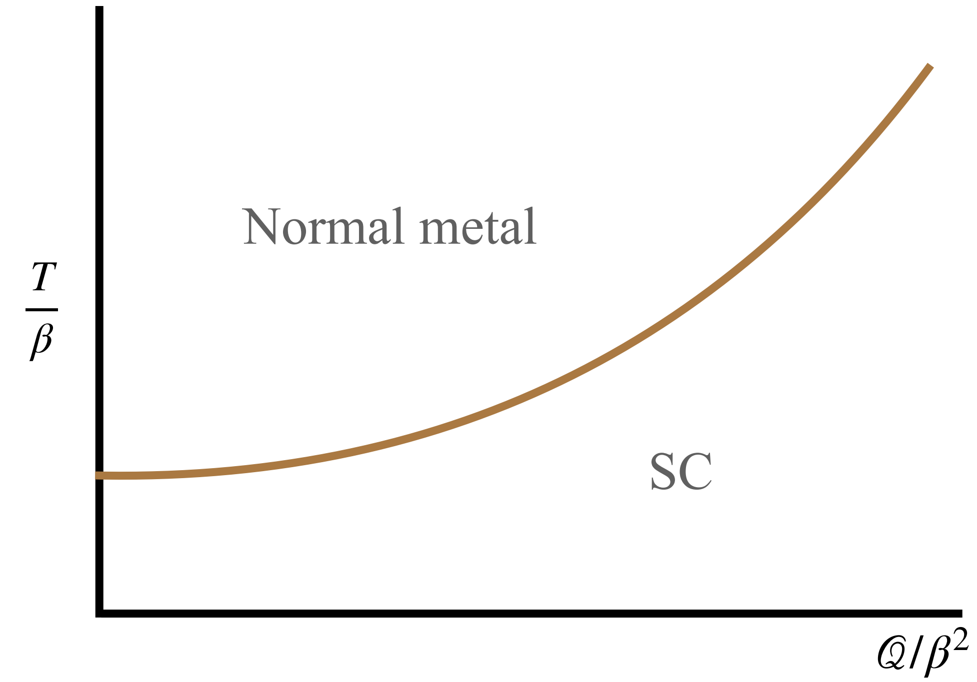
Building a Holographic Superconductor #3
Sean A. Hartnoll (Santa Barbara, KITP), Christopher P. Herzog (Princeton U.), Gary T. Horowitz (UC, Santa Barbara) (Mar, 2008)
Published in: *Phys.Rev.Lett.* 101 (2008) 031601 • e-Print: 0803.3295 [hep-th]
pdf DOI cite claim reference search 1,535 citations

$$S_B = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R+\frac{6}{L^2}-\frac{1}{2}\sum_{i=1}^{2}(\partial\chi^i)-\frac{1}{4}F^2-|D\phi|^2-m^2|\phi|^2\right)+\cancel{S_{NED}}\,,$$

A Simple Holographic Superconductor with Momentum Relaxation #58
Keun-Young Kim (GIST, Gwangju), Kyung Kiu Kim (GIST, Gwangju), Miok Park (Korea Inst. Advanced Study, Seoul) (Jan 2, 2015)
Published in: *JHEP* 04 (2015) 152 • e-Print: 1501.00446 [hep-th]
pdf DOI cite claim reference search 50 citations

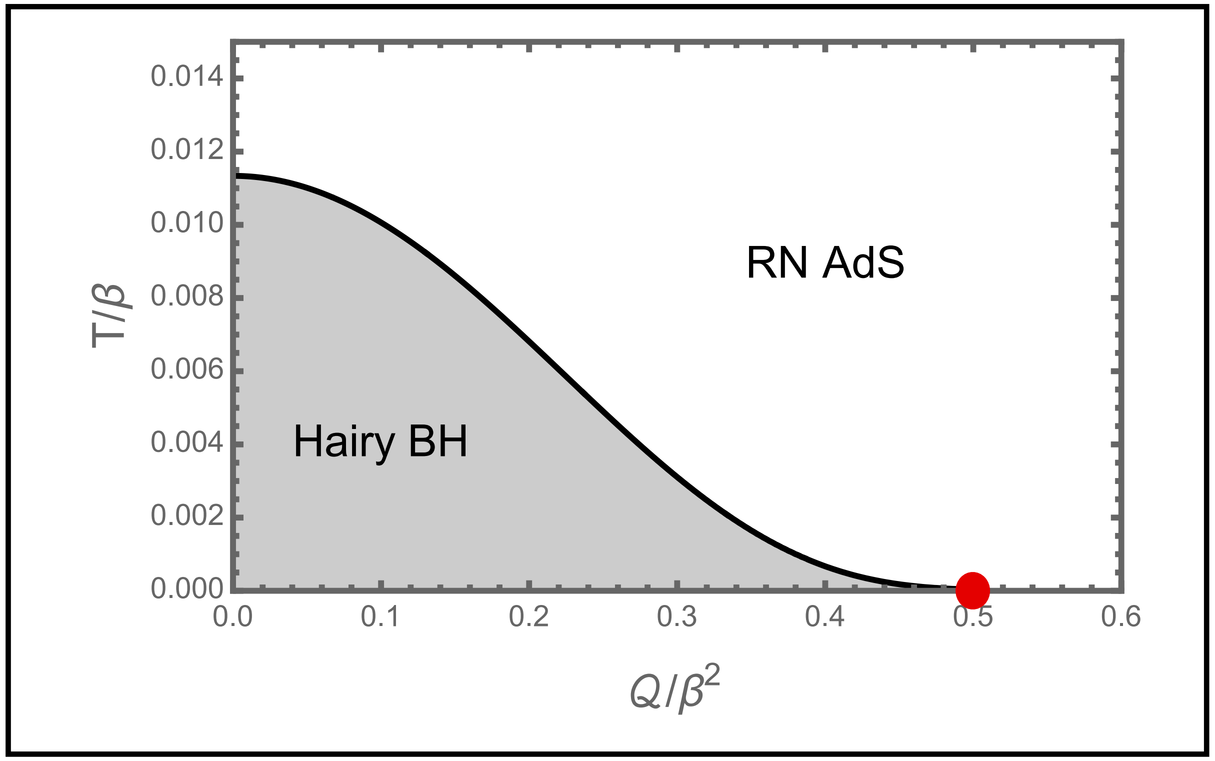






Background Geometry

■ Quantum phase transition



Impurity-driven metal-insulator transitions in holography #1

Yunseok Seo (Seoul, Kookmin U.), Youngjun Ahn (GIST, Gwangju), Keun-Young Kim (GIST, Gwangju), Sang-Jin Sin (Hanyang U.), Kyung Kiu Kim (Seoul, Kookmin U.) (Feb 15, 2023)

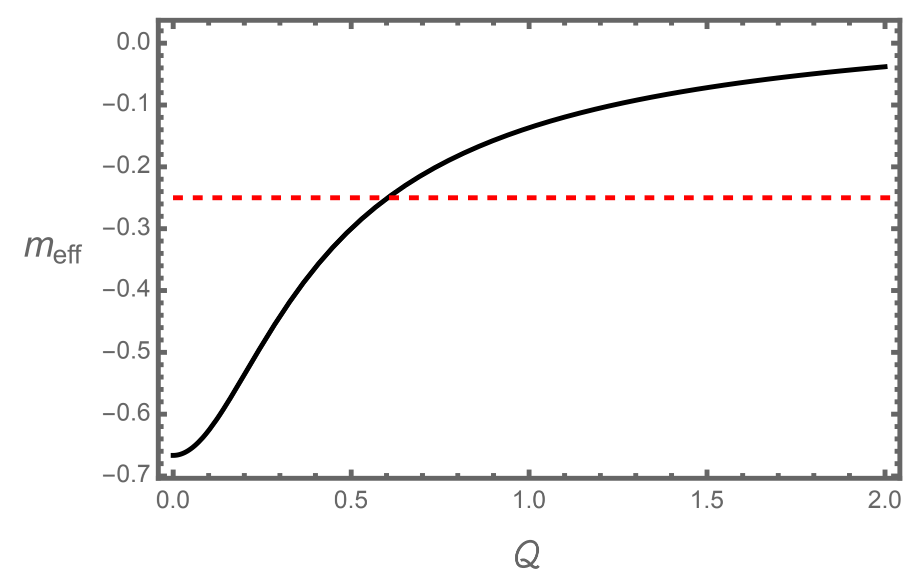
Published in: *JHEP* 06 (2023) 112 • e-Print: 2302.07539 [hep-th]

$$m^2_{\text{eff}} = \left(m^2 + \frac{1}{2}\gamma_2 F^2\right)\bigg|_{r=r_h} = m^2 - \gamma_2 \frac{Q^2 L^6}{r_h^4}$$

$$S_{tot} = S_0 + S_{int} + S_{bd},$$

$$S_0 = \int d^4x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}(\partial\chi)^2 - \frac{1}{4}F^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right)$$

$$S_{int} = -\int \sqrt{-g}\frac{\gamma_2}{4}\phi^2 F^2.$$



Background Geometry

- Action in 3+1 dim. (arXiv:2312.06321)

$$S_B = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R+\frac{6}{L^2}-\frac{1}{2}\sum_{i=1}^{2}(\partial\chi^i)-\frac{1}{4}F^2-|D\phi|^2-m^2|\phi|^2\right)+S_{NED}\,,$$

$$S_{NED} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\frac{1}{4}|\phi|^2\left(\gamma_2 F^2+\gamma_4 L^2(F^2)^2\right)\,.$$

- Ansatz

$$\phi=\phi_1(r)e^{i\phi_2(r)} \qquad A=A_t dt \qquad \chi^I=(\beta x,\beta y)$$

$$ds^2=-U(r)e^{2w(r)}dt^2+\frac{r^2}{L^2}(dx^2+dy^2)+\frac{1}{U(r)}dr^2$$

Holographic Superconductor

■ BF bound analysis

$$\left(D^2 - m^2 - \frac{1}{4}\left(\gamma_2 F^2 + \gamma_4 L^2 \left(F^2\right)^2\right)\right)\phi = 0$$

$$\lim_{r\to r_h} m_{\text{eff}}^2 = m_h^2 = m^2 - 2e^2\left(1 - \frac{1}{\mathbf{p}}\right) - \frac{6\gamma_2}{L^2}\frac{(\mathbf{p}-1)}{(\mathbf{p}+1)} + \frac{12^2\gamma_4}{L^2}\frac{(\mathbf{p}-1)^2}{(\mathbf{p}+1)^2}$$

$$\mathbf{p} \equiv \sqrt{1 + 12\frac{\mathcal{Q}^2}{\beta^4}}$$

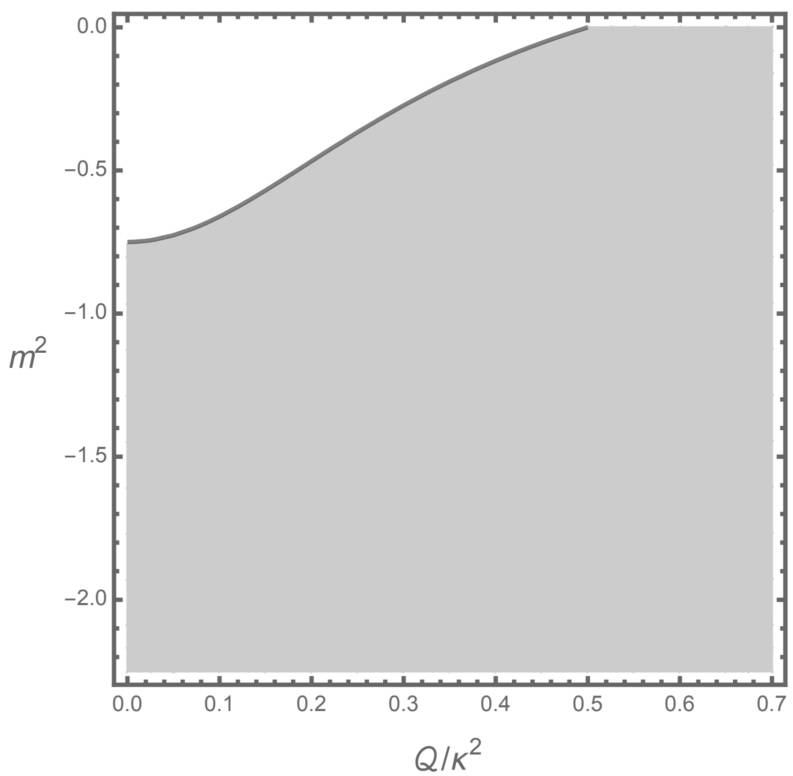
○ Near horizon stability condition

$$m_h^2 L_{\text{eff}}^2 > -1/4$$

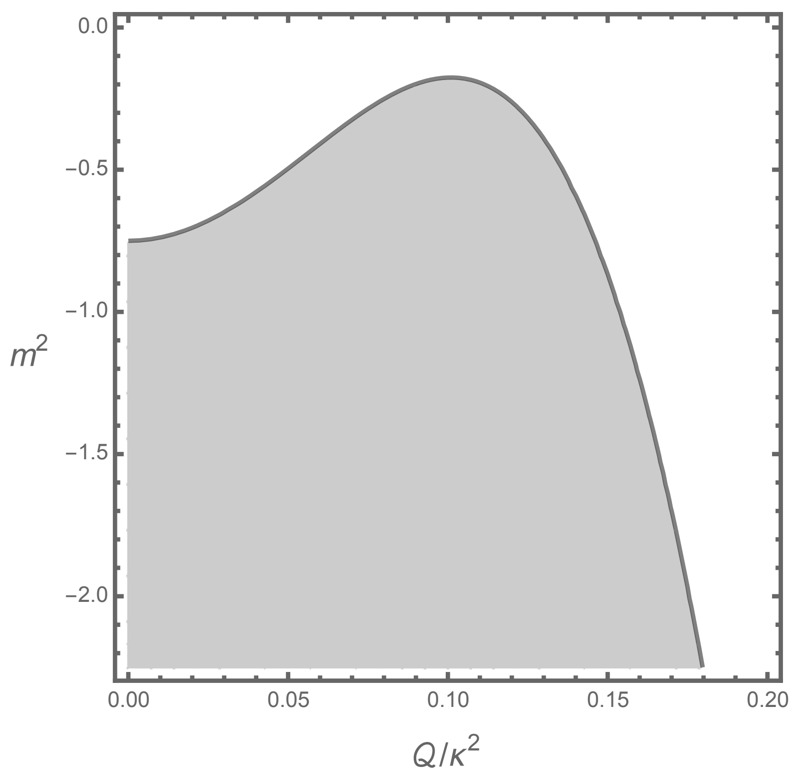
$$\mathbb{H} = \frac{1}{6}\left(1 + \frac{1}{\mathbf{p}}\right)\left\{\tilde{m}^2 - 2\tilde{e}^2\left(1 - \frac{1}{\mathbf{p}}\right) - \frac{6\gamma_2}{L^2}\frac{(\mathbf{p}-1)}{(\mathbf{p}+1)} + \frac{12^2\gamma_4}{L^2}\frac{(\mathbf{p}-1)^2}{(\mathbf{p}+1)^2}\right\} > -\frac{1}{4}$$

Holographic Superconductor

■ BF bound analysis



(a) $\gamma_2 = \gamma_4 = 0$



(b) $\gamma_2 = 6, \ \gamma_4 = 1.5$

Background Geometry

■ Equations of motion

$$
\begin{aligned}
&R_{MN} \\
&-\frac{1}{2}g_{MN}\left(R+\frac{6}{L^2}-\frac{1}{2}\sum_{i=1}^{2}\left(\partial\chi^i\right)-\frac{1}{4}\left(1+\gamma_2|\phi|^2\right)F^2-|D\phi|^2-m^2|\phi|^2-\frac{L^2}{4}\gamma_4\,|\phi|^2(F^2)^2\right) \\
&-\frac{1}{2}\sum_{i=1}^{2}\partial_M\chi^i\partial_N\chi^i-\frac{1}{2}\left(1+\gamma_2|\phi|^2\right)F_M{}^PF_{NP}-\frac{1}{2}\left(D_M\phi D_N\phi^*+D_N\phi D_M\phi^*\right) \\
&-\gamma_4L^2\,|\phi|^2F_M{}^PF_{NP}F^2=0\,, \qquad (18) \\
&\nabla^2\chi^i=0\,, \qquad (19) \\
&\left(D^2-m^2-\frac{1}{4}\left(\gamma_2F^2+\gamma_4L^2\left(F^2\right)^2\right)\right)\phi=0\,, \qquad (20) \\
&\nabla_M\left(1+\gamma_2|\phi|^2+2\gamma_4L^2\,|\phi|^2F^2\right)F^{MN}=ie\left(\phi^*D^N\phi-\phi D^N\phi^*\right)\,. \qquad (21)
\end{aligned}
$$

■ Horizon regularity condition + Source free condition for scalar field

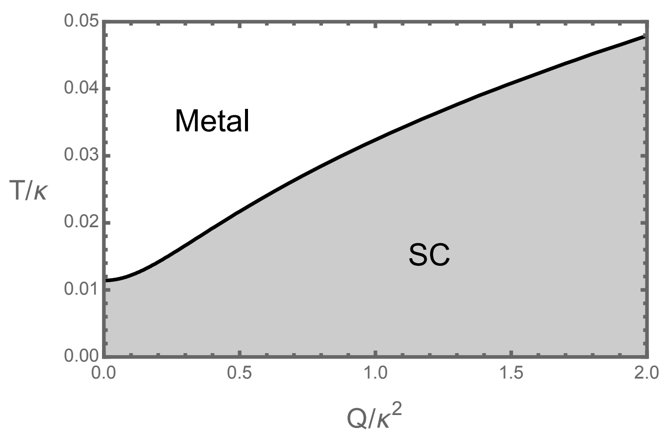
$$
U(r)\Big|_{r\to r_h}\sim(r-r_h)U'(r_h)
$$

$$
a_t(r)\Big|_{r\to r_h}\sim(r-r_h)a_t'(r_h),
$$

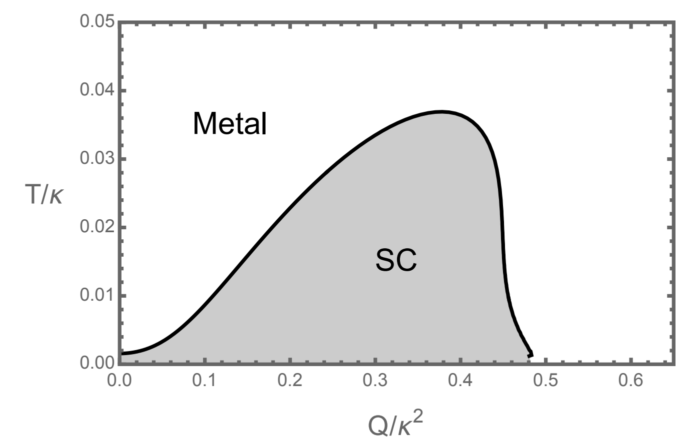
$$
\phi(r)\Big|_{r\to\infty}\sim\frac{J_\phi}{r^{\Delta_-}}+\frac{\mathcal{O}_\phi}{r^{\Delta_+}},\qquad \Delta_\pm=\frac{3}{2}\pm\sqrt{\frac{9}{4}+m^2}
$$

Holographic Superconductor

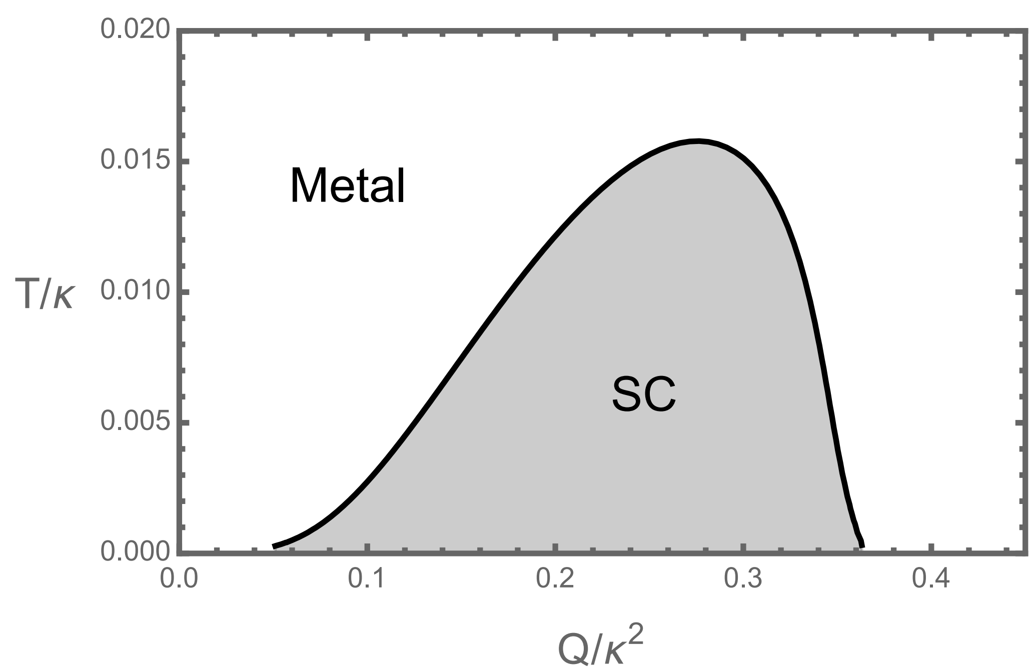
■ Phase diagram



(a) $m^2 = -2,\ \gamma_2 = \gamma_4 = 0$



(b) $m^2 = -1.5,\ \gamma_2 = 6,\ \gamma_4 = 1.5$

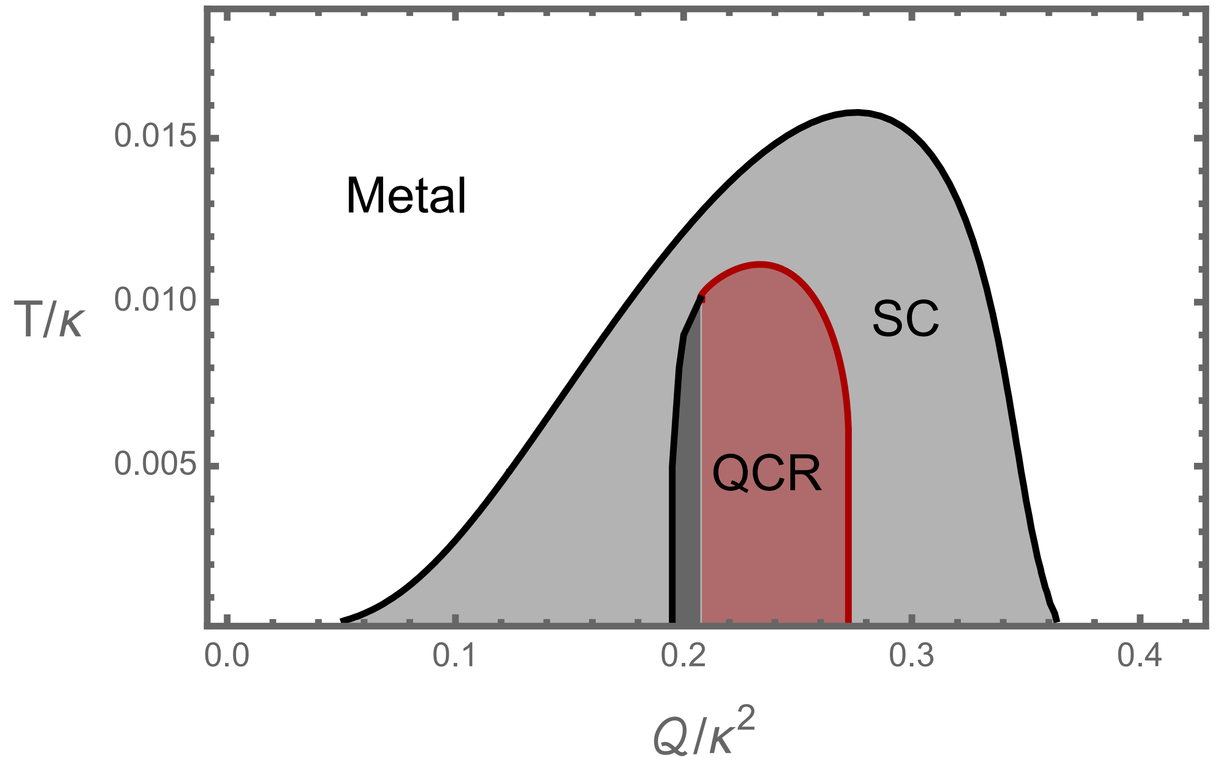


(c) $m^2 = -1,\ \gamma_2 = 6,\ \gamma_4 = 1.5$

Phase Diagram

■ Black hole horizon at $r = r_h$

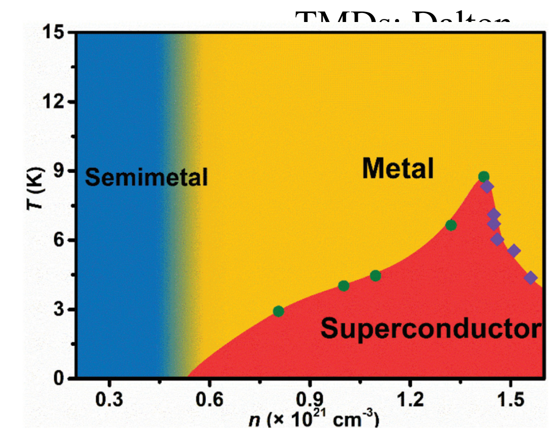
$$U(r)\Big|_{r\to r_h} \sim (r - r_h)U'(r_h)$$
$$a_t(r)\Big|_{r\to r_h} \sim (r - r_h)a_t'(r_h),$$

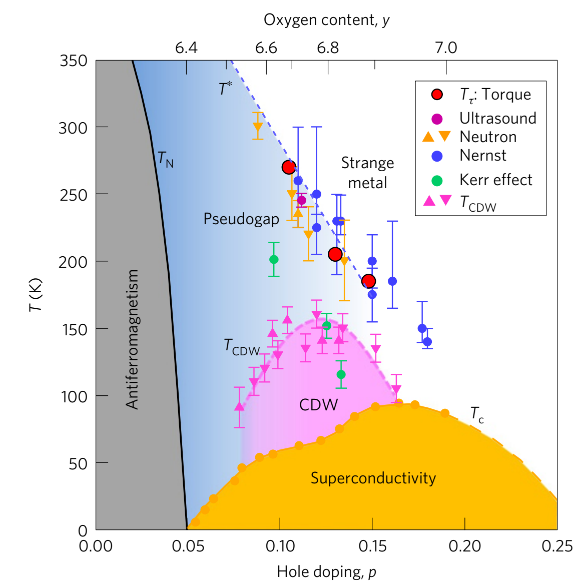


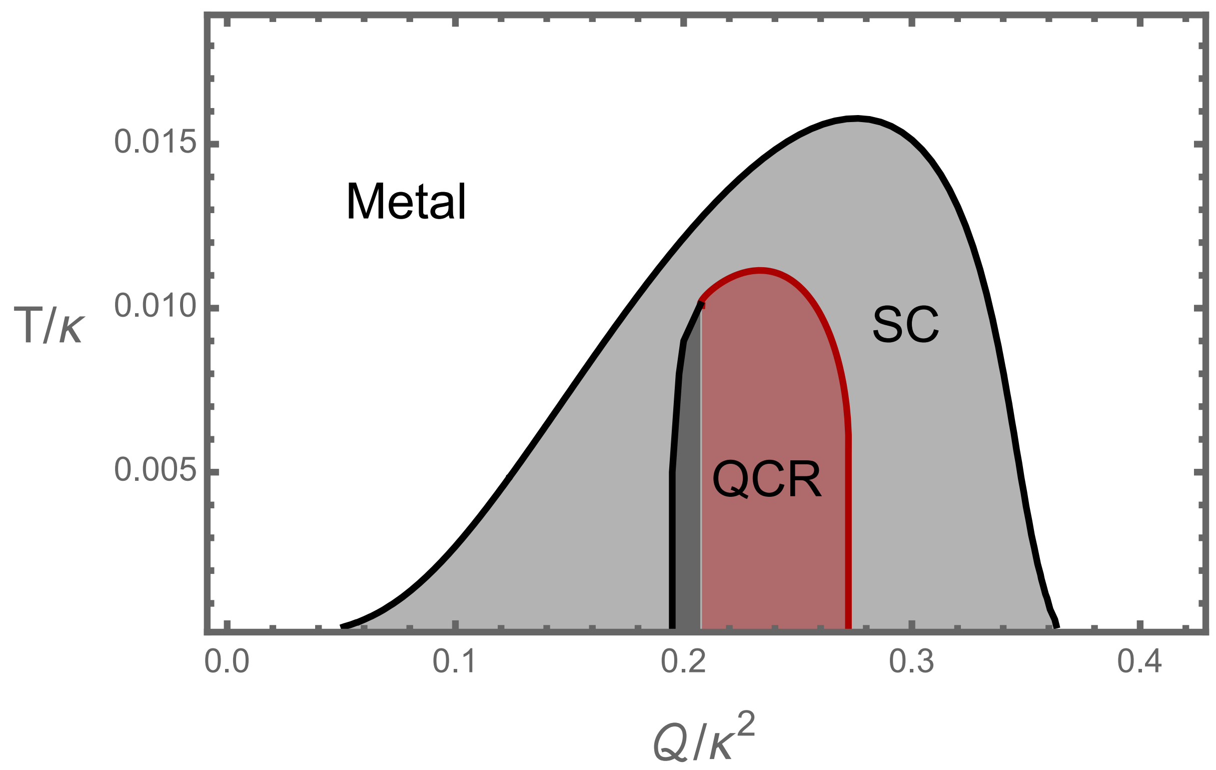
Phase Diagram

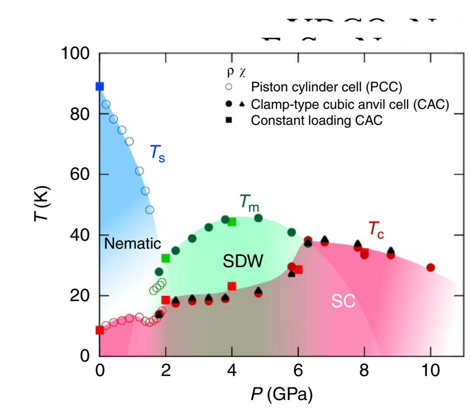
■ Black hole horizon at $r = r_h$

$$U(r)\Big|_{r\to r_h} \sim (r-r_h)U'(r_h)$$
$$a_t(r)\Big|_{r\to r_h} \sim (r-r_h)a_t'(r_h),$$







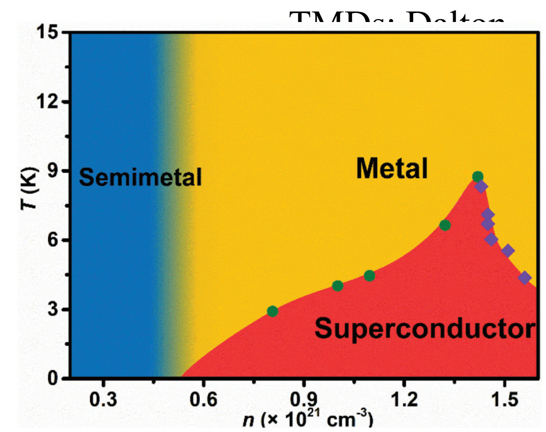


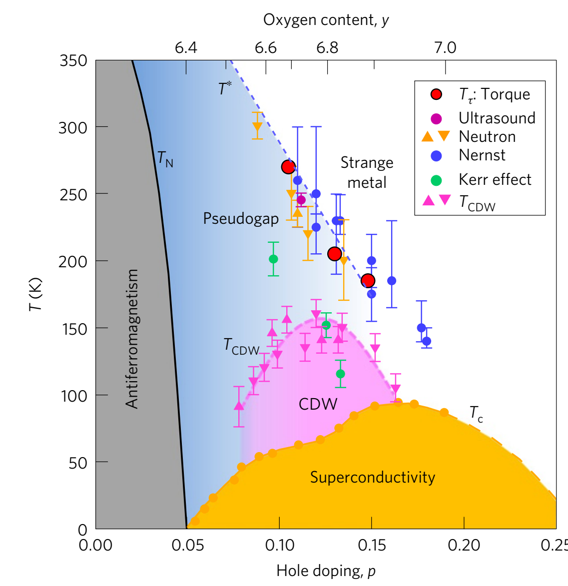
Phase Diagram

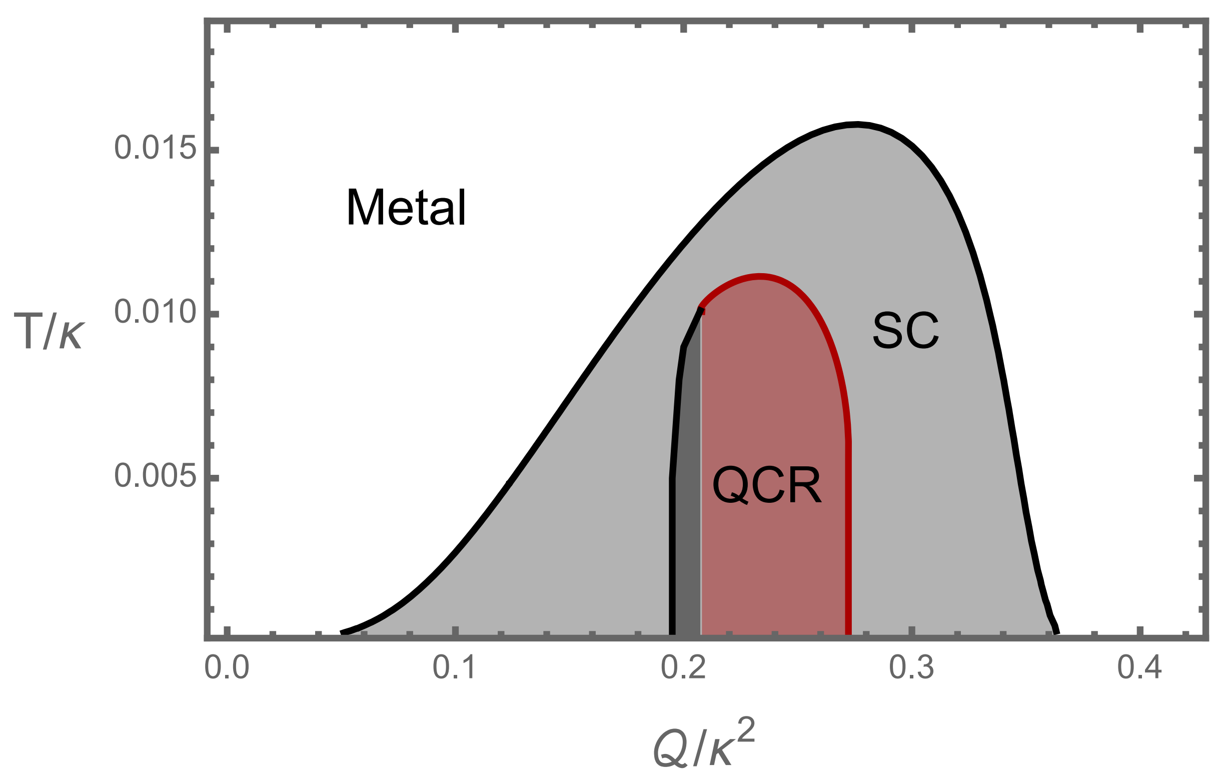
■ Black hole horizon at $r = r_h$

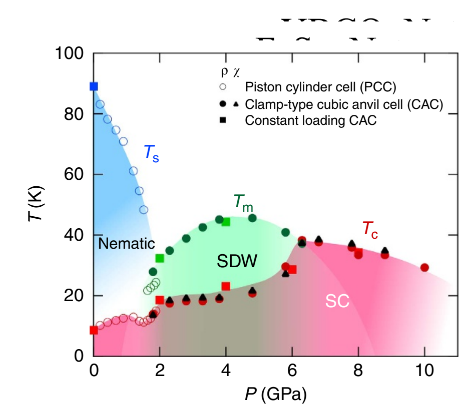
$$U(r)\Big|_{r\to r_h} \sim (r - r_h)U'(r_h)$$
$$a_t(r)\Big|_{r\to r_h} \sim (r - r_h)a_t'(r_h),$$

Sejin's talk!





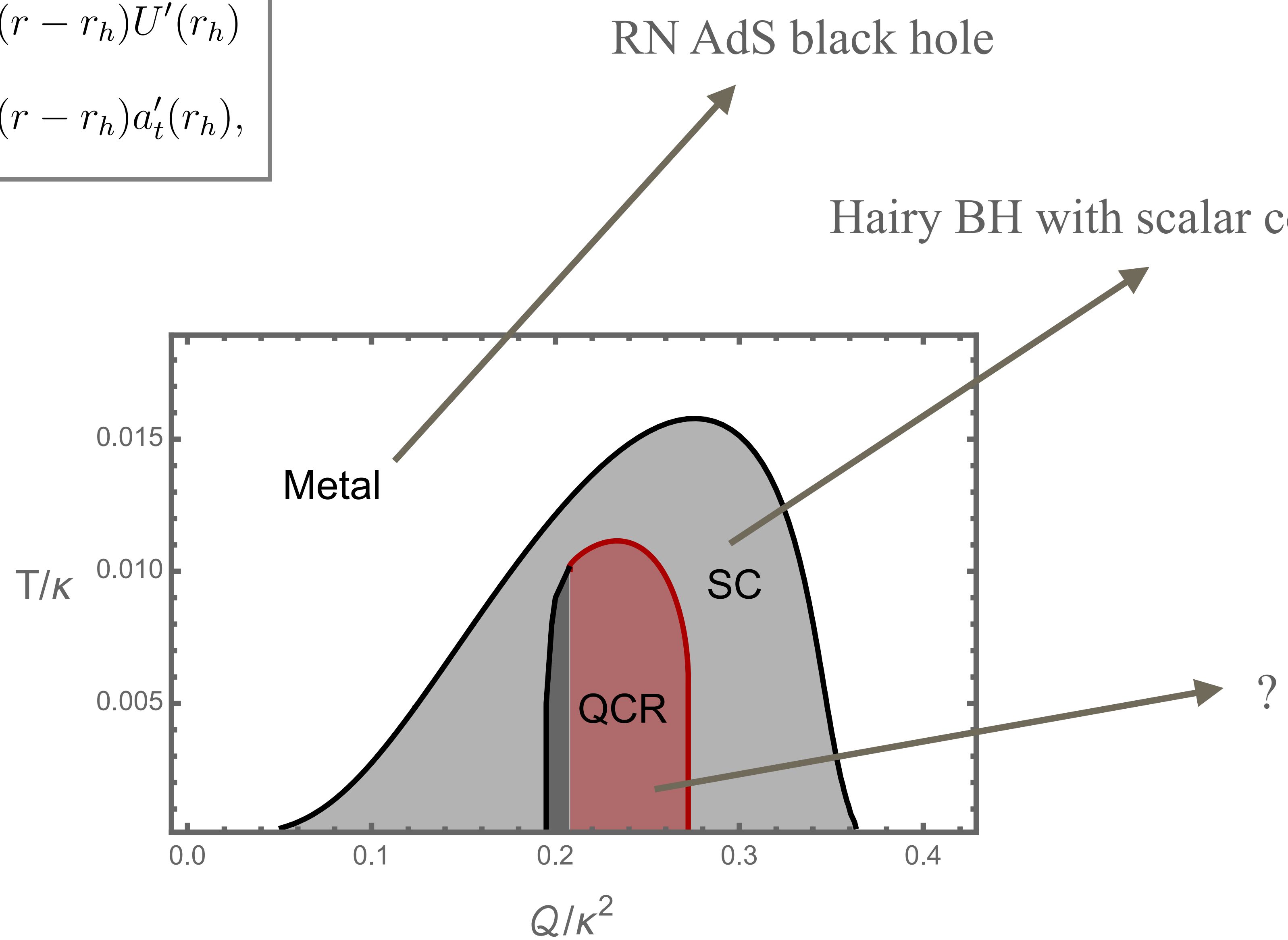




Phase Diagram

■ Black hole horizon at $r = r_h$

$$U(r)\Big|_{r\to r_h} \sim (r - r_h)U'(r_h)$$
$$a_t(r)\Big|_{r\to r_h} \sim (r - r_h)a_t'(r_h),$$

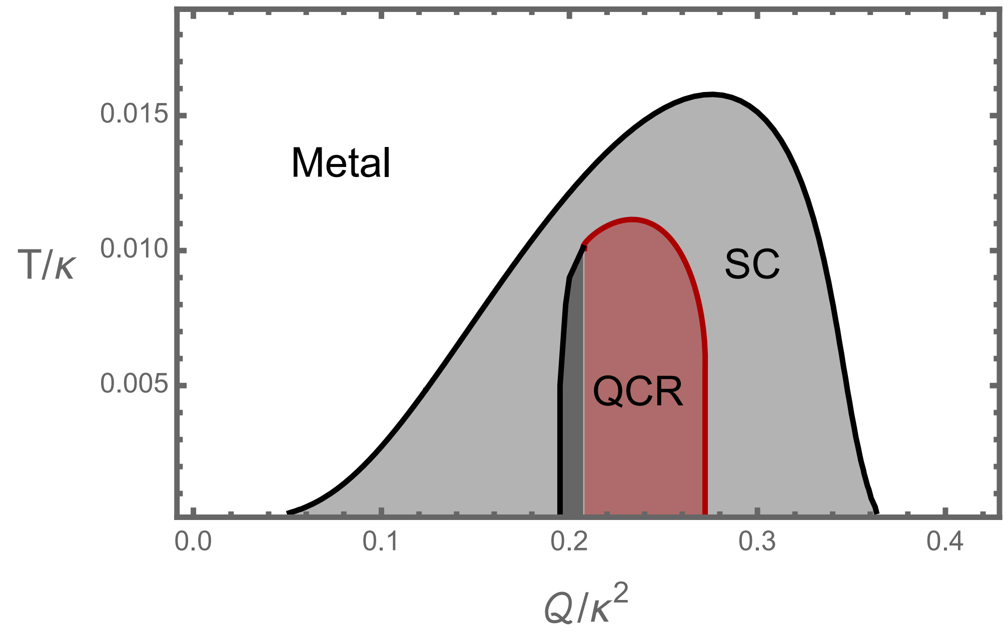


Quantum Critical Region

■ Asymptotic solution

$$\phi(r) \to \phi_0, \qquad U(r) \to r^2,$$

$$A_t(r) \sim A_0\, r^z, \qquad W(r) \sim \frac{1}{2}\log\left(\frac{e^2 A_0^2 \phi_0^2 L_0^4\, r^{2(z-1)}}{2(z-1)}\right)$$

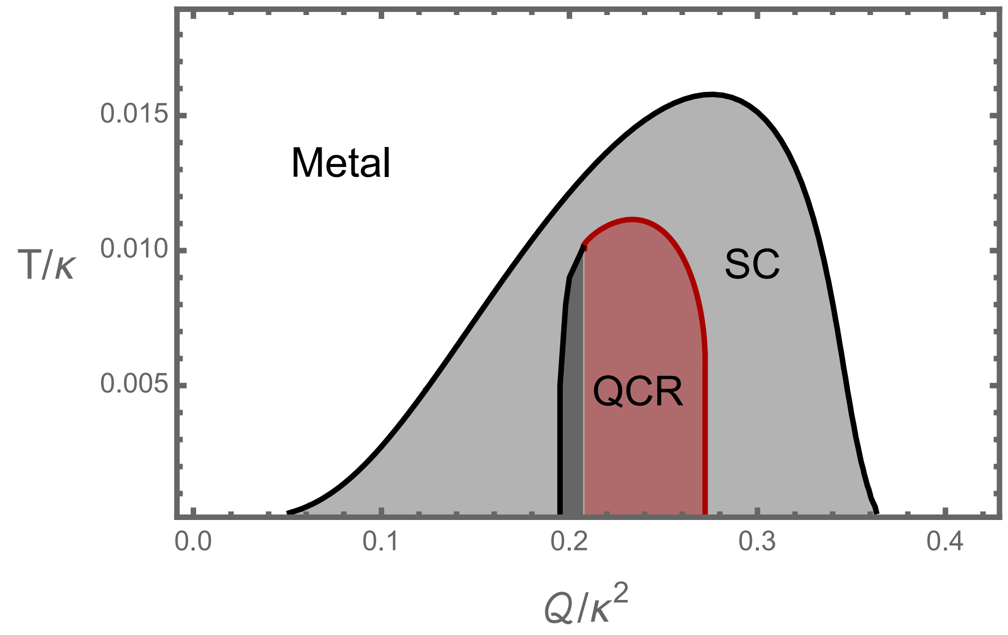


Quantum Critical Region

■ Asymptotic solution

$$\phi(r) \to \phi_0, \qquad U(r) \to r^2,$$

$$A_t(r) \sim A_0\, r^z, \qquad W(r) \sim \frac{1}{2}\log\left(\frac{e^2 A_0^2 \phi_0^2 L_0^4\, r^{2(z-1)}}{2(z-1)}\right)$$



$$ds^2 = -\frac{A_0^2 e^2 \phi_0^2}{z-1}\, r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2(dx^2 + dy^2)$$

Lifshitz symmetry   Quantum Critical Region

$$\frac{(z-1)z^2}{e^2 L_0^4}\gamma_2 - \frac{12(z-1)^2 z^4}{e^4 L_0^8 \phi_0^2}\gamma_4 - 6 + m^2\phi_0^2 + \frac{2(2+z)}{L_0^2} + \frac{(z-1)z^2}{e^2 L_0^4 \phi_0^2} = 0$$

$$e^2 L_0^4 \phi_0^2 (z-1) z^2 \gamma_2 - 4(z-1)^2 z^4 \gamma_4 - e^4 L_0^6 \phi_0^2 (2 - 2z + m^2 L_0^2 \phi_0^2) = 0$$

$$e^2 L_0^4 \phi_0^2 z \gamma_2 - 8(z-1) z^3 \gamma_4 + e^2 L_0^4 (z - e^2 L_0^2 \phi_0^2) = 0.$$

Quantum Critical Region

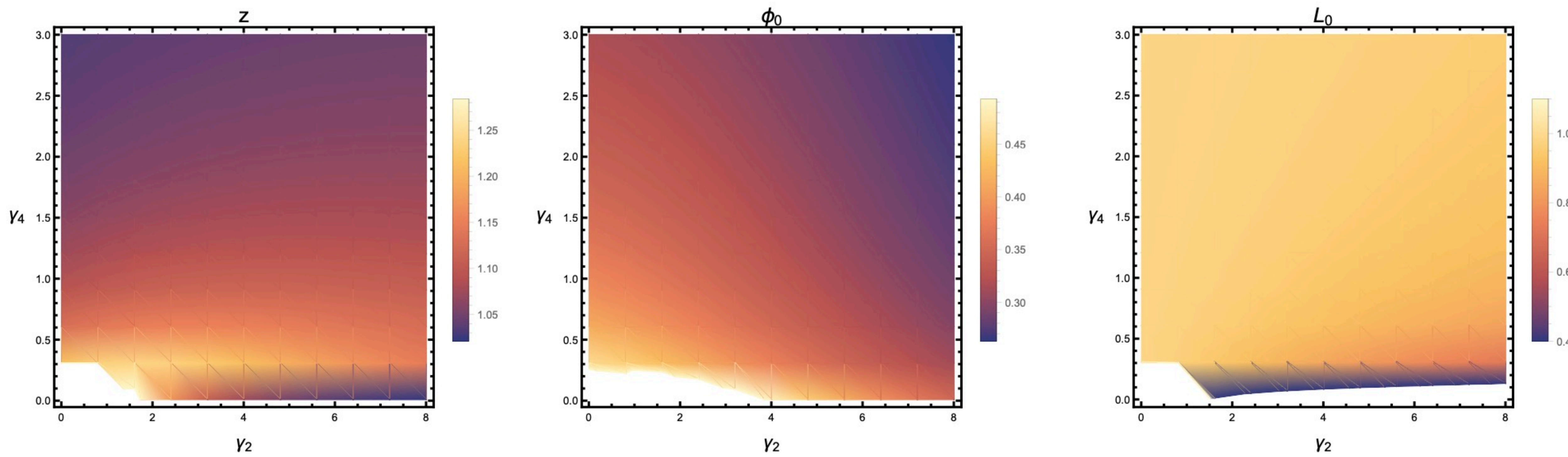
■ Asymptotic solution III

$$\phi(r) \to \phi_0, \qquad U(r) \to r^2,$$

$$A_t(r) \sim A_0 r^z, \qquad W(r) \sim \frac{1}{2}\log\left(\frac{A_0^2 e^2 \phi_0^2\, e^2 (z-1)}{2(z-1)}\right)$$

Lifshitz symmetry Quantum Critical Region

$$ds^2 = -\frac{A_0^2 e^2 \phi_0^2}{z-1} r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2(dx^2 + dy^2)$$



Quantum Critical Region

- On-shell actioin

$$S_B = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} - \frac{1}{2}\sum_{i=1}^{2}(\partial\chi^i) - \frac{1}{4}F^2 - |D\phi|^2 - m^2|\phi|^2\right) + S_{NED}\,,$$

$$S_{NED} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\frac{1}{4}|\phi|^2\left(\gamma_2 F^2 + \gamma_4 L^2 (F^2)^2\right)\,.$$

$$S_{on} \sim \int d^4x\sqrt{-g}\left(R + \frac{6}{L} - \frac{1}{2}(\partial\chi)^2 - \frac{1}{4}F^2 - \phi_0^2 A_\mu A^\mu - M(F^2)\phi_0^2\right)$$

Quantum Critical Region

■ On-shell actioin

$$S_B = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} - \frac{1}{2}\sum_{i=1}^{2}(\partial\chi^i) - \frac{1}{4}F^2 - |D\phi|^2 - m^2|\phi|^2\right) + S_{NED}\,,$$

$$S_{NED} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\frac{1}{4}|\phi|^2\left(\gamma_2 F^2 + \gamma_4 L^2 (F^2)^2\right)\,.$$

$$S_{on} \sim \int d^4x\sqrt{-g}\left(R + \frac{6}{L} - \frac{1}{2}(\partial\chi)^2 - \frac{1}{4}F^2 - \phi_0^2 A_\mu A^\mu - M(F^2)\phi_0^2\right)$$

- Massive gauge field: Lifshitz solution
- A_0 is free parameter: Charge can be defined independently
- Thermodynamics of boundary theory

Thermodynamics

- Thermodynamic quantities

$$T_H = \frac{1}{4\pi} e^{W(r_H)} U'(r_H)$$

$$s = 4\pi r_H^2$$

$$Q^{rt} = \sqrt{-g}\left(1 + \gamma_2 |\phi|^2 + 2\gamma_4 |\phi^2| F^2\right) F^{rt}$$

Thermodynamics

■ Asymptotic AdS geometry

$$S_{tot} = S_{bulk} + S_c$$
$$= -\int d^3x \int_{r_H}^{\Lambda} dr \sqrt{-g}\,\mathscr{L}_{bulk} - \int_{r=\Lambda} d^3x \sqrt{-\gamma}\left(-2K - 4 + \frac{1}{2}\gamma^{ij}\partial_i\psi^I\partial_j\psi^I\right)$$

Thermodynamics

■ Asymptotic AdS geometry

$$S_{tot} = S_{bulk} + S_c$$
$$= -\int d^3x \int_{r_H}^{\Lambda} dr \sqrt{-g}\,\mathscr{L}_{bulk} - \int_{r=\Lambda} d^3x \sqrt{-\gamma}\left(-2K - 4 + \frac{1}{2}\gamma^{ij}\partial_i\psi^I\partial_j\psi^I\right)$$

○ Einstein tensor + Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}g_{\mu\nu}\left(\mathscr{L}_{bulk} - R\right) - T_{\mu\nu}$$

$$S_{bulk} = -V_3\int_{r_H}^{\Lambda}\sqrt{-g}\left(-G_t^t - G_r^r + 2\frac{T_{xx}}{r^2}\right) = -V_3\int_{r_H}^{\Lambda}\left(-\beta^2 e^{W(r)} - 2(e^{W(r)} rU(r))'\right)$$
$$= -V_3\int_{r_H}^{\Lambda}\left(-\beta^2 e^{W(r)}\right) + 2V_3\,\Lambda\,U(\Lambda)$$

$$S_c = -V_3\left[\sqrt{U(\Lambda)}\left(-4\Lambda^2 + \beta^2\right) + \Lambda^2 U'(\Lambda) + 4\Lambda\,U(\Lambda)\right]$$

Thermodynamics

- Asymptotic AdS geometry

$$\frac{S_{on}}{V_3} = -M_0 - \beta^2 \Lambda - \int_{r_H}^{\Lambda} (-\beta^2 e^{W(r)})$$

$$U(r) \sim r^2 - \frac{\beta^2}{2} - \frac{M_0}{r} \cdots$$

$$A_t(r) \sim \mu - \frac{\mathcal{Q}}{r} + \cdots$$

Thermodynamics

■ Asymptotic AdS geometry

$$\frac{S_{on}}{V_3} = -M_0 - \beta^2 \Lambda - \int_{r_H}^{\Lambda} (-\beta^2 e^{W(r)})$$

$$U(r) \sim r^2 - \frac{\beta^2}{2} - \frac{M_0}{r} \cdots$$

$$A_t(r) \sim \mu - \frac{\mathcal{Q}}{r} + \cdots$$

- Trace of the Einstein equation

$$-\beta^2 e^{W(r)} = \partial_r \left[4rU(r)e^{W(r)}\right] - \partial_r \left[r^2 U(r)(e^{W(r)})'\right] + \partial_r \left[(r^2 U(r) e^{W(r)})'\right] - \partial_r \left[Q^{rt}(r) A_t(r)\right]$$

$$\frac{S_{on}}{V_3} = 2M_0 - \mu\mathcal{Q} - r_H^2 U'(r_H) e^{W(r_H)}$$

Thermodynamics

■ Asymptotic AdS geometry

$$\frac{S_{on}}{V_3} = -M_0 - \beta^2 \Lambda - \int_{r_H}^{\Lambda} (-\beta^2 e^{W(r)})$$

$$U(r) \sim r^2 - \frac{\beta^2}{2} - \frac{M_0}{r} \cdots$$

$$A_t(r) \sim \mu - \frac{\mathcal{Q}}{r} + \cdots$$

- Trace of the Einstein equation

$$-\beta^2 e^{W(r)} = \partial_r \left[4rU(r)e^{W(r)}\right] - \partial_r \left[r^2 U(r)(e^{W(r)})'\right] + \partial_r \left[(r^2 U(r) e^{W(r)})'\right] - \partial_r \left[Q^{rt}(r)A_t(r)\right]$$

$$\frac{S_{on}}{V_3} = 2M_0 - \mu\mathcal{Q} - r_H^2 U'(r_H) e^{W(r_H)}$$

- Boundary energy momentum tensor

$$\Pi_{\mu\nu} \equiv \frac{\delta S_{\text{ren}}}{\delta \gamma^{\mu\nu}} = \sqrt{-\gamma}\left(K_{\mu\nu} - K\gamma_{\mu\nu} - 2\gamma_{\mu\nu} + G_{\mu\nu}[\gamma] - \frac{1}{2}\partial_\mu \chi_I \partial_\nu \chi_I + \frac{1}{4}\gamma_{\mu\nu} \nabla\chi_I \cdot \nabla\chi_I\right)$$

$$\langle T_{\mu\nu}\rangle = \lim_{r\to\infty} \frac{2r}{\sqrt{-\gamma}} \Pi_{\mu\nu} = \begin{pmatrix} 2m_0 & 0 & 0 \\ 0 & m_0 & 0 \\ 0 & 0 & m_0 \end{pmatrix}$$

Thermodynamics

■ Asymptotic AdS geometry

$$\frac{S_{on}}{V_3} = -M_0 - \beta^2\,\Lambda - \int_{r_H}^{\Lambda} (-\beta^2 e^{W(r)})$$

$$U(r) \sim r^2 - \frac{\beta^2}{2} - \frac{M_0}{r}\cdots$$

$$A_t(r) \sim \mu - \frac{\mathcal{Q}}{r} + \cdots$$

- Trace of the Einstein equation

$$-\beta^2 e^{W(r)} = \partial_r\left[4rU(r)e^{W(r)}\right] - \partial_r\left[r^2U(r)(e^{W(r)})'\right] + \partial_r\left[(r^2U(r)e^{W(r)})'\right] - \partial_r\left[Q^{rt}(r)A_t(r)\right]$$

$$\frac{S_{on}}{V_3} = 2M_0 - \mu\mathcal{Q} - r_H^2U'(r_H)e^{W(r_H)} = \epsilon - \mu\mathcal{Q} - s\,T \equiv -\mathcal{P}$$

$$\mathcal{P} + \epsilon = s\,T + \mu\mathcal{Q}$$

Thermodynamics

■ Asymptotic Lifshitz geometry

$$S_{tot} = S_{bulk} + S_c$$
$$= -\int d^3x \int_{r_H}^{\Lambda} dr \sqrt{-g}\,\mathscr{L}_{bulk} - \int_{r=\Lambda} d^3x \sqrt{-\gamma} \left(-2K - 4\alpha_1 + \alpha_2 \gamma^{ij} \partial_i \psi^I \partial_j \psi^I + \alpha_3 \frac{1}{\sqrt{-\gamma}} A_t Q^{rt} \right)$$

Thermodynamics

■ Asymptotic Lifshitz geometry

$$S_{tot} = S_{bulk} + S_c$$
$$= -\int d^3x \int_{r_H}^{\Lambda} dr \sqrt{-g}\,\mathcal{L}_{bulk} - \int_{r=\Lambda} d^3x \sqrt{-\gamma} \left(-2K - 4\alpha_1 + \alpha_2 \gamma^{ij} \partial_i \psi^I \partial_j \psi^I + \alpha_3 \frac{1}{\sqrt{-\gamma}} A_t Q^{rt} \right)$$

○ Einstein tensor + Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} g_{\mu\nu} \left(\mathcal{L}_{bulk} - R \right) - T_{\mu\nu}$$

$$S_{bulk} = -V_3 \int_{r_H}^{\Lambda} \sqrt{-g} \left(-G^t_t - G^r_r + 2\frac{T_{xx}}{r^2} \right) = -V_3 \int_{r_H}^{\Lambda} \left(-\beta^2 e^{W(r)} - 2(e^{W(r)} r U(r))' \right)$$
$$= -V_3 \int_{r_H}^{\Lambda} \left(-\beta^2 e^{W(r)} \right) + 2V_3 \Lambda\, U(\Lambda)$$

Thermodynamics

■ Asymptotic Lifshitz geometry

○ Trace of the Einstein equation

$$-\beta^2 e^{W(r)} = \partial_r \left[4rU(r)e^{W(r)}\right] - \partial_r \left[r^2 U(r)(e^{W(r)})'\right] + \partial_r \left[(r^2 U(r)e^{W(r)})'\right] - \partial_r \left[Q^{rt}(r)A_t(r)\right]$$

$$\frac{S_{on}}{V_3} = 2\,\Lambda\, U(\Lambda) + \left| 4rU(r)e^{W(r)} - r^2 U(r)(e^{W(r)})' + (r^2 U(r)e^{W(r)})' - Q^{rt}(r)A_t(r) \right|_{r_H}^{\Lambda} + S_c \Big|_{\Lambda}$$

$$U(r) \sim r^2 - \frac{\beta^2}{2(2-z)} - \frac{M_0}{r^z}\cdots \qquad \phi(r) \sim \phi_0 + \frac{\phi_1}{r}\cdots$$

$$A_t(r) \sim A_0\, r^z + \frac{A_1}{r} + \cdots \qquad e^{2W(r)} \sim \frac{A_0^2 e^2 L_0^4 \phi_0^2}{2(z-1)}\, r^{2z-2} + \cdots$$

Thermodynamics

■ Asymptotic Lifshitz geometry

○ Trace of the Einstein equation

$$-\beta^2 e^{W(r)} = \partial_r \left[4rU(r)e^{W(r)}\right] - \partial_r \left[r^2 U(r)(e^{W(r)})'\right] + \partial_r \left[(r^2 U(r)e^{W(r)})'\right] - \partial_r \left[Q^{rt}(r)A_t(r)\right]$$

$$\frac{S_{on}}{V_3} = 2\,\Lambda\, U(\Lambda) + \left|4rU(r)e^{W(r)} - r^2 U(r)(e^{W(r)})' + (r^2 U(r)e^{W(r)})' - Q^{rt}(r)A_t(r)\right|_{r_H}^{\Lambda} + S_c\Big|_{\Lambda}$$

To be Continued……

Thank you !!