Thermodynamic Stability Versus Chaos Bound Violation in Charged Black Hole

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Based on the work in cooperation with Xian-Hui Ge* and Surojit Dalui. (arXiv:2404.18193)

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Quantum Chaos and the upper bound on Lyapunov exponent

Quantum Chaos

Out-of-time-ordered correlator, OTOC

$$F(t) = \langle V(0)\, W(t)\, V(0)\, W(t) \rangle_\beta \sim 1 - \epsilon_c e^{\lambda t}$$

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The upper bound of Lyapunov exponent, **chaos bound**

$\lambda \leq \frac{2\pi T}{\hbar}$ (Depend on the temperature of system T)

*J. Maldacena, S. H. Shenker and D. Stanford, JHEP **08** (2016), 106*

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The research of Shock wave and Pole-skipping in black holes.

V. Jahnke, K. Y. Kim and J. Yoon, JHEP ***05*** *(2019), 037.*

Y. Liu and A. Raju, JHEP ***12*** *(2020), 027.*

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Chaos bound in classical motion

The static equilibrium of test particles with **external repulsive force** near black holes.

K. Hashimoto and N. Tanahashi, Phys. Rev. D ***95*** *(2017) no.2, 024007*

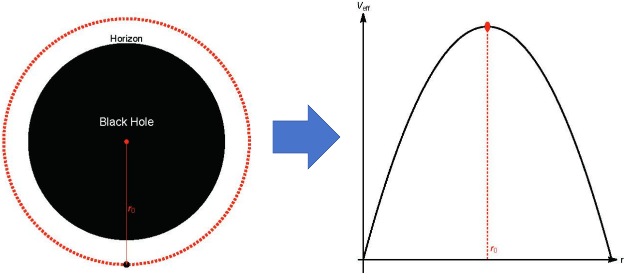

Figure 1: Schematic diagram of particle motion and radial effective potential V_{eff}.

Chaos bound in classical motion

The unstable equilibrium of particles near black holes

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \lambda^2 (r - r_0) \quad \rightarrow \quad r = r_0 + C_1 e^{\lambda t} + C_2 e^{-\lambda t}$$

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Near the horizon $\lambda = \kappa = 2\pi T$ $\rightarrow$ **Chaos bound in particle motion** $\lambda \leq \kappa$

A different physical background from QFT.

Unstable equilibrium $\sim$ The chaos bound

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S. Dalui, B. R. Majhi and P. Mishra, Phys. Lett. B ***788*** *(2019), 486-493*

M. Čubrović, JHEP ***12*** *(2019), 150*

D. Z. Ma, F. Xia, D. Zhang, G. Y. Fu and J. P. Wu, Eur. Phys. J. C ***82*** *(2022) no.4, 372*

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D. Z. Ma, F. Xia, D. Zhang, G. Y. Fu and J. P. Wu, Eur. Phys. J. C ***82*** *(2022) no.4, 372*

S. Das, S. Dalui and R. Samanta, arxiv: 2405.09945.

Near-horizon expansion and the **violation** of chaos bound

With the near-horizon expansion, the Lyapunov exponent λ satisfies

$$\lambda^2 = \kappa^2 + \gamma(r - r_h) + \mathcal{O}\left((r - r_h)^2\right),$$

There is a **violation** for $\lambda \leq \kappa$ when $\gamma > 0$, and the value of γ depends on the black hole metric and the potential function.

The effects from the high order terms of near-horizon expansion!

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The violation of chaos bound in static equilibrium

- **Non-violation cases**: Reissner-Nordström (RN)BH and Reissner-Nordström Anti-de Sitter (RN-AdS) BH.
- **Violation cases**: Reissner-Nordström de Sitter (RN-dS) BH, BHs in Einstein-Maxwell-dilaton gravity, Einstein-Born-Infeld gravity and Einstein-Gauss-Bonnet-Maxwell gravity.

More research on particle motion and chaos bound

- Minimal length effect:

*F. Lu, J. Tao and P. Wang, JCAP **12** (2018), 036*

*X. Guo, K. Liang, B. Mu, P. Wang and M. Yang, Chin. Phys. C **45** (2021) no.2, 023115*

- The influence of angular momentum:

*N. Kan and B. Gwak, Phys. Rev. D **105** (2022) no.2, 026006.*

*B. Gwak, N. Kan, B. H. Lee and H. Lee, JHEP **09** (2022), 026.*

*Y. Q. Lei and X. H. Ge, Phys. Rev. D **105** (2022) no.8, 084011*

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The upper bound of Lyapunov exponent and its violation in particle motion.

The physical meaning ?

The Lyapunov exponent

D-dimensional spherically symmetrically charged black holes

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{g(r)} + d\Omega_{D-2}^2.$$

Test particles moving on the equatorial plane

$$\mathcal{L} = \frac{1}{2}\left(-f(r)\dot{t}^2 + \frac{\dot{r}^2}{g(r)} + r^2\dot{\phi}^2\right) - qA_t(r)\dot{t}, \quad q = \frac{e}{m},$$

" $\cdot$ " denotes the derivative of the proper time τ.

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Generalized momentum:

$$\pi_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} = -E,\ \pi_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{\dot{r}}{g(r)},\ \pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\dot{\phi} = L,\ \cdots$$

The Hamiltonian of particles

$$\mathcal{H} = \pi_\mu\dot{x}^\mu - \mathcal{L} = \frac{\pi_\phi^2 f + r^2(\pi_r^2 f^2 - (\pi_t + qA_t)^2)}{2r^2 f}.$$

The Lyapunov exponent

With the equation of motion in the proper time τ

$$\dot{x}^{\mu} = \frac{\partial \mathcal{H}}{\partial \pi_{\mu}}, \qquad \dot{\pi}_{\mu} = -\frac{\partial \mathcal{H}}{\partial x^{\mu}}.$$

The radical motion in coordinate time t

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = \frac{\pi_r f^2}{E - qA_t} := F_1,$$

$$\frac{d\pi_r}{dt} = \frac{\dot{\pi}_r}{\dot{t}} = \frac{f'\left(qA_t - E\right)}{2f} + \frac{f\left(2L^2 - r^3\pi_r^2 f'\right)}{2r^3\left(E - qA_t\right)} - qA_t' := F_2.$$

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Taking (r, π_r) as the phase space variables, the Jacobian matrix is

$$K_{ij} = \begin{pmatrix} \frac{\partial F_1}{\partial r} & \frac{\partial F_1}{\partial \pi_r} \\ \frac{\partial F_2}{\partial r} & \frac{\partial F_2}{\partial \pi_r} \end{pmatrix}.$$

The Lyapunov exponent

With the conditions $\pi_r = \frac{d\pi_r}{dt} = 0$ and $\dot{x}^\mu \dot{x}_\mu = -1$ near the position r_0 , the Jacobian matrix can be reduced

$$K_{ij} = \left.\begin{pmatrix} 0 & -\frac{f^2}{qA_t - E} \\ \left(\frac{(qA_t - E)f'}{2f}\right)' - qA_t'' - \left(\frac{L^2 f}{r^3(qA_t - E)}\right)' & 0 \end{pmatrix}\right|_{r=r_0} .$$

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Calculated by **the Jacobian matrix**, the Lyapunov exponent λ of the unstable circular orbit satisfies

$$\lambda^2 = \frac{1}{4}\left[f'^2 - \frac{4L^2 f^2 \left(A_t'\left(2L^2 + 3r^2\right) + rA_t''\left(L^2 + r^2\right)\right)}{r^2\left(L^2 + r^2\right)^2 A_t'} + f\left\{f'\left(\frac{4L^2}{rL^2 + r^3} + \frac{2A_t''}{A_t'}\right)\right\} - 2f''\right]\Bigg|_{r=r_0}$$

Near-horizon expansion

For non-extremal BHs , we can consider the Taylor expansion near the horizon $r = r_+$

$$\lambda^2 = \kappa^2 + \gamma_1 (r_0 - r_+) + \gamma_2 (r_0 - r_+)^2 + \mathcal{O}\left((r - r_+)^3\right),$$

$$\gamma_1 = \frac{1}{4}(f_1 g_2 - f_2 g_1) + 4\kappa^2 \left(\frac{L^2}{L^2 r_+ + r_+^3} + \frac{A_t''(r_+)}{2A_t'(r_+)} \right)$$

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The influence of angular momentum L:

The increase in $L \Rightarrow$ More larger $\gamma_1 \Rightarrow$ More likely to violate the chaos bound $\lambda \leq \kappa$

Black hole Thermodynamics

For D-dimensional RN black hole

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + d\Omega^2_{D-2},$$

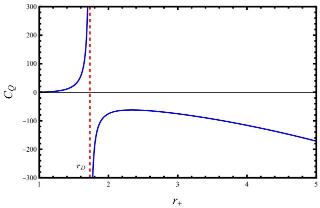
where $f(r) = 1 - \frac{2\bar{M}}{r^{D-3}} + \frac{\bar{Q}}{r^{2(D-3)}}$. The corresponding potential is $A_t(r) = \frac{Q}{(D-3)r^{D-3}}$. The parameters $\bar{M}$ and $\bar{Q}$ are related to the ADM mass M and the charge Q of black holes, which satisfy

$$M = \frac{(D-2)\omega_{D-2}}{8\pi}\bar{M}, \qquad Q^2 = \frac{(D-2)(D-3)\omega_{D-2}}{8\pi}\bar{Q}^2.$$

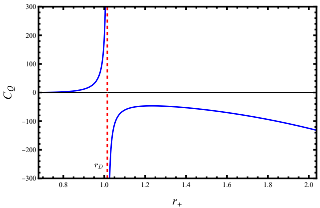
The heat capacity at constant charge C_Q is

$$C_Q = \frac{2(D-2)^2\pi^{D+\frac{1}{2}}r_+^{3D}T}{4(2D-5)\pi^{\frac{3}{2}}Q^2r_+^7\Gamma^2(\frac{D-1}{2}) - (D-2)(D-3)\pi^{\frac{D}{2}}r_+^{2D+1}\Gamma(\frac{D-1}{2})}.$$

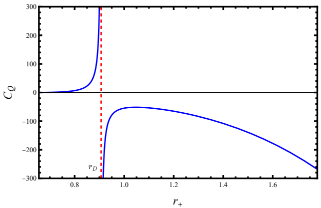
Black hole Thermodynamics



(a) $D = 4$



(b) $D = 5$



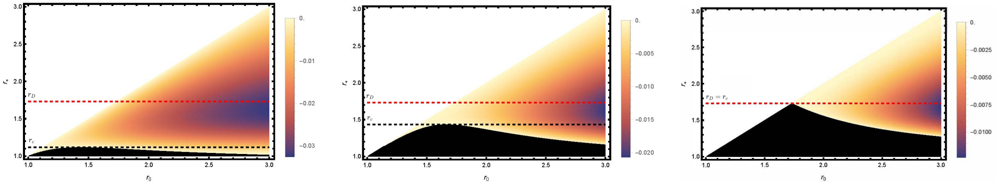
(c) $D = 6$

Figure 2: Heat capacity C_Q at constant charge $Q = 1$ with r_+ for D dimensional RN black holes. The figure (a), (b), (c) correspond to $D = 4,\ 5,\ 6$, respectively.

Thermodynamic phase transition point r_D (Davies points)

$$r_D = \left(\frac{\sqrt{D-2}\,\pi^{\frac{D+1}{4}}}{2\sqrt{(2D-5)\,Q^2\Gamma(\frac{D-3}{2})}} \right)^{\frac{1}{3-D}}$$

Numerical result of $\lambda^2 - \kappa^2$



(a) $D = 4$ and $L = 1$.

(b) $D = 4$ and $L = 5$.

(c) $D = 4$ and the large angular momentum limit.

Figure 3: The numerical results of $\lambda^2 - \kappa^2$ in 4-dimensional RN black holes with $Q = 1$.

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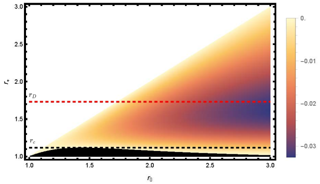

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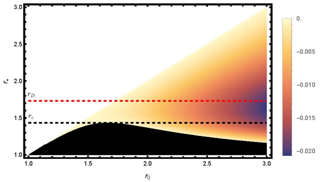

(b) $D = 4$ and $L = 5$.

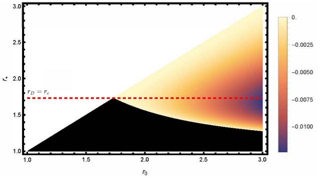

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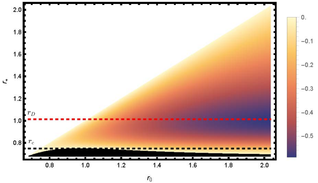
Figure 3: The numerical results of $\lambda^2 - \kappa^2$ in 4-dimensional RN black holes with $Q = 1$.

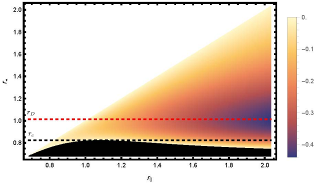
$$\bar{\lambda}^2 = \kappa^2 + \gamma_1 (r - r_h) + \gamma_2 (r - r_h)^2 + \mathcal{O}\left((r - r_h)^3\right), \quad \gamma_1 = 0,\ \gamma_2 = -\frac{6(Mr_h - 2Q^2)(Mr_h - Q^2)}{r_h^8}$$

$$\gamma_2 = 0 \quad \rightarrow \quad \bar{r}_c = \sqrt{3} \simeq 1.732$$

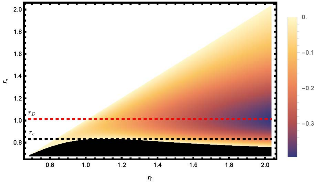
Numerical result of $\lambda^2 - \kappa^2$



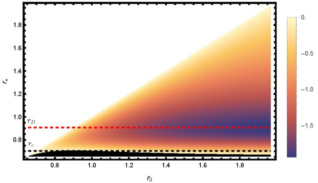
(a) $D = 5$ and $L = 1$.



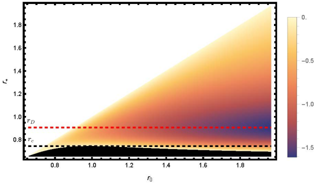
(b) $D = 5$ and $L = 5$.



(c) $D = 5$ and the large angular momentum limit.



(d) $D = 6$ and $L = 1$.



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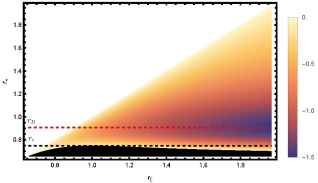


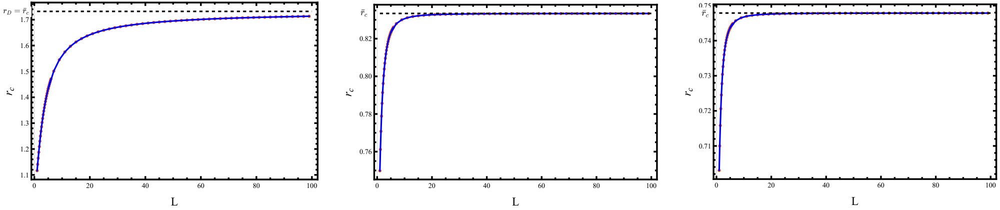
(f) $D = 6$ and the large angular momentum limit.

Numerical result of $\lambda^2 - \kappa^2$

DimensionAngular momentum L	$L = 1$	$L = 5$	Large L limit
D=4	$r_c = 1.116$	$r_c = 1.433$	$r_c = 1.732$
D=5	$r_c = 0.750$	$r_c = 0.823$	$r_c = 0.833$
D=6	$r_c = 0.703$	$r_c = 0.744$	$r_c = 0.748$

Table 1: The values of threshold parameter r_c for violating the chaos bound in different cases.

More discussion



(a) $D = 4$. (b) $D = 5$. (c) $D = 6$.

Figure 5: The threshold value r_c for violating chaos bound as a function of the angular momentum L in D-dimensional RN black holes. The figures (a), (b), (c) correspond to $D = 4,\ 5,\ 6$, respectively.

Summary

Black Hole Thermodynamic Stability $\sim$ Upper Bound of Lyapunov Exponent

1. For 4-dimensional RN black holes: Thermodynamically stable black holes correspond to scenarios where the Lyapunov exponent upper bound is exceeded.

2. For higher-dimensional RN black holes: The bound is only violated in the context of thermodynamically stable black holes.

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Research on Classical Aspects

Outlook

1. Investigate the universality of the relationship between black hole thermodynamic stability and the upper bound of the Lyapunov exponent.

2. Explore connections between the chaos bound and other properties of black holes.

3. Go back to quantum chaos.

And more $\cdots$

Thank You !